VARIATIONAL ITERATION METHOD TO SOLVE MOVING BOUNDARY PROBLEM WITH TEMPERATURE DEPENDENT PHYSICAL PROPERTIES

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In this paper, variational iteration method is used to solve a moving boundary problem arising during melting or freezing of a semi infinite region when physical properties (thermal conductivity and specific heat) of the two regions are temperature dependent. The Result is compared with result obtained by exact method (when thermal conductivity and specific heat in two regions are temperature independent) and semi analytical method (When thermal conductivity and specific heat are temperature dependent) and are in good agreement. We obtain the solution in the form of continuous functions. The method performs extremely well in terms of efficiency and simplicity and effective for solving the moving boundary problems.

Key Words**:** *Moving boundary problem, phase change process, temperature distribution, thermal conductivity and specific heat, freezing, Stefan number, Variational iteration method*

1. Introduction

Moving boundary problems involving heat phenomena (melting or freezing) occur in numerous important areas of science ,engineering and industry [1- 3] and have been of special interest due to its non linearity and unknown position of moving interface [4- 7]. The exact solution of these problems is limited which are well documented in the literature [8, 9]. During a melting or freezing process, the special situations which arise are temperature dependent thermal conductivity and specific heat of materials in two regions. The resulting Moving boundary problem can not be solved exactly. Thus, semi analytical [10- 12], and numerical methods [13, 14] have been used to solve them. Semi analytical method such as heat balance integral method [15, 16], variable space grid method [17], Galerkin's method [18], Non integral method [19, 20], regular perturbation method [21, 22] strained coordinate method [23] exist in the literature. Most of these methods have difficulty in accommodating time dependent applied surface condition, choice of a suitable approximate profile for the transfer potentials of heat and their applicability is restricted to short times.

J.H. He [24, 25] first proposed Variational iteration method to solve non linear differential and integral equations. This method was successfully applied to solve initial spherical growth during equiaxed solidification by Yao [26], reliable treatment of heat equation with Ill-defined initial data by Chun [27].In 2009, Cao et al. [28] solved the nonlinear Ill-posed operator equations by Homotopy perturbation technique. Recently, He et al. [29] discussed new algorithm of variation iteration method. This method is applicable wide range of non linear problem. Many authors [30- 34] applied this method in various different problems of non linear differential and integral equations. D. Slota [35] used this method in direct and inverse one-phase Stefan problem.

To the best of author knowledge solution of the two phase moving boundary problem with temperature dependent thermal conductivity and specific heat have not been solved yet using variational iteration method. In this Paper, the proposed method is used to obtain a semi analytical solution to a two phase moving boundary problem when thermal conductivity and specific heat of the two regions are temperature dependent.

NOMENCLATURE	
a = thermal diffusivity [$M^2 S^{-1}$]	y = dimensionless coordinate
c = specific heat [k J kg^{-1} K^{-1}]	θ = dimensionless temperature
H = latent heat of fusion [k J kg^{-1}]	λ = dimensionless phase change front
k = thermal conductivity [$Wm^{-1}K^{-1}$]	**Greek symbols**
s(t) = Moving interface [M]	α=specific heat coefficient
Ste = Stefan number	β = thermal conductivity coefficient
T = temperature [^{0}C]	**Subscripts**
ρ = density [kg M^{-3}]	1 and 2 phase 1and 2 respectively
t = time [S]	i= initial , f = freezing, m=melting
x=spatial coordinate [M]	o= at the surface x=0

2. Formulation of the problem

A Semi-infinite medium consisting of a solid/melt is initially at a temperature T_i which is slightly below/above the melting/freezing temperature of the solid/melt, T_m /T_f. . At time t=0, the surface x=0 is subjected to a temperature T_0. As a result, melting/freezing starts at the surface x=0 and the liquid solid/solid liquid interface x = s(t) moves in the positive x direction. Temperature in the two phases is unknown. Hence the problem is a two phase problem. In given problem, we take the freezing process only. The Dynamics of freezing can be described by the following equations. The basic equation for phase 1 and phase 2 are respectively.

$$(\rho_1 c_1(T_1))\frac{\partial T_1}{\partial t}=\frac{\partial}{\partial x}(k_1(T_1)\frac{\partial T_1}{\partial x}),\ 0<x<s(t),\ \ t>0. \qquad (1)$$

$$(\rho_2 c_2(T_2))\frac{\partial T_2}{\partial t}+(\rho_1-\rho_2)c_2(T_2)\frac{ds}{dt}\frac{\partial T_2}{\partial x}=\frac{\partial}{\partial x}(k_2(T_2)\frac{\partial T_2}{\partial x}),\ x>s(t),\ \ t>0. \qquad (2)$$

The associated initial and boundary conditions are

$$T_1(x,0)=T_i, \qquad (3)$$

$$T_1(0,t)=T_0. \qquad (4)$$

The condition of temperature continuity and the Stefan condition on the moving interface

$$T_1(s(t),t)=T_2(s(t),t)=T_f, \qquad (5)$$

and interface equation

$$k_1(T_1)\frac{\partial T_1}{\partial x}-k_2(T_2)\frac{\partial T_2}{\partial x}=\rho_1 H\frac{ds}{dt}, \qquad at\ \ x=s(t). \tag{6}$$

As $x \to \infty$ the material is at initial temperature i.e.

$$s(0)=0, \tag{7}$$

$$\lim_{x\to\infty} T_2(x,t)=T_i. \tag{8}$$

Introducing the dimensionless variable and similarity criteria

$$\theta_k=\frac{T_k-T_0}{T_i-T_0}, \qquad k=1,\ 2,\ f \tag{9}$$

$$\left.\begin{aligned}
&a_{01}=\frac{k_{01}}{c_{01}\,\rho_1},\\
&a_{02}=\frac{k_{02}}{c_{02}\,\rho_2},\\
&a_{12}=\frac{a_{02}}{a_{01}}\ ,\ \rho_{12}=\frac{\rho_2}{\rho_1}\ \ and\ \ \ k_{12}=\frac{k_{02}}{k_{01}},\\
&\lambda=\frac{s(t)}{2\sqrt{a_{01}t}},\ (\lambda\ is\ calculated\ Parameter),\\
&s(t)=\frac{g_1(\theta_f)\,c_{01}\left|T_i-T_0\right|}{H},
\end{aligned}\right\} \tag{10}$$

and using transformation

$$y=\frac{x}{2\sqrt{a_{01}t}}. \tag{11}$$

The system of equation (1) to (8) reduces to the following non-dimensional form

$$\frac{d}{dy}\left[g_1(\theta_1)\frac{d\theta_1}{dy}\right]+2\,y\ \ f_1(\theta_1)\frac{d\theta_1}{dy}=0, \tag{12}$$

$$a_{12}\ \frac{d}{dy}\left[g_2(\theta_2)\frac{d\theta_2}{dy}\right]+2\left[y+(\frac{1}{\rho_{12}}-1)\,\lambda\right]f_2(\theta_2)\frac{d\theta_2}{dy}=0, \tag{13}$$

$$\theta_1(0)=0, \tag{14}$$

$$\theta_1(\lambda)=\theta_2(\lambda)=\theta_f. \tag{15}$$

Interface equation is

$$\frac{d\theta_1}{dy}-K_{12}\frac{g_2(\theta_f)}{g_1(\theta_f)}\frac{d\theta_2}{dy}=\frac{2\,\lambda}{ste}, \qquad at\ \ y=\lambda, \tag{16}$$

$$\lim_{y\to\infty}\ \theta_2(y)=1. \tag{17}$$

The thermal conductivity and specific heat in two regions varies with temperature and is assumed to be

$$\left.\begin{aligned} k_1 &= k_{01} g_1(\theta_1), \\ k_2 &= k_{02} g_2(\theta_2), \\ c_1 &= c_{01} f_1(\theta_1), \\ c_2 &= c_{02} f_2(\theta_2). \end{aligned}\right\} \tag{18}$$

3. Variational Iteration Method

To clarify the basic ideas of He's Variational iteration method, we consider the differential equation.

$$Lu + Nu = g(t) \tag{19}$$

Where L and N are linear and nonlinear operators respectively and g (t) is the source of inhomogeneous term. A correction functional for equation (19) can be written as

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(Lu_n(\xi) + N\tilde{u}_n(\xi) - g(\xi))\,d\xi, \quad n \ge 0. \tag{20}$$

Where λ is a general Lagrange multiplier, which can be identified optimally via the variational theory. The Subscript n indicate the nth approximation and $\tilde{u}_n$ is a restricted variation, which means $\delta\tilde{u}_n = 0$ .Therefore, we first determine the Lagrange multiplier λ that will be identified optimally via integration by parts. The successive approximations $u_{n+1}(t)\ \ n \ge 0,$ of the solution $u(t)$ will be readily obtained upon using the Lagrange multiplier obtained and by using selective function $u_0(t)$.

Consequently, the exact solution may be obtained by using

$$u(t) = \lim_{n\to\infty} u_n(t). \tag{21}$$

4. Solution of the problem

In order to simplify the solution of the problem we have to consider two cases

Case 1: When thermal conductivity and specific heat varies exponentially i.e.

$$\left.\begin{aligned} f_1(\theta_1) &= e^{\alpha_1\theta_1}, \\ f_2(\theta_2) &= e^{\alpha_2\theta_2}, \\ g_1(\theta_1) &= e^{\beta_1\theta_1}, \\ g_2(\theta_2) &= e^{\beta_2\theta_2}. \end{aligned}\right\} \tag{22}$$

We construct a correction functional, for equation (12) restrict as follows

$$\theta_{1,n+1}(y) = \theta_{1,n}(y) + \int_0^y \lambda(\xi)\left[e^{\beta_1\theta_{1,n}(\xi)}\frac{d^2\theta_{1,n}(\xi)}{d\xi^2} + e^{\beta_1\theta_{1,n}(\xi)}\beta_1\left(\frac{d\tilde{\theta}_{1,n}(\xi)}{d\xi}\right)^2 + 2\,\xi\,e^{\alpha_1\theta_{1,n}(\xi)}\frac{d\tilde{\theta}_{1,n}(\xi)}{d\xi}\right]d\xi, \tag{23}$$

taking variation both sides and using

$$\delta\tilde{\theta}_{1,n} = 0, \tag{24}$$

$$\delta\theta_{1,n+1}(y) = \delta\theta_{1,n}(y) + \delta\int_0^y \lambda(\xi)\, e^{\beta_1\theta_{1,n}(\xi)}\frac{d^2\theta_{1,n}}{d\xi^2}d\xi. \tag{25}$$

To find the optimal value of $\lambda(\xi)$, we integrate the equation by parts and obtain stationary condition as follows (expanding exponential series and restrict all $\theta_{1,n}(\xi)$

$$\lambda''(\xi)\big|_{\xi=y} = 0,\ 1-\lambda'(\xi)\big|_{\xi=y} = 0,\ \lambda(\xi)\big|_{\xi=y} = 0. \tag{26}$$

The Lagrangian multiplier can identified as solving above equation

$$\lambda(\xi) = (\xi - y). \tag{27}$$

Putting this value in eq (23), as a result, we obtain the following iterative formula

$$\theta_{1,n+1}(y) = \theta_{1,n}(y) + \int_0^y (\xi - y)\left[e^{\beta_1\theta_{1,n}(\xi)}\frac{d^2\theta_{1,n}(\xi)}{d\xi^2} + \beta_1 e^{\beta_1\theta_{1,n}(\xi)}\left(\frac{d\theta_{1,n}(\xi)}{d\xi}\right)^2 + 2\ \xi\, e^{\alpha_1\theta_{1,n}(\xi)}\frac{d\theta_{1,n}(\xi)}{d\xi}\right]d\xi. \tag{28}$$

Taking the initial approximation according to initial and boundary condition

$$\theta_{1,0}(y) = \frac{\theta_f}{\lambda}y, \tag{29}$$

using the above variational iteration formula (28) we have

$$\theta_{1,1}(y) = \frac{1}{\beta_1} - \frac{e^{\frac{\beta_1 y\theta_f}{\lambda}}}{\beta_1} + \frac{2y\theta_f}{\lambda} - \frac{2y\lambda}{\alpha_1{}^2\theta_f} - \frac{2e^{\frac{\alpha_1 y\theta_f}{\lambda}}y\lambda}{\alpha_1{}^2\theta_f} - \frac{4\lambda^2}{\alpha_1{}^3\theta_f{}^2} + \frac{4e^{\frac{\alpha_1 y\theta_f}{\lambda}}\lambda^2}{\alpha_1{}^3\theta_f{}^2}, \tag{30}$$

and so on.

Similarly we can find $\theta_2(y)$ for differential equation (13), now we construct correctional functional

$$\theta_{2,n+1}(y) = \theta_{2,n}(y) + \int_0^y \lambda(\xi)\left[a_{12}e^{\beta_2\theta_{2,n}(\xi)}\frac{d^2\theta_{2,n}(\xi)}{d\xi^2} + \beta_2\, a_{12}\, e^{\beta_2\theta_{2,n}(\xi)}\left(\frac{d\tilde{\theta}_{2,n}(\xi)}{d\xi}\right)^2 + \{2\ \xi + 2\lambda(\frac{1}{\rho_{12}} - 1)\}e^{\alpha_2\theta_{2,n}(\xi)}\frac{d\tilde{\theta}_{2,n}(\xi)}{d\xi}\right]d\xi. \tag{31}$$

Its stationary condition (expanding exponential series and restrict all $\theta_{2,n}(\xi)$

$$\lambda''(\xi)\big|_{\xi=y} = 0, 1 - a_{12}\,\lambda'(\xi)\big|_{\xi=y} = 0,\ \lambda(\xi)\big|_{\xi=y} = 0. \tag{32}$$

The langrangian multiplier obtained

$$\lambda(\xi) = \frac{1}{a_{12}}(\xi - y). \tag{33}$$

The following iteration formula becomes

$$\theta_{2,n+1}(y) = \theta_{2,n}(y) + \int_0^y \frac{1}{a_{12}}(\xi - y)\left[a_{12}e^{\beta_2\theta_{2,n}(\xi)}\frac{d^2\theta_{2,n}(\xi)}{d\xi^2} + \beta_2\, a_{12}\, e^{\beta_2\theta_{2,n}(\xi)}\left(\frac{d\tilde{\theta}_{2,n}(\xi)}{d\xi}\right)^2 + \{2\ \xi + 2\lambda(\frac{1}{\rho_{12}} - 1)\}e^{\alpha_2\theta_{2,n}(\xi)}\frac{d\tilde{\theta}_{2,n}}{d\xi}\right]d\xi. \tag{34}$$

Now we start with initial approximation $\theta_{2,0}(y) = 1 + (\theta_f - 1)\,e^{(\lambda - y)}$,

and we get the first approximation

$$\begin{aligned}\theta_{2,1}(y) = {} & 1 - e^{(\lambda-y)} + e^{\beta_2 + e^{\lambda}\beta_2(\theta_f - 1) + \lambda}y + e^{(\lambda-y)}\theta_f - e^{\beta_2 + e^{\lambda}\beta_2(\theta_f - 1) + \lambda}y\theta_f \\ & - \frac{1}{a_{12}}\int_0^y\left\{2e^{\alpha_2(1 + e^{(\lambda-\xi)}(\theta_f - 1)) + (\lambda - \xi)}(\theta_f - 1)(\xi - y)((\frac{1}{\rho_{12}} - 1)\lambda + \xi)\right\}d\xi.\end{aligned} \tag{35}$$

Similarly, we find other approximation, After getting expression of $\theta_{1,n}$ *and* $\theta_{2,n}$ we have,

$$\theta_1(y) = \lim_{n\to\infty} \theta_{1,n}(y)\,, \tag{36}$$

$$\theta_2(y) = \lim_{n\to\infty} \theta_{2,n}(y)\,. \tag{37}$$

Next we consider,

Case 2: when thermal conductivity and specific heat varies linearly with temperature [36],

$$\left.\begin{aligned} f_1(\theta_1) &= 1+\alpha_1\theta_1, \\ f_2(\theta_2) &= 1+\alpha_2\theta_2, \\ g_1(\theta_1) &= 1+\beta_1\theta_1, \\ g_2(\theta_2) &= 1+\beta_2\theta_2. \end{aligned}\right\} \tag{38}$$

Now we construct a correction functional, for equation (12) restrict as follows,

$$\theta_{1,n+1}(y) = \theta_{1,n}(y) + \int_0^y \lambda(\xi)\left[(1+\beta_1\tilde{\theta}_{1,n}(\xi))\frac{d^2\theta_{1,n}(\xi)}{d\xi^2} + \beta_1\left(\frac{d\tilde{\theta}_{1,n}(\xi)}{d\xi}\right)^2 + 2\ \xi(1+\alpha_1\tilde{\theta}_1(\xi))\frac{d\tilde{\theta}_{1,n}(\xi)}{d\xi}\right]d\xi. \tag{39}$$

Taking variation both sides and using $\delta\tilde{\theta}_{1,n} = 0,$

$$\delta\theta_{1,n+1}(y) = \delta\theta_{1,n}(y) + \delta\int_0^y \lambda(\xi)\ \frac{d^2\theta_{1,n}}{d\xi^2}\,d\xi. \tag{40}$$

Its stationary condition can be obtained as follow,

$$\lambda''(\xi)\big|_{\xi=y} = 0, \quad 1-\lambda'(\xi)\big|_{\xi=y} = 0, \quad \lambda(\xi)\big|_{\xi=y} = 0. \tag{41}$$

The Lagrangian multiplier can identified as solving above equations (41),

$$\lambda(\xi) = (\xi - y). \tag{42}$$

Putting this value in eq (39), as a result, we obtain the following formula,

$$\theta_{1,n+1}(y) = \theta_{1,n}(y) + \int_0^y (\xi-y)\left[(1+\beta_1\theta_{1,n}(\xi))\frac{d^2\theta_{1,n}(\xi)}{d\xi^2} + \beta_1\left(\frac{d\theta_{1,n}(\xi)}{d\xi}\right)^2 + 2\ \xi(1+\alpha_1\theta_{1,n}(\xi))\frac{d\theta_{1,n}(\xi)}{d\xi}\right]d\xi. \tag{43}$$

Taking the initial approximation,

$$\theta_{1,0}(y) = \frac{\theta_f}{\lambda}\,y, \tag{44}$$

using the above variational correctional formula (43) we have,

$$\theta_{1,1}(y) = \frac{\eta\,\theta_f}{\lambda} - \frac{\beta_1\eta^2\theta_f^{\,2}}{2\lambda^2} - \frac{\alpha_1\eta^4\theta_f^{\,2}}{6\lambda^2} - \frac{\eta^3\theta_f}{3\lambda}, \tag{45}$$

$$\begin{aligned} \theta_{1,2}(\eta) = {} & \frac{\eta\,\theta_f}{\lambda} - \frac{\beta_1\eta^2\theta_f^{\,2}}{2\,\lambda^2} - \frac{\eta^3\theta_f}{3\lambda} + \frac{\beta_1^{\,2}\eta^3\theta_f^{\,3}}{2\lambda^3} + \frac{\beta_1\eta^4\theta_f^{\,2}}{2\lambda^2} - \frac{\alpha_1\eta^4\theta_f^{\,2}}{.6\lambda^2} - \frac{\beta_1^{\,3}\eta^4\theta_f^{\,4}}{8\lambda^4} + \frac{\eta^5\theta_f}{10\lambda} \\ & - \frac{\beta_1^{\,2}\eta^5\theta_f^{\,3}}{6\lambda^3} + \frac{19\alpha_1\beta_1\eta^5\theta_f^{\,3}}{60\lambda^3} - \frac{\beta_1\eta^6\theta_f^{\,2}}{18\lambda^2} + \frac{2\alpha_1\eta^6\theta_f^{\,2}}{15\lambda^2} - \frac{7\alpha_1\beta_1^{\,2}\eta^6\theta_f^{\,4}}{60\lambda^4} + \frac{5\alpha_1^{\,2}\eta^7\theta_f^{\,3}}{126\lambda^3} \\ & - \frac{2\alpha_1\beta_1\eta^7\theta_f^{\,3}}{21\lambda^3} - \frac{\alpha_1\eta^8\theta_f^{\,2}}{84\lambda^2} - \frac{2\alpha_1^{\,2}\beta_1\eta^8\theta_f^{\,4}}{63\lambda^4} - \frac{7\alpha_1^{\,2}\eta^9\theta_f^{\,3}}{648\lambda^3} - \frac{\alpha_1^{\,3}\eta^{10}\theta_f^{\,4}}{405\lambda^4}. \end{aligned} \tag{46}$$

and so on.

For differential equation (13) we construct correctional functional

$$\theta_{2,n+1}(y)=\theta_{2,n}(y)+\int_0^y \lambda(\xi)\left[a_{12}(1+\beta_2\tilde{\theta}_{2,n}(\xi))\frac{d^2\theta_{2,n}(\xi)}{d\xi^2}+a_{12}\beta_2\left(\frac{d\tilde{\theta}_{2,n}(\xi)}{d\xi}\right)^2+\{2\ \xi+2\lambda(\frac{1}{\rho_{12}}-1)\}(1+\alpha_2\tilde{\theta}_{2,n}(\xi))\frac{d\tilde{\theta}_{2,n}}{d\xi}\right]d\xi. \quad (47)$$

Its stationary condition,

$$\lambda''(\xi)\big|_{\xi=y}=0,\ \ 1-a_{12}\,\lambda'(\xi)\big|_{\xi=y}=0,\ \ \ \lambda(\xi)\big|_{\xi=y}=0. \quad (48)$$

The langrangian multiplier obtained,

$$\lambda(\xi)=\frac{1}{a_{12}}(\xi-y). \quad (49)$$

The following iteration formula eq. (47) becomes,

$$\theta_{2,n+1}(y)=\theta_{2,n}(y)+\int_0^y \frac{1}{a_{12}}(\xi-y)\left[a_{12}(1+\beta_2\theta_{2,n}(\xi))\frac{d^2\theta_{2,n}(\xi)}{d\xi^2}+a_{12}\beta_2\left(\frac{d\theta_{2,n}(\xi)}{d\xi}\right)^2+\{2\ \xi+2\lambda(\frac{1}{\rho_{12}}-1)\}(1+\alpha_2\theta_{2,n}(\xi))\frac{d\theta_{2,n}}{d\xi}\right]d\xi. \quad (50)$$

Now we start with initial approximation $\theta_{2,0}(\eta)=1+(\theta_f-1)\,e^{(\lambda-y)}$,

$$\begin{aligned}\theta_{2,1}(\eta)=&1-e^{\lambda}+\frac{4e^{\lambda}}{a_{12}}-\frac{4e^{(\lambda-y)}}{a_{12}}+\frac{4e^{\lambda}\alpha_2}{a_{12}}-\frac{e^{2\lambda}\alpha_2}{2a_{12}}-\frac{4e^{(\lambda-y)}\alpha_2}{a_{12}}+\frac{e^{2(\lambda-y)}\alpha_2}{2a_{12}}-e^{\lambda}\beta_2+\frac{1}{2}e^{2\lambda}\beta_2+\beta_2e^{(\lambda-y)}-\frac{1}{2}e^{2(\lambda-y)}\beta_2 \quad (51)\\
&+e^{\lambda}y-\frac{2e^{\lambda}y}{a_{12}}-\frac{2e^{(\lambda-\eta)}y}{a_{12}}-\frac{2e^{\lambda}\alpha_2y}{a_{12}}+\frac{e^{2\lambda}\alpha_2y}{2a_{12}}-\frac{2e^{(\lambda-y)}\alpha_2y}{a_{12}}+\frac{e^{2(\lambda-y)}\alpha_2y}{2a_{12}}+e^{\lambda}\beta_2y-e^{2\lambda}\beta_2y+e^{\lambda}\theta_f-\frac{4\theta_fe^{\lambda}}{a_{12}}\\
&+\frac{4}{a_{12}}e^{(\lambda-y)}\theta_f-\frac{4e^{\lambda}\alpha_2\theta_f}{a_{12}}+\frac{e^{2\lambda}\alpha_2\theta_f}{a_{12}}+\frac{4e^{(\lambda-y)}\alpha_2\theta_f}{a_{12}}-\frac{e^{2(\lambda-y)}\alpha_2\theta_f}{a_{12}}+e^{\lambda}\beta_2\theta_f-e^{2\lambda}\beta_2\theta_f-e^{(\lambda-y)}\beta_2\theta_f+e^{2(\lambda-y)}\beta_2\theta_f\\
&-e^{\lambda}\eta\theta_f+\frac{2e^{\lambda}y\theta_f}{a_{12}}+\frac{2e^{(\lambda-y)}\eta\theta_f}{a_{12}}+\frac{2e^{\lambda}\alpha_2y\theta_f}{a_{12}}-\frac{e^{2\lambda}\alpha_2y\theta_f}{a_{12}}+\frac{2e^{(\lambda-y)}\alpha_2y\theta_f}{a_{12}}-\frac{e^{2(\lambda-y)}\alpha_2y\theta_f}{a_{12}}-e^{\lambda}\beta_2y\theta_f+2e^{2\lambda}\beta_2y\theta_f\\
&-\frac{e^{2\lambda}\alpha_2\theta_f^{\,2}}{2a_{12}}+\frac{e^{2(\lambda-y)}\alpha_2\theta_f^{\,2}}{2a_{12}}+\frac{1}{2}e^{2\lambda}\beta_2\theta_f^{\,2}-\frac{1}{2}e^{2(\lambda-y)}\beta_2\theta_f^{\,2}+\frac{e^{2\lambda}\alpha_2y\theta_f^{\,2}}{2a_{12}}+\frac{e^{2(\lambda-y)}\alpha_2y\theta_f^{\,2}}{2a_{12}}-e^{2\lambda}\beta_2y\theta_f^{\,2}+\frac{2e^{\lambda}k\lambda}{a_{12}}\\
&-\frac{2e^{(\lambda-y)}k\lambda}{a_{12}}+\frac{2e^{\lambda}k\alpha_2\lambda}{a_{12}}-\frac{e^{2\lambda}k\alpha_2\lambda}{2a_{12}}-\frac{2e^{(\lambda-y)}k\alpha_2\lambda}{a_{12}}+\frac{e^{2(\lambda-y)}k\alpha_2\lambda}{2a_{12}}-\frac{2e^{\lambda}ky\lambda}{a_{12}}-\frac{2e^{\lambda}k\alpha_2y\lambda}{a_{12}}+\frac{e^{2\lambda}k\alpha_2y\lambda}{a_{12}}-\frac{2e^{\lambda}k\theta_f\lambda}{a_{12}}\\
&+\frac{2e^{(\lambda-y)}k\theta_f\lambda}{a_{12}}-\frac{2e^{\lambda}k\alpha_2\theta_f\lambda}{a_{12}}+\frac{e^{2\lambda}k\alpha_2\theta_f\lambda}{a_{12}}+\frac{2e^{(\lambda-y)}k\alpha_2\theta_f\lambda}{a_{12}}-\frac{e^{2(\lambda-y)}k\alpha_2\theta_f\lambda}{a_{12}}+\frac{2e^{\lambda}ky\theta_f\lambda}{a_{12}}+\frac{2e^{\lambda}k\alpha_2y\theta_f\lambda}{a_{12}}-\frac{2e^{2\lambda}k\alpha_2y\theta_f\lambda}{a_{12}}\\
&-\frac{e^{2\lambda}k\alpha_2\theta_f^{\,2}\lambda}{2a_{12}}+\frac{e^{2(\lambda-y)}k\alpha_2\theta_f^{\,2}\lambda}{2a_{12}}+\frac{e^{2\lambda}k\alpha_2y\theta_f^{\,2}\lambda}{2a_{12}}.\end{aligned}$$

We get the first approximation, in this expression $k=(\frac{1}{\rho_{12}}-1)$,

and so on.

In same manner we find the other approximation, After find out the expression of $\theta_{1,n}$ *and* $\theta_{2,n}$ we have,

$$\theta_1(y)=\lim_{n\to\infty}\theta_{1,n}(y), \quad (52)$$

$$\theta_2(y)=\lim_{n\to\infty}\theta_{2,n}(y). \quad (53)$$

5. Numerical computation and discussion

Substituting, $\theta_1(y)$ *and* $\theta_2(y)$ for both cases in the interface equation (16), we obtain a transcendental equation in the term of $\alpha_1, \alpha_2, \beta_1, \beta_2$ *and ste*. The computation has been made and the results are presented in five figures. On the figures presented in this study, only the parameters whose values different from the reference values are indicated.

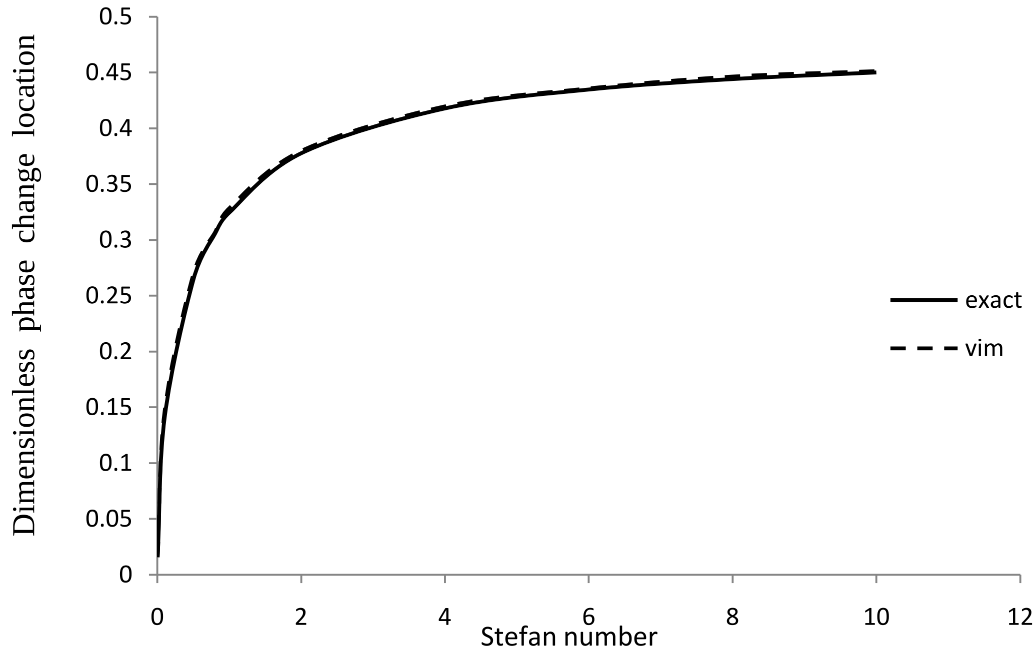
In Fig (1), the dimensionless phase change location is plotted as a function of the Stefan number. It has been observed that the result obtained by VIM is exactly the same as that obtained by exact method. The dimensionless temperature θ_1 and θ_2 as a function of space coordinate are shown in Fig (2).

On Fig (3), the dimensionless phase change location is plotted as a function of Stefan number, when thermal conductivity and specific heat in two regions varies linearly with temperature. It has been observed that as α_1, α_2 increase, dimensionless phase change location decreases. This result is same as that obtained by Oliver and Sunderland [36].

It is clear from Fig (4) that as β_1, β_2 increases, dimensionless phase change location decreases. It is clear from Fig (3) and Fig (4) that at higher values of heat of fusion (low values of ste), the variation in thermal conductivity are important, but the effect of variable specific heat diminishes. On Fig (5) the dimensionless phase change location is plotted as a function of Stefan number, when thermal conductivity and specific heat in two regions varies exponentially with temperature (case 1). It has been observed that as α_1, α_2 increases, dimensionless phase change location decreases.

Conclusion

The variational iteration method is very powerful in finding the solution of moving boundary problem in freezing process. Sharing its application to moving boundary problem of temperature dependent thermal conductivity and specific heat of two regions, we may conclude that this method will be very much useful for solving moving and other many physical problems. The advantage of this method consists in obtaining the interface position and temperature distribution in the form of continuous function, instead of discrete form. Moreover, No linearization is needed and it avoids the accuracy of finding the temperature distribution by the numerical techniques.



Fig(1) Graph between exact solution of linear equation and VIM

$$\alpha_1 = \alpha_2 = 0, \beta_1 = \beta_2 = 0, \theta_f = 0.5, \rho_{12} = 1.0, a_{12} = 1.0$$

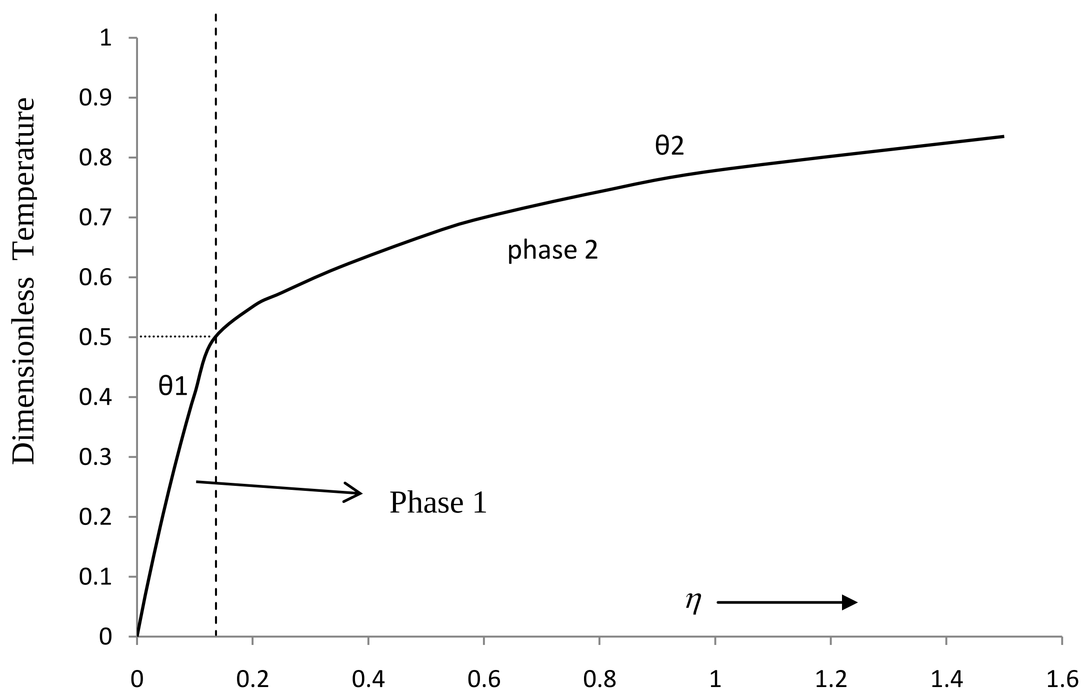


Fig (2) Dimensionless temperature distribution of Phase 1 and Phase 2 (case 2)

$$\alpha_1 = \alpha_2 = 1.0, \beta_1 = \beta_2 = 1.0, \theta_f = 0.5, \rho_{12} = 1.0, a_{12} = 1.0, \lambda = 0.13$$

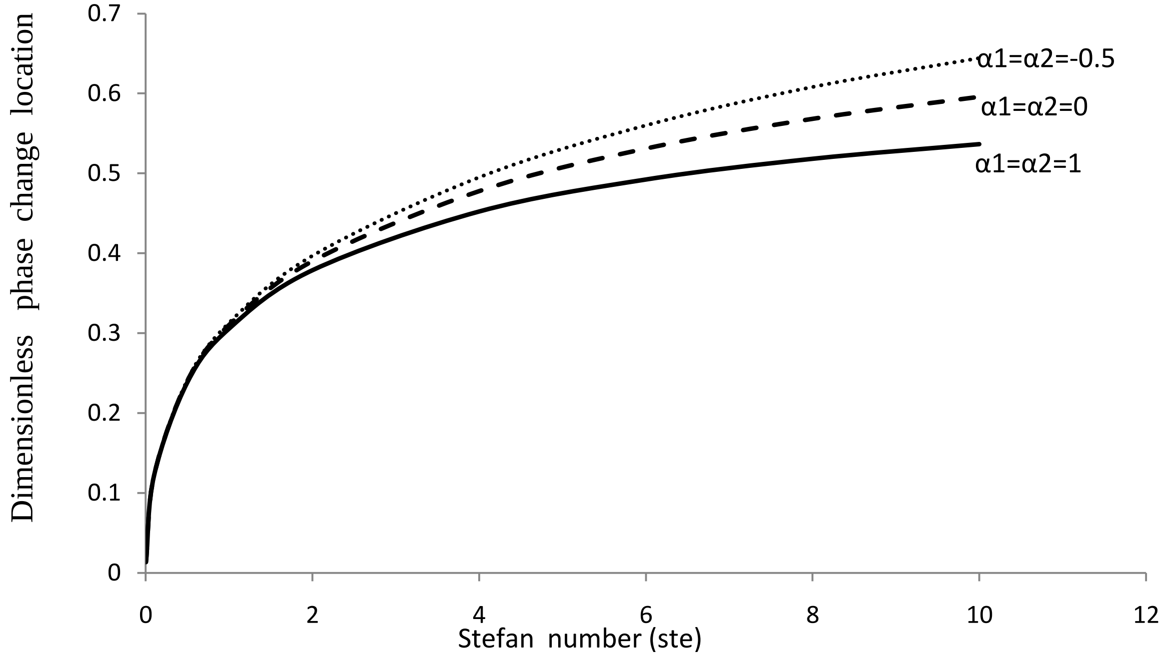


Fig (3) Dimensionless phase change location vs Stefan number in (case 2)

$\beta_1 = \beta_2 = 1.0,\ \theta_f = 0.5,\ \rho_{12} = 1.0,\ a_{12} = 1.0$

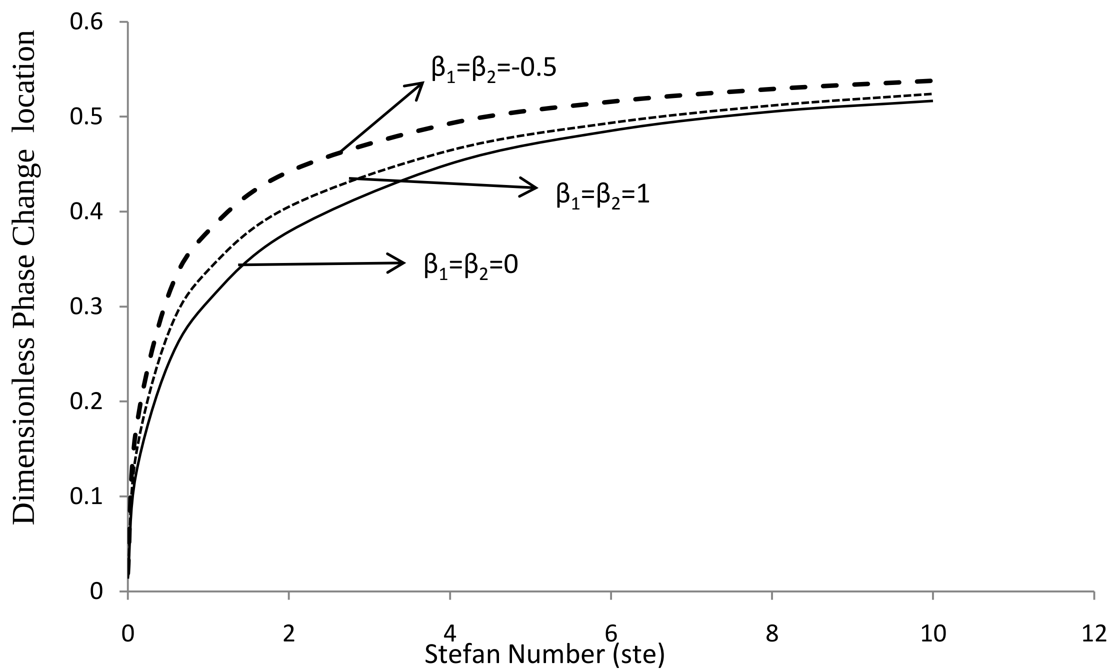


Fig (4) Dimensionless Phase Change location vs. Stefan number in (case 2)

$\alpha_1 = \alpha_2 = 1.0,\ \theta_f = 0.5,\ \rho_{12} = 1.0,\ a_{12} = 1.0$

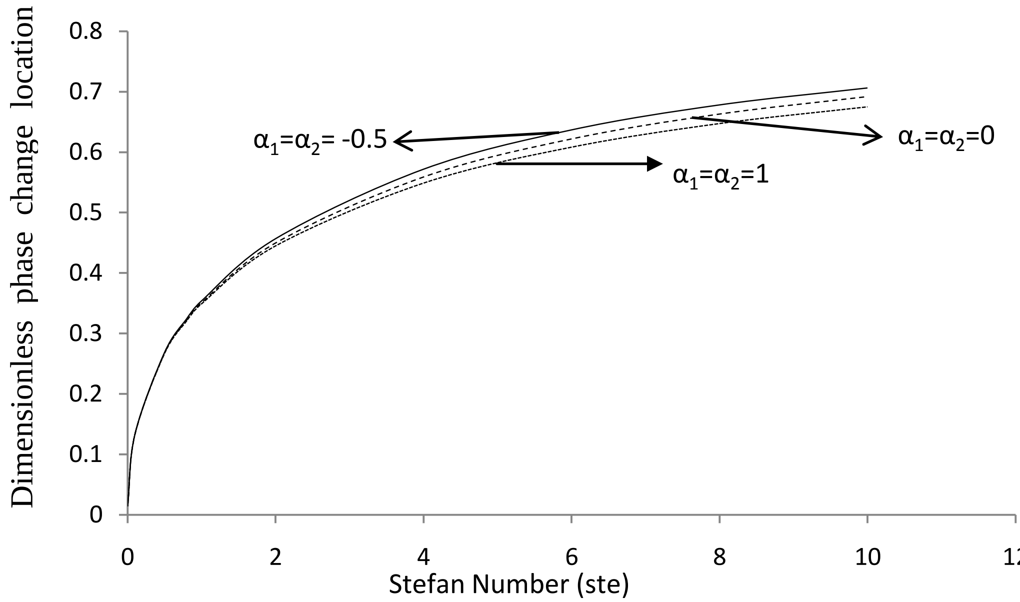


Fig (5) Dimensionless phase change location vs. Stefan number in (case 1)
$\beta_1 = \beta_2 = 1.0,\ \theta_f = 0.5,\ \rho_{12} = 1.0,\ a_{12} = 1.0$

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