EXACT SOLUTIONS FOR ROTATIONAL FLOW OF A FRACTIONAL MAXWELL FLUID IN A CIRCULAR CYLINDER

by

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This paper deals with the rotational flow of a fractional Maxwell fluid in an infinite circular cylinder, due to the torsional variable time-dependent shear stress that is prescribed on the boundary of the cylinder. The fractional calculus approach in the constitutive relationship model of a Maxwell fluid is introduced. The velocity field and the resulting shear stress are determined by means of the Laplace and finite Hankel transforms to satisfy all imposed initial and boundary conditions. The solutions corresponding to ordinary Maxwell fluids as well as those for Newtonian fluids, performing the same motion, are obtained as limiting cases of our general solutions. Finally, the influence of the relaxation time and the fractional parameter on the velocity of the fluid is analyzed by graphical illustrations.

Keywords: *Fractional Maxwell fluid, Fractional Calculus, Velocity field, Shear stress, Exact solutions.*

1. Introduction

The non-Newtonian fluids, such as polymer solutions, paints, grease, blood, oils, glycerin etc, are frequently encountered in many disciplinary fields, such as chemical engineering, biomedicine, food stuff etc., and also are closely related to many industrial processes. Typical non-Newtonian characteristics include shear thinning, viscoplasticity, viscoelasticity and shear thickening behavior. Because of these complexities, there are several models of non-Newtonian fluids in the literature.

In the category of non-Newtonian fluids, the fluids of differential type have acquired special status as well as much controversy [1]. These fluids cannot describe the influence of relaxation and retardation times. Among non-Newtonian fluid models that are capable of describing these effects are the rate type models [2, 3]. The first viscoelastic rate type model is due to Maxwell [4] and this model has had some success in describing the response of some polymeric liquids.

Moreover, in order to describe the viscoelasticity [5, 6] the fractional calculus approach is very important. Recently, Fractional Calculus has encountered much success in the description of viscoelasticity. In particular, it has been proved to be a valuable tool in handle viscoelastic properties. The starting point of the fractional derivative model of viscoelastic

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viscoelastic fluids is usually a classic differential equation which is modified by replacing the time derivative of an integer order by the so-called Riemann-Liouville fractional calculus operator. This generalization allows us to define precisely non-integer order integrals or derivatives [7, 8]. So, due to the importance of viscoelasticity, many exact solutions corresponding to different motions of non-Newtonian fluids with fractional derivatives have been established, but we mention here only a few in cylindrical domain [9-19]. Furthermore, the one-dimensional fractional derivative Maxwell model has been found very useful in modeling the linear viscoelastic response of some polymers in the glass transition and the glass state [20]. In other cases it has been shown that the governing equations employing fractional derivatives are also linked to molecular theories [21].

The aim of this paper is to present exact solutions for the unsteady rotational flow of an incompressible generalized Maxwell fluid with a fractional derivative model within an infinite circular cylinder of radius R. Generally, in one dimension the constitutive equation of a generalized Maxwell fluid with fractional derivative can be expressed as [21, 22]

$$\tau(t) + \lambda^\beta D_t^\beta \tau(t) = \mu \frac{d\varepsilon(t)}{dt}, \tag{1}$$

where $\tau(t)$ is the shear stress, $\varepsilon(t)$ is the shear strain, λ is the relaxation time, μ is the dynamic viscosity of the fluid and β is the fractional coefficient such that $0 \le \beta \le 1$. Also, D_t^β is the Riemann-Liouville fractional differential operator defined as [7, 8].

$$D_t^\beta [f(t)] = \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_0^t \frac{f(u)}{(t-u)^\beta}\, du, \quad 0 < \beta < 1 \tag{2}$$

where $\Gamma(\cdot)$ is the Gamma function. This model can be reduced to the ordinary Maxwell model when $\beta \to 1$, because in this case $D_t^1 f = df(t)/d(t)$. Furthermore, this model can be reduce to the classical Newtonian model for $\beta \to 1$ and $\lambda \to 0$. The flow of the fluid is generated by the shear stress which is prescribed on the surface of the cylinder in the form

$$\tau(R,t) = \frac{f}{\lambda} \int_0^\infty (t-s)^a\, G_{\beta,0,1}\left(-\frac{1}{\lambda}, s\right) ds, \quad t > 0 \tag{3}$$

where f is a constant, $a \ge 0$ and $G_{a,b,c}(\cdot,\cdot)$ is the generalized G-function [23]. The velocity fields and the resulting shear stresses, obtained by means of the Laplace and finite Hankel transforms, are presented in terms of generalized G-functions. The solutions corresponding to ordinary Maxwell fluids or Newtonian fluids are obtained as special cases of our solutions. Finally, for comparison, the profiles of the velocity $\omega(r,t)$, for Newtonian, Maxwell and generalized Maxwell fluids, for different values of the fractional coefficient β and for a constant shear stress on the boundary, are plotted as functions of cylindrical coordinate r.

2. Governing equations

Let us consider an incompressible fractional Maxwell fluid at rest in an infinite circular cylinder of radius R. At time $t = 0^+$ the cylinder begins to rotate around its axis due to a time-dependent torque (3). Obviously the motion of fractional Maxwell fluid is axial symmetric, so we choose the cylindrical coordinates (r,θ,z), and the components of velocity are $v_r = 0,\ v_\theta = \omega(r,t),\ v_z = 0.$ Under the above assumptions, the constitutive equation of fractional Maxwell fluid is [10].

$$\tau(r,t) + \lambda^\beta D_t^\beta \tau(r,t) = \mu\left(\frac{\partial}{\partial r} - \frac{1}{r}\right)\omega(r,t) \tag{4}$$

where $\tau(r,t) = \tau_{r\theta}(r,t)$ is the component of shear stress which is different of zero and (for simplicity, we take $\lambda = \lambda^\beta$ hereinafter).

In absence of the pressure gradient and neglecting the body forces, the balance of linear momentum leads to the partial differential equation [10]

$$\rho\frac{\partial\omega(r,t)}{\partial t} = \left(\frac{\partial}{\partial r} - \frac{2}{r}\right)\tau(r,t), \tag{5}$$

where ρ is the constant density of the fluid.

By eliminating τ between eqs. (4) and (5), we obtain the following motion equation of fractional Maxwell fluid:

$$\left(1+\lambda D_t^\beta\right)\frac{\partial\omega(r,t)}{\partial t} = \nu\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right)\omega(r,t),\ \ r\in(0,R),\ \ t>0, \tag{6}$$

where $\nu = \mu/\rho$ is the kinematic viscosity of the fluid. The appropriate initial and boundary conditions are

$$\omega(r,0) = \frac{\partial\omega(r,0)}{\partial t} = 0,\ \ r\in[0,R], \tag{7}$$

$$\left(1+\lambda D_t^\beta\right)\tau(R,t) = \nu\left[\frac{\partial\omega(r,t)}{\partial r} - \frac{\omega(r,t)}{r}\right]_{r=R} = ft^a,\ \ t>0, \tag{8}$$

where f is a constant and $a \ge 0.$

Of course, $\tau(R,t)$ given by eq. (3) is just the solution of the fractional differential equation $(8)_1$. To solve this problem we shall use as in [10, 14, 15], the Laplace and finite Hankel transforms.

3. Analytical solution of the model

Analytical solution of the aforementioned problem (6)-(8) is obtained by using the finite Hankel transform with respect to r and the Laplace transform with respect to variable t. Applying the Laplace transform to eqs. (6) and $(8)_2$, using $(7)_{1,2}$ and formulae [8]

$$L\left[D_t^\beta f(t)\right] = q^\beta L\left[f(t)\right],\ \ L[t^a] = \frac{\Gamma(a+1)}{q^{a+1}},\ \ a>-1, \tag{9}$$

We obtain the following problem with boundary condition

$$\left(q+\lambda q^{\beta+1}\right)\overline{\omega}(r,q)=\nu\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}\right)\overline{\omega}(r,q), \tag{10}$$

$$\left[\frac{\partial\overline{\omega}(r,q)}{\partial r}-\frac{\overline{\omega}(r,q)}{r}\right]_{r=R}=\frac{f}{\mu}\frac{\Gamma(a+1)}{q^{a+1}}, \tag{11}$$

Where $\overline{\omega}(r,q)=\int_0^\infty \omega(r,t)\exp(-qt)dt$ is the Laplace transform of function $\omega(r,t)$ and q is the transform parameter.

In the following let us denote by [24]

$$\overline{\omega}_H(r_n,q)=\int_0^R r\overline{\omega}(r,q)J_1(rr_n)dr \tag{12}$$

the finite Hankel transform of $\overline{\omega}(r,q)$, where $r_n,\ n=1,\,2,\,3\cdots$ are the positive roots of the transcendental equation $J_2(Rr)=0$. In the above relations, $J_\nu(\cdot)$ is the first-kind, ν-order Bessel function.

By using the following formulae [25, 26]

$$\begin{aligned}\frac{d}{dr}J_1\left[u(r)\right]&=\left[\frac{1}{u(r)}J_1\left[u(r)\right]-J_2\left[u(r)\right]\right]u'(r),\\ \frac{d}{dr}J_2\left[u(r)\right]&=\left[J_1\left[u(r)\right]-\frac{2}{u(r)}J_2\left[u(r)\right]\right]u'(r),\end{aligned} \tag{13}$$

we obtain that

$$\int_0^R r\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}\right)\overline{\omega}(r,q)J_1(rr_n)dr=RJ_1(Rr_n)\left[\frac{\partial\overline{\omega}(r,q)}{\partial r}-\frac{\overline{\omega}(r,q)}{r}\right]_{r=R}-r_n^2\,\overline{\omega}_H(r_n,q). \tag{14}$$

Applying the Hankel transform to eq. (10) and taking into account eqs. (11) and (14), we find that

$$\overline{\omega}_H(r_n,q)=RfJ_1(Rr_n)\Gamma(a+1)\frac{1}{\rho q^{a+1}\left(q+\lambda q^{\beta+1}+\nu r_n^2\right)} \tag{15}$$

or equivalently,

$$\overline{\omega}_H(r_n,q)=\frac{RfJ_1(Rr_n)}{\mu r_n^2}\frac{\Gamma(a+1)}{q^{a+1}}-\frac{RfJ_1(Rr_n)}{\mu r_n^2}\frac{\Gamma(a+1)}{q^a}\frac{1+\lambda q^\beta}{q+\lambda q^{\beta+1}+\nu r_n^2}. \tag{16}$$

The inverse Hankel transform of the function $\bar{\omega}_H(r_n, q)$ is [24]

$$\bar{\omega}_H(r_n, q) = \frac{fr^3}{2\mu R^2}\frac{\Gamma(a+1)}{q^{a+1}} - \frac{2f}{\mu R}\sum_{n=1}^{\infty}\frac{J_1(rr_n)}{r_n J_1^2(Rr_n)}\frac{\Gamma(a+1)}{q^a}\frac{1+\lambda q^\beta}{q+\lambda q^{\beta+1}+\nu r_n^2}. \tag{17}$$

In order to determine the inverse Laplace transform of function $\bar{\omega}(r,q)$ we introduce the following functions:

$$F_1(q) = \frac{\Gamma(a+1)}{q^{a+1}}, \quad F_2(q) = \frac{\Gamma(a+1)}{q^a}, \tag{18}$$

$$F_3(r_n, q) = \frac{1+\lambda q^\beta}{q+\lambda q^{\beta+1}+\nu r_n^2} = \frac{q^{\beta-1}+\frac{1}{\lambda}q^{-1}}{\left(q^\beta+\frac{1}{\lambda}\right)+\frac{\nu r_n^2}{\lambda}q^{-1}} = \sum_{k=0}^{\infty}\left(\frac{-\nu r_n^2}{\lambda}\right)^k \frac{q^{\beta-k-1}+\frac{1}{\lambda}q^{-k-1}}{\left(q^\beta+\frac{1}{\lambda}\right)^{k+1}}. \tag{19}$$

Using $(9)_2$ and the known result [23, eq. (97)]

$$L^{-1}\left\{\frac{q^b}{(q^a-d)^c}\right\} = G_{a,b,c}(d,t), \quad \mathrm{Re}(ac-b)>0, \quad \mathrm{Re}(q)>0, \quad |d|>|q^a|, \tag{20}$$

where the generalized $G_{a,b,c}(\cdot,t)$ function is defined by [23, eqs. (99) and (101)]

$$G_{a,b,c}(d,t) = \sum_{j=0}^{\infty}\frac{d^j\Gamma(c+j)}{\Gamma(c)\Gamma(j+1)}\frac{t^{(c+j)a-b-1}}{\Gamma[(c+j)a-b]}, \tag{21}$$

we find the inverse Laplace transforms of functions F_i, $i=1,2,3$ are [27]

$$f_1(t) = L^{-1}[F_1(q)] = t^a, \ a \ge 0, \quad f_2(t) = L^{-1}[F_2(q)] = at^{a-1}, \ a>0, \tag{22}$$

$$f_3(r_n,t) = L^{-1}[F_3(r_n,q)] = \sum_{k=0}^{\infty}\left(\frac{-\nu r_n^2}{\lambda}\right)^k\left[G_{\beta,\beta-k-1,k+1}\left(-\frac{1}{\lambda},t\right)+\frac{1}{\lambda}G_{\beta,-k-1,k+1}\left(-\frac{1}{\lambda},t\right)\right]. \tag{23}$$

Now, in order to avoid the tedious calculations of residues and contour integrals, we apply the discrete inverse Laplace transform method [10–15] to eq. (17), using (20), $(22)_2$, (23) and the convolution theorem we find the velocity field $\omega(r,t)$ in the following forms.

If $a>0$, then

$$\omega(r,t)=\frac{fr^3}{2\mu R^2}t^a-\frac{2af}{\mu R}\sum_{n=1}^{\infty}\frac{J_1(rr_n)}{r_n{}^2J_1(Rr_n)}\sum_{k=0}^{\infty}\left(\frac{-\nu r_n{}^2}{\lambda}\right)^k\times$$
$$\times\int_0^t(t-s)^{a-1}\left[G_{\beta,\beta-k-1,k+1}\left(-\frac{1}{\lambda},s\right)+\frac{1}{\lambda}G_{\beta,-k-1,k+1}\left(-\frac{1}{\lambda},s\right)\right]ds, \tag{24}$$

and if $a=0,$ then

$$\omega(r,t)=\frac{fr^3}{2\mu R^2}-\frac{2f}{\mu R}\sum_{n=1}^{\infty}\frac{J_1(rr_n)}{r_n{}^2J_1(Rr_n)}\times$$
$$\times\sum_{k=0}^{\infty}\left(\frac{-\nu r_n{}^2}{\lambda}\right)^k\left[G_{\beta,\beta-k-1,k+1}\left(-\frac{1}{\lambda},t\right)+\frac{1}{\lambda}G_{\beta,-k-1,k+1}\left(-\frac{1}{\lambda},t\right)\right], \tag{25}$$

where

$$h(t)*g(t)=\int_0^t h(t-z)g(z)dz=\int_0^t h(z)g(t-z)dz, \tag{26}$$

represents the convolution of functions h and g.

3.1. Calculation of the shear stress

Applying the Laplace transforms to eq. (4), and using the initial condition $(7)_3$, we find that

$$(1+\lambda q^{\beta})\bar{\tau}(r,q)=\mu\left(\frac{\partial}{\partial r}-\frac{1}{r}\right)\bar{\omega}(r,q). \tag{27}$$

Differentiating eq. (17) with respect to r and using the identity $(13)_2$ we find that

$$\bar{\tau}(r,q)=\frac{fr^2}{R^2}\frac{\Gamma(a+1)}{q^{a+1}}\frac{1}{1+\lambda q^{\beta}}+\frac{2f}{R}\sum_{n=1}^{\infty}\frac{J_2(rr_n)}{r_nJ_1(Rr_n)}\frac{\Gamma(a+1)}{q^a}\frac{1}{q+\lambda q^{\beta+1}+\nu r_n{}^2}. \tag{28}$$

To determine the inverse Laplace transform of function $\bar{\tau}(r,q),$ we introduce the following functions:

$$F_4(q)=\frac{1}{1+\lambda q^{\beta}}=\frac{1}{\lambda}\frac{1}{q^{\beta}+\frac{1}{\lambda}}, \tag{29}$$

$$F_5(r_n,q)=\frac{1}{q+\lambda q^{\beta+1}+\nu r_n{}^2}=\frac{1}{\lambda q}\frac{1}{\left(q^{\beta}+\frac{1}{\lambda}\right)+\frac{\nu r_n{}^2}{\lambda}q^{-1}}=\sum_{k=0}^{\infty}\left(\frac{-\nu r_n{}^2}{\lambda}\right)^k\frac{q^{-k-1}}{\left(q^{\beta}+\frac{1}{\lambda}\right)^{k+1}}. \tag{30}$$

The inverse Laplace transform of the above functions by using (20) are [27]

$$f_4(t) = L^{-1}\left[F_4(q)\right] = \frac{1}{\lambda} G_{\beta,0,1}\left(-\frac{1}{\lambda}, t\right), \qquad (31)$$

$$f_5(r_n, t) = L^{-1}\left[F_5(r_n, q)\right] = \sum_{k=0}^{\infty} \left(\frac{-\nu r_n^2}{\lambda}\right)^k G_{\beta,-k-1,k+1}\left(-\frac{1}{\lambda}, t\right), \qquad (32)$$

Applying again the discrete inverse Laplace transform to eq. (28), using $(22)_1$, (31), (32), the convolution theorem (26) and the identity

$$\int_0^t G_{a,b,c}\left(d,s\right) ds = G_{a,b-1,c}\left(d,t\right), \qquad (33)$$

we find the shear stress $\tau(r,t)$ in the following forms.

If $a > 0$, then

$$\tau(r,t) = \frac{fr^2}{\lambda R^2} \int_0^t (t-s)^a G_{\beta,0,1}\left(-\frac{1}{\lambda}, s\right) ds + \frac{2af}{\lambda R} \sum_{n=1}^{\infty} \frac{J_2(rr_n)}{r_n J_1(Rr_n)} \sum_{k=0}^{\infty} \left(\frac{-\nu r_n^2}{\lambda}\right)^k \times$$
$$\times \int_0^t (t-s)^{a-1} G_{\beta,-k-1,k+1}\left(-\frac{1}{\lambda}, s\right) ds, \qquad (34)$$

and if $a = 0$, then

$$\tau(r,t) = \frac{fr^2}{\lambda R^2} G_{\beta,-1,1}\left(-\frac{1}{\lambda}, t\right) + \frac{2f}{\lambda R} \sum_{n=1}^{\infty} \frac{J_2(rr_n)}{r_n J_1(Rr_n)} \sum_{k=0}^{\infty} \left(\frac{-\nu r_n^2}{\lambda}\right)^k G_{\beta,-k-1,k+1}\left(-\frac{1}{\lambda}, t\right). \qquad (35)$$

4. Limiting Cases

4.1. Flow of a Maxwell fluid due to torsional constant shear stress $(\beta = 1,\ a = 0)$

For $\beta \to 1$ our model is reduced to the ordinary Maxwell fluid and for $a = 0$ the shear stress on the boundary of the cylinder is constant, equal with f.

Making $\beta \to 1$ into eqs. (25) and (35), and by using the following identity

$$G_{1,-1,1}\left(-\frac{1}{\lambda}, t\right) = \lambda\left[1 - \exp\left(-\frac{t}{\lambda}\right)\right],$$

We obtain the velocity field

$$\omega(r,t)=\frac{fr^3}{2\mu R^2}-\frac{2f}{\mu R}\sum_{n=1}^{\infty}\frac{J_1(rr_n)}{r_n^{\,2}J_1(Rr_n)}\sum_{k=0}^{\infty}\left(\frac{-\nu r_n^{\,2}}{\lambda}\right)^k\times$$
$$\times\left[G_{1,-k,k+1}\left(-\frac{1}{\lambda},t\right)+\frac{1}{\lambda}G_{1,-k-1,k+1}\left(-\frac{1}{\lambda},t\right)\right], \tag{36}$$

and the associated shear stress

$$\tau(r,t)=\frac{fr^2}{R^2}\left[1-\exp\left(-\frac{t}{\lambda}\right)\right]+\frac{2f}{\lambda R}\sum_{n=1}^{\infty}\frac{J_2(rr_n)}{r_nJ_1(Rr_n)}\sum_{k=0}^{\infty}\left(\frac{-\nu r_n^{\,2}}{\lambda}\right)^k G_{1,-k-1,k+1}\left(-\frac{1}{\lambda},t\right), \tag{37}$$

corresponding to the ordinary Maxwell fluid, performing the same motion.

4.2. Flow of a Newtonian fluid due to torsional constant shear stress ($\beta=1,\ a=0,\ \lambda\to 0$)

Assuming $\lambda\to 0$ in eqs. (36) and (37), the known solutions

$$\omega(r,t)=\frac{fr^3}{2\mu R^2}-\frac{2f}{\mu R}\sum_{n=1}^{\infty}\frac{J_1(rr_n)}{r_n^{\,2}J_1(Rr_n)}\exp(-\nu r_n^{\,2}t), \tag{38}$$

and

$$\tau(r,t)=\frac{fr^2}{R^2}+\frac{2f}{R}\sum_{n=1}^{\infty}\frac{J_2(rr_n)}{r_nJ_1(Rr_n)}\exp(-\nu r_n^{\,2}t), \tag{39}$$

corresponding to the Newtonian fluid are recovered [28, eqs. (4.9) and (4.10)].

4.3. Flow of a Maxwell fluid due to torsional time-variable shear stress ($\beta=1,\ a>0$)

Assuming $\beta\to 1$ in eqs. (24) and (34) we obtain the velocity field

$$\omega(r,t)=\frac{fr^3}{2\mu R^2}t^a-\frac{2af}{\mu R}\sum_{n=1}^{\infty}\frac{J_1(rr_n)}{r_n^{\,2}J_1(Rr_n)}\sum_{k=0}^{\infty}\left(\frac{-\nu r_n^{\,2}}{\lambda}\right)^k\times$$
$$\times\int_0^t(t-s)^{a-1}\left[G_{1,-k,k+1}\left(-\frac{1}{\lambda},s\right)+\frac{1}{\lambda}G_{1,-k-1,k+1}\left(-\frac{1}{\lambda},s\right)\right]ds, \tag{40}$$

and the associated shear stress

$$\tau(r,t)=\frac{fr^2}{\lambda R^2}\int_0^t(t-s)^a\exp\left(-\frac{s}{\lambda}\right)ds+\frac{2af}{\lambda R}\sum_{n=1}^{\infty}\frac{J_2(rr_n)}{r_nJ_1(Rr_n)}\times$$
$$\times\sum_{k=0}^{\infty}\left(\frac{-\nu r_n^{\,2}}{\lambda}\right)^k\int_0^t(t-s)^{a-1}G_{1,-k-1,k+1}\left(-\frac{1}{\lambda},s\right)ds. \tag{41}$$

For $a = 1$, the expressions (40) and (41), by using the identity (33) can be written in the simplified forms

$$\omega(r,t) = \frac{fr^3}{2\mu R^2} t - \frac{2f}{\mu R}\sum_{n=1}^{\infty}\frac{J_1(rr_n)}{r_n^{\,2} J_1(Rr_n)}\sum_{k=0}^{\infty}\left(\frac{-\nu r_n^{\,2}}{\lambda}\right)^k \times$$
$$\times\left[G_{1,-k-1,k+1}\left(-\frac{1}{\lambda},t\right) + \frac{1}{\lambda}G_{1,-k-2,k+1}\left(-\frac{1}{\lambda},t\right)\right], \tag{42}$$

and

$$\tau(r,t) = \frac{fr^2}{R^2}\left[t - \lambda\left(1 - \exp\left(-\frac{t}{\lambda}\right)\right)\right] + \frac{2f}{\lambda R}\sum_{n=1}^{\infty}\frac{J_2(rr_n)}{r_n J_1(Rr_n)} \times$$
$$\times\sum_{k=0}^{\infty}\left(\frac{-\nu r_n^{\,2}}{\lambda}\right)^k G_{1,-k-2,k+1}\left(-\frac{1}{\lambda},t\right), \tag{43}$$

respectively, obtained in [11, eqs. (19) and (20)] by different techniques.

4.4. Flow of a Newtonia fluid due to torsional time-variable shear stress $(\beta = 1,\ a > 0,\ \lambda \to 0)$

Assuming $\lambda \to 0$ in eqs. (40) and (41) or (42) and (43), the solutions

$$\omega(r,t) = \frac{fr^3}{2\mu R^2} t^a - \frac{2af}{\mu R}\sum_{n=1}^{\infty}\frac{J_1(rr_n)}{r_n^{\,2} J_1(Rr_n)}\int_0^t (t-s)^{a-1}\exp(-\nu r_n^{\,2} s)ds, \tag{44}$$

and

$$\tau(r,t) = \frac{fr^2}{\lambda R^2} t^a + \frac{2af}{\lambda R}\sum_{n=1}^{\infty}\frac{J_2(rr_n)}{r_n J_1(Rr_n)}\int_0^t (t-s)^{a-1}\exp(-\nu r_n^{\,2} s)ds, \tag{45}$$

corresponding to the Newtonian fluid are obtained.

For $a = 1$ in (44) and (45), the simple solutions

$$\omega(r,t) = \frac{fr^3}{2\mu R^2} t - \frac{2f}{\nu\mu R}\sum_{n=1}^{\infty}\frac{J_1(rr_n)}{r_n^{\,4} J_1(Rr_n)}\left[1 - \exp(-\nu r_n^{\,2} t)\right], \tag{46}$$

$$\tau(r,t) = \frac{fr^2}{R^2} t + \frac{2f}{\nu R}\sum_{n=1}^{\infty}\frac{J_2(rr_n)}{r_n^{\,3} J_1(Rr_n)}\left[1 - \exp(-\nu r_n^{\,2})\right], \tag{47}$$

for a Newtonian fluid are recovered [11, eqs. (29) and (37)].

5. Conclusion

In this paper, the velocity field and the shear stress corresponding to the unsteady rotational flow of a fractional Maxwell fluid induced by an infinite circular cylinder subject to a time dependent shear stress are determined by means of the Laplace and finite Hankel transforms. The obtained solutions are presented in integral and series form in terms of the generalized G-functions, satisfy all imposed initial and boundary conditions.

In special cases, for $\beta = 1$, the model of the fluid with fractional derivatives reduce to the classical model of the Maxwell fluid and for $\beta = 1$, $\lambda \to 0$, our generalized model reduce to the model of the Newtonian fluid. The solution corresponding to these cases are also presented.

Finally, in order to reveal some relevant physical aspects of the obtained results, the diagrams of the velocity field $\omega(r,t)$ are presented for different values of the time t and of the fractional parameter β. Figs. (1) and (2) clearly shows that the velocity of the generalized Maxwell fluid increases when the fractional coefficient β increases. The influence of fractional coefficient on the velocity is more meaningful for the low values of the time t. When the time t increases, the differences between the velocity fields of Maxwell, generalized Maxwell and Newtonian fluids disappear.

The influence of the relaxation time λ and the fractional parameter β, at time $t = 1.5$ s is shown by fig. 3. We see that the influence of the relaxation time on the velocity $\omega(r,t)$ is more meaningful for the low values of the relaxation time λ. For $\beta \to 1$ the diagrams corresponding to the generalized Maxwell fluid tend to those for an obtained Maxwell fluid. In all figures we used $R = 0.04,\ f = 2,\ \rho = 1260,\ \nu = 0.0011746$.

The units of the material parameters are SI units and the roots r_n have been approximated by $(4n+3)\pi / 4R$.

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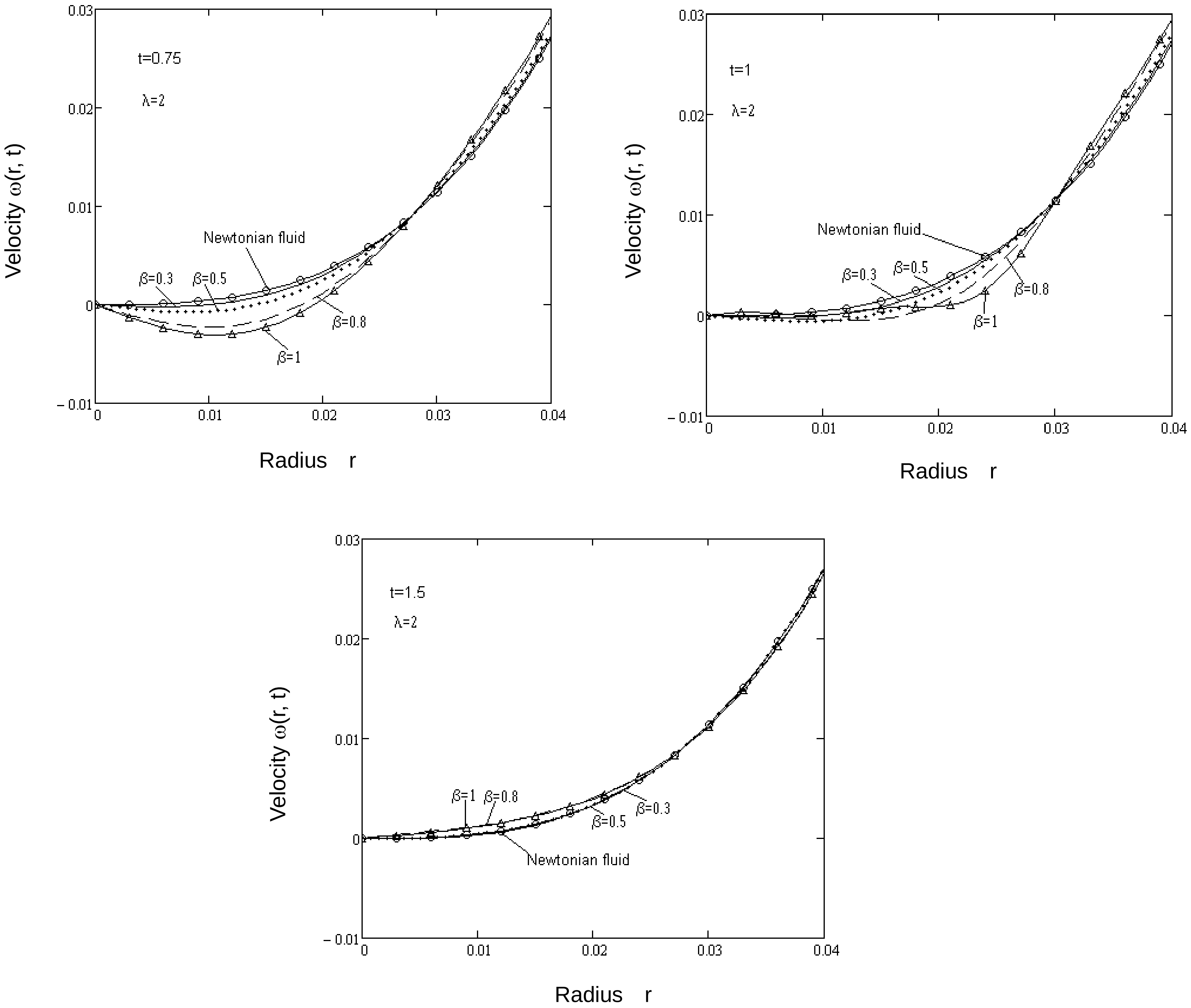
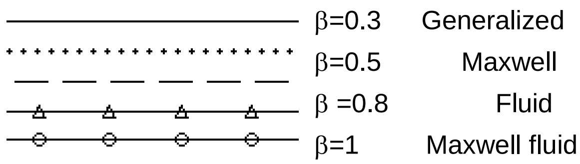


fig. 1. Profiles of velocity ω(r, t) for ν=0.001176, μ=1.48 and for different values of the fractional coefficient β and of the time t



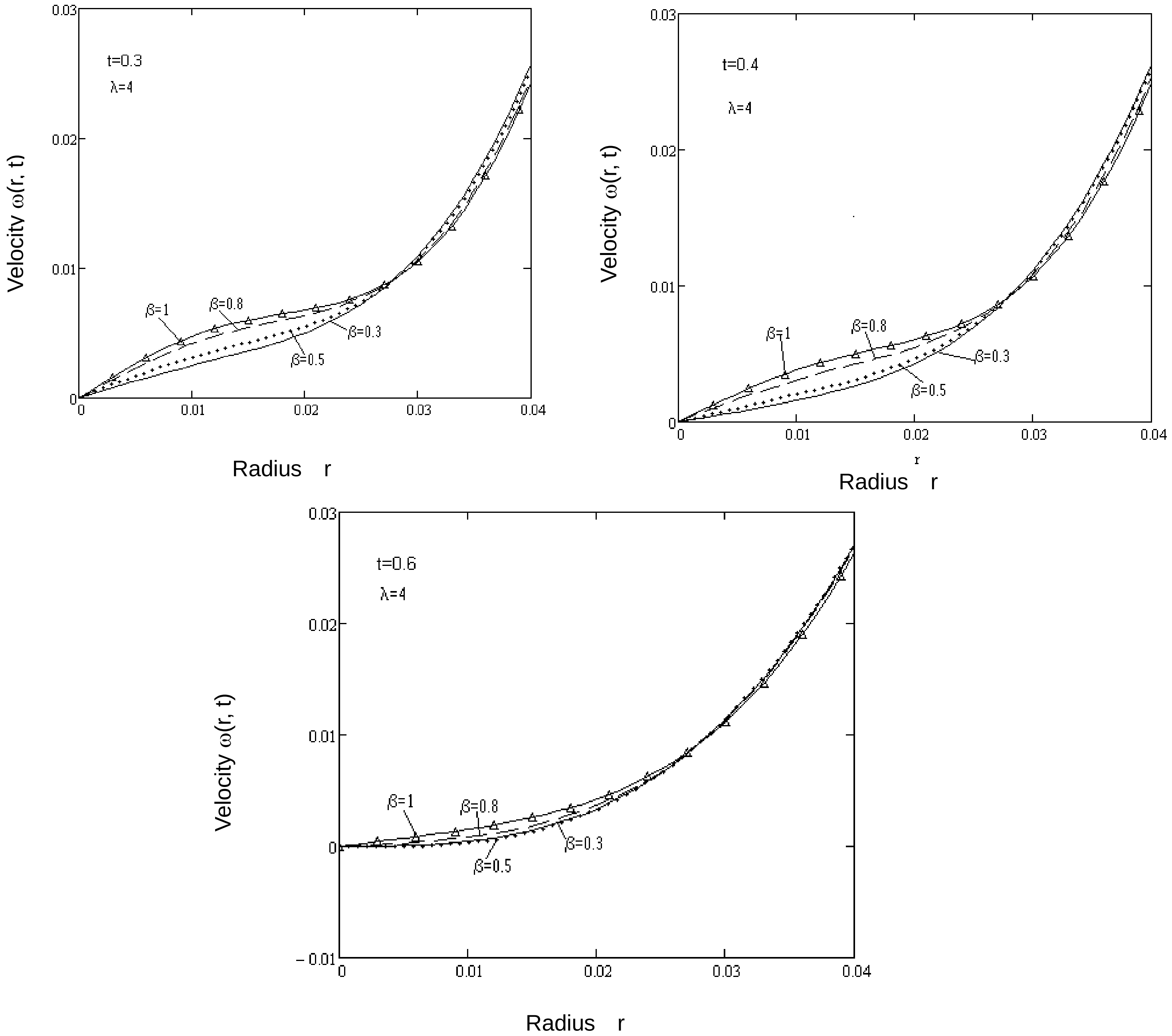
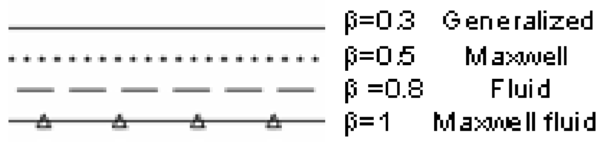


fig. 2. Profiles of velocity ω(r, t) for ν=0.001176, μ=1.48 and for different values of the fractional coefficient β and of the time t



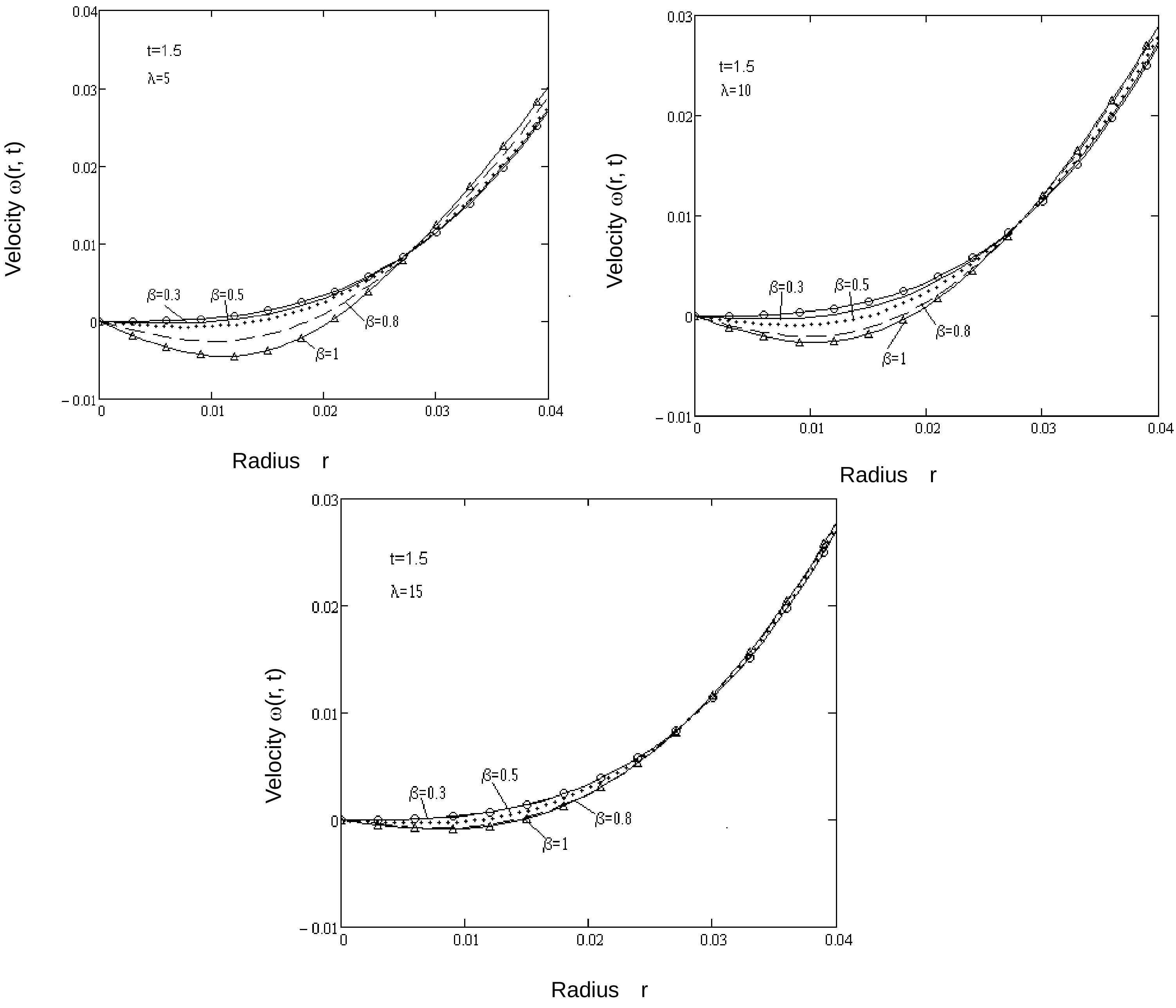


fig. 3. Profiles of velocity $\omega(r, t)$ for $\nu=0.001176$, $\mu=1.48$ and for different values of the fractional coefficient β and of the relaxation time λ