NUMERICAL SIMULATION OF THE FRACTIONAL LANGEVIN EQUATION

by

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In this paper, we study the fractional Langevin equation, whose derivative is in Caputo sense. By using the derived numerical algorithm, we obtain the displacement and the mean square displacement which describe the dynamic behaviors of the fractional Langevin equation.

Key words: *fractional Langevin equation, Riemann-Liouville derivative, Caputo derivative, mean square displacement, numerical simulation.*

1. Introduction

The fractional calculus has been studied for more than three hundred years. For a long time the fractional calculus has been studied only in the pure mathematical field. In history, Euler, Riemann, Liouville, Grünwald, Letnikov, Lebnitiz, L'Hospital, *et al.*, contributed to the fractional calculus [1-3]. It has not attracted more attention because of not finding applications. However, in recent few decades, the fractional calculus has been widely used in many fields such as chaotic dynamics, viscoelasticity, acoustics, physical chemistry, electromagnetics, signal processing, earthquake prediction, *etc.*. The study of fractional differential equations shifts from pure theory to real applications [2], [4-15]. Especially, the stochastic fractional differential equations have attracted increasing interests due to the potential applications. In effect, human evolutions are deterministic in local situations and in short terms but are random globally and in a long run. The stochastic fractional differential equation can reflect dual characters-stochasticity and globality. Hence, the stochastic fractional differential equation is possibly the better choice for characterizing human evolutions.

In this paper we will study the fractional Langevin equation where the fractional derivative is in Caputo sense. In 1908 the French physicist Langevin introduced the concept of the equation of motion with a random variable, which reads as,

$$m\frac{d^2x(t)}{dt^2} = -\gamma\frac{dx(t)}{dt} + F(x) + \xi(t), \tag{1}$$

where m is the mass of the particle, γ is the coefficient of viscosity, $F(x)$ is the external force, $\xi(t)$ is the random force. The Langevin equation is always regarded as the first stochastic differential equation. Although the classical Langevin equation has a fundamental role in so many areas such as physics, chemistry, signal processing, financial market, *etc.* There are still some dynamics, for example the anomalous diffusion (sub-diffusion and super-diffusion), power-law phenomena, long-tail character, long-range interaction, *etc.*, which can not be described by the classical Langevin equation. Therefore, the generalized Langevin equations have been introduced to model the above behaviors [16-23].

Among the generalized Langevin equations, the fractional version is often used, which is in the following form,

$$\frac{d^2x(t)}{dt^2} = -\gamma\int_0^t (t-\tau)^{-\alpha} x'(\tau)d\tau + F(x) + \xi(t), \tag{2}$$

where $0<\alpha<1$, γ is a constant, $F(x)$ is an external force field, $\xi(t)$ is a random force, and $\langle \xi(t)\rangle = 0$, $\langle \xi(t)\xi(s)\rangle = \delta(t-s)$. Here, $\int_0^t (t-\tau)^{-\alpha} x'(\tau)d\tau = \Gamma(1-\alpha)\,{}_C D_{o,t}^{\alpha} x(t)$, the fractional derivative ${}_C D_{o,t}^{\alpha} x(t)$ will be defined in the following section.

2. Preliminaries

First of all, we give some basic definitions. In general, fractional calculus includes both fractional integral (integration) and fractional differentiation. The fractional integral mainly means the Riemann-Liouville integral. For fractional differentiation, however there exist more than six kinds of fractional derivatives. Among them, the Riemann-Liouville derivative and the Caputo derivative are mostly utilized. In the following, we only introduce the fractional integral, the Riemann-Liouville derivative and the Caputo derivative [2], [3], [8-10].

Definition 2.1 *Fractional integral of function* $f(x)$ *with order* $\alpha>0$ *is defined by*

$$D_{a,x}^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-\tau)^{\alpha-1} f(\tau)d\tau. \tag{3}$$

Definition 2.2 *Riemann-Liouville fractional derivative of function* $f(x)$ *with order* $\alpha>0$ *is defined by*

$$ {}_{RL}D_{a,t}^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \int_a^x (x-\tau)^{m-\alpha-1} f(\tau) d\tau, \quad m-1 < \alpha < m. \tag{4} $$

Definition 2.3 *Caputo fractional derivative of function* $f(x)$ *with order* $\alpha > 0$ *is defined by*

$$ {}_{C}D_{a,t}^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_a^x (x-\tau)^{m-\alpha-1} f^{(m)}(\tau) d\tau, \quad m-1 < \alpha < m. \tag{5} $$

The above two fractional derivatives are not equivalent, whose important properties are listed below.

Property 2.1 *Assume* $m-1 < \alpha < m \in Z^+$, $\beta > 0$, *and that suitable smooth conditions are satisfied,* $n \in Z^+$ *.Then the following equalities hold:*

1. $ {}_{C}D_{a,x}^{\alpha} f(x) = {}_{RL}D_{a,x}^{\alpha} \left[f(x) - \sum_{k=0}^{m-1} \frac{f^{(k)}(a)}{\Gamma(k+1)} (x-a)^k \right]; $

2. $ D_{a,x}^{-\alpha} \cdot {}_{RL}D_{a,x}^{\alpha} f(x) = f(x) - \sum_{k=0}^{m-1} \frac{[{}_{RL}D_{a,x}^{\alpha-k} f(x)](a)}{\Gamma(\alpha-k+1)} (x-a)^k; $

3. $ {}_{RL}D_{a,x}^{\alpha} \cdot {}_{RL}D_{a,x}^{-\beta} f(x) = {}_{RL}D_{a,x}^{\alpha-\beta} f(x); $

4. $ {}_{C}D_{a,x}^{\alpha} f^{(n)}(x) = {}_{C}D_{a,x}^{\alpha+n} f(x), $

where $[{}_{RL}D_{a,x}^{\alpha-k} f(x)](a) = [{}_{RL}D_{a,x}^{\alpha-k} f(x)]_{x=a}$.

3. Algorithm for the fractional Langevin equation

In this paper we will give the numerical simulation for the fractional Langevin equation (2).

3.1. Without external force

In the following, we consider the force free type, i.e. $F(x) = 0$. The above equation can be written as the following form

$$ \frac{d^2 x(t)}{dt^2} = -\gamma \int_0^t (t-\tau)^{-\alpha} x'(\tau) d\tau + \xi(t), \tag{6} $$

then integrating the both sides of the above equation yields

$$x'(t_{k+1}) - x'(t_k) = -\frac{\gamma}{\Gamma(1-\alpha)}\int_0^{t_{k+1}} (t_{k+1}-\tau)^{-\alpha} x(\tau)d\tau + \frac{\gamma}{\Gamma(1-\alpha)}\int_0^{t_k} (t_k-\tau)^{-\alpha} x(\tau)d\tau$$
$$+\frac{\gamma x(0)}{\Gamma(1-\alpha)}(t_{k+1}^{1-\alpha} - t_k^{1-\alpha}) + \int_{t_k}^{t_{k+1}} dW(t). \tag{7}$$

The first integral term and the second one in (7) are approximated by the rectangle formula. For the first integral part in the right hand side

$$\int_0^{t_{k+1}} (t_{k+1}-\tau)^{-\alpha} x(\tau)d\tau \approx \sum_{j=0}^{k} b_{j,k+1} x(t_j), \tag{8}$$

then we can calculate the coefficients

$$b_{j,k+1} = \frac{h^{1-\alpha}}{1-\alpha}[(k+1-j)^{1-\alpha} - (k-j)^{1-\alpha}]. \tag{9}$$

Similarly, for the second term in the right hand side

$$\int_0^{t_k} (t_k-\tau)^{-\alpha} x(\tau)d\tau \approx \sum_{j=0}^{k-1} c_{j,k} x(t_j), \tag{10}$$

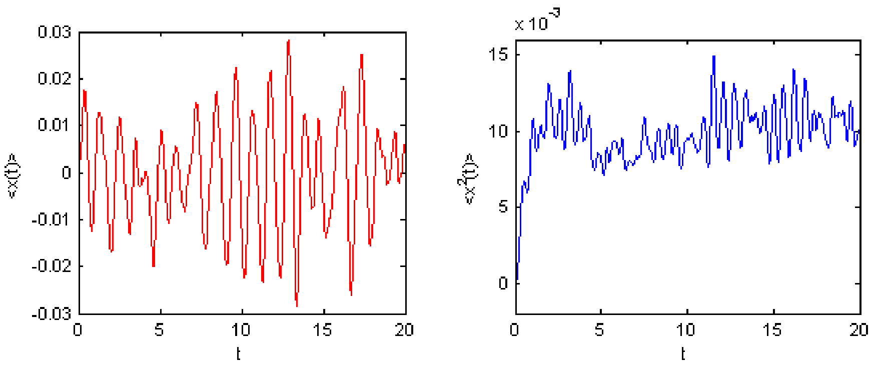
we can also get the coefficients

$$c_{j,k} = \frac{h^{1-\alpha}}{1-\alpha}[(k-j)^{1-\alpha} - (k-j-1)^{1-\alpha}]. \tag{11}$$

For the left hand side of equation (7), we use the finite difference method to approximate it. Therefore we get the following discrete formula.

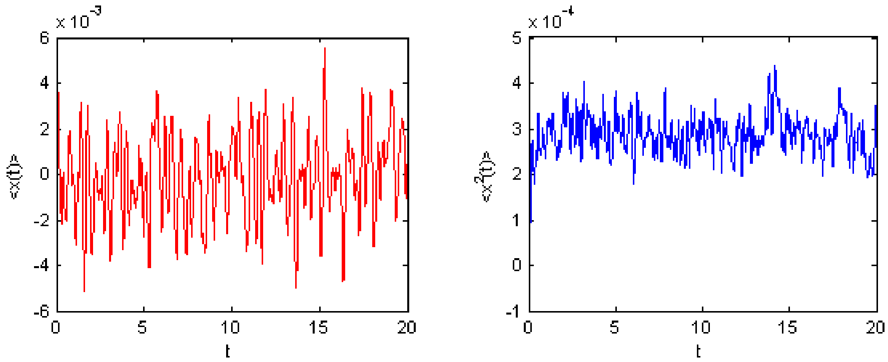
$$x_{k+1} = 2x_k - x_{k-1} - \frac{\gamma h}{\Gamma(1-\alpha)}\sum_{j=0}^{k} b_{k-j} y_j + \frac{\gamma h}{\Gamma(1-\alpha)}\sum_{j=0}^{k-1} b_{k-j-1} y_j$$
$$-\frac{\gamma h x(0)}{\Gamma(1-\alpha)}(t_k^{1-\alpha} - t_{k+1}^{1-\alpha}) + h(W(t_{k+1}) - W(t_k)), \quad k \ge 2, \tag{12}$$

where $W(t)$ is a Wiener process with mean zero, and $dW(t)$ is in Itô sense, i.e., an independent increment random. With the above algorithm we simulate its dynamic behaviors. The following figures give the displacement and the mean square displacement under different parameters. Here we take $x(0) = 0, \quad x(1) = 0.001.$



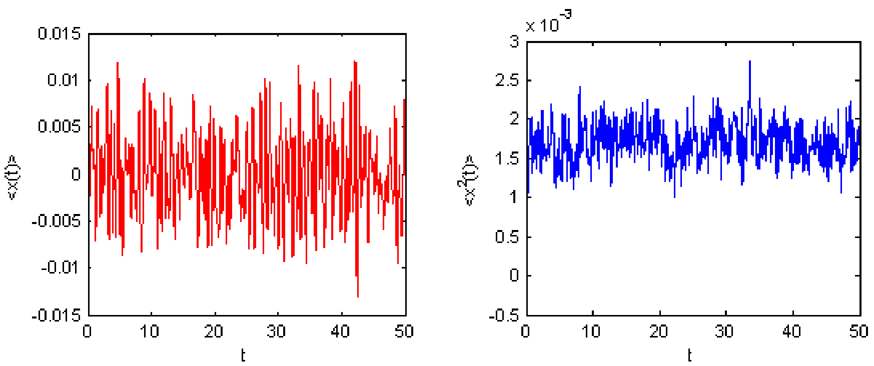
(a1) (a2)

Figure a. The numerical simulation of equation (2), where $\alpha = 0.9$, $\gamma = 0.1$. (a1) The displacement, (a2) The mean square displacement.



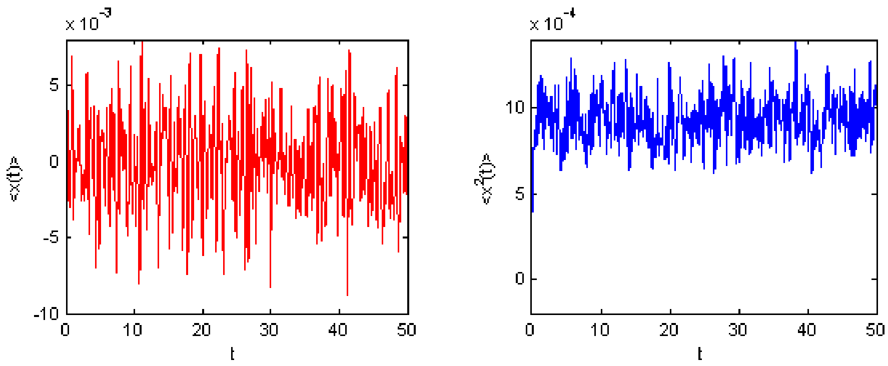
(b1) (b2)

Figure b. The numerical simulation of equation (2), where $\alpha = 0.9$, $\gamma = 0.6$. (b1) The displacement, (b2) The mean square displacement.



(c1) (c2)

Figure c. The numerical simulation of equation (2), where $\alpha = 0.8$, $\gamma = 0.7$. (c1) The displacement, (c2) The mean square displacement.



(d1) (d2)

Figure d. The numerical simulation of equation (2), where $\alpha = 0.8$, $\gamma = 0.95$. (d1) The displacement, (d2) The mean square displacement.

The Figs. a1, b1, c1, and d1 indicate the displacement. Figs. a2, b2, c2, and d2 indicate the mean square displacement. From the above figures we find that the displacement depends heavily on the damping coefficient. If the damping coefficient is relatively smaller, the displacement is bigger; if the damping coefficient is relatively bigger, the displacement is smaller.

3.2. With constant external force

In the following part, we consider the fractional Langevin equation with a constant external force, i.e. $F(x) = F \neq 0$.

$$\frac{d^2x(t)}{dt^2} = -\gamma \int_0^t (t-\tau)^{-\alpha} x'(\tau) d\tau + F + \xi(t). \tag{13}$$

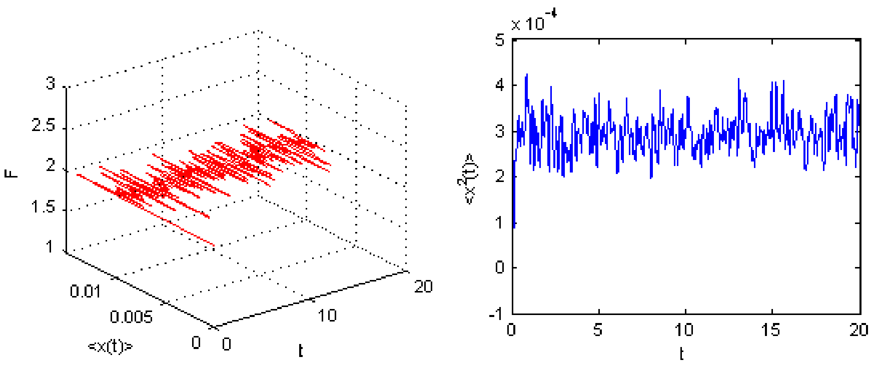
The algorithm is almost similar to that for the fractional Langevin equation without external force.

$$\begin{aligned} x_{k+1} &= 2x_k - x_{k-1} - \frac{\gamma h}{\Gamma(1-\alpha)} \sum_{j=0}^{k} b_{k-j} x_j + \frac{\gamma h}{\Gamma(1-\alpha)} \sum_{j=0}^{k-1} b_{k-j-1} x_j \\ &- \frac{\gamma h x(0)}{\Gamma(1-\alpha)} (t_k^{1-\alpha} - t_{k+1}^{1-\alpha}) + h(W(t_{k+1}) - W(t_k)) + F_k h^2, \quad k \geq 2, \end{aligned} \tag{14}$$

where

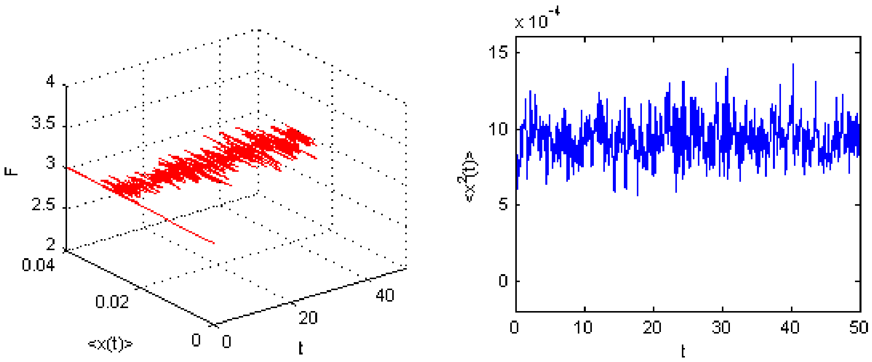
$$b_{j,k+1} = \frac{h^{1-\alpha}}{1-\alpha} [(k+1-j)^{1-\alpha} - (k-j)^{1-\alpha}]. \tag{15}$$

Then we display the figures with different parameters.



(e1) (e2)

Figure e. The numerical simulation of equation (2), where $F=2$, $\alpha=0.9$, $\gamma=0.6$. **(e1) The displacement, (e2) The mean square displacement.**



(f1) (f2)

Figure f. The numerical simulation of equation (2), where $F=3$, $\alpha=0.8$, $\gamma=0.95$. **(f1) The displacement, (f2) The mean square displacement.**

4. Conclusion

In this paper we study the generalized Langevin equation with a memory kernel in the damping item, i.e. the fractional Langevin equation in Caputo sense. We study two cases of the fractional Langevin equation (i.e., without force and with constant external force). Finally we give an algorithm and numerical experiments with different parameters. From the numerical simulations, we find that the displacement is bigger if the coefficient of the damping item is relatively smaller and vice versa.

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