# SIMILARITY SOLUTION FOR NATURAL CONVECTION FROM A MOVING VERTICAL PLATE WITH INTERNAL HEAT GENERATION AND A CONVECTIVE BOUNDARY CONDITION

by

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Original scientific paper
UDC: 532.517.2/.4:536.33:517.91/.95
DOI: 10.2298/TSCI11S1137M

*Steady laminar natural convection flow over a semi-infinite moving vertical plate in the presence of internal heat generation and a convective surface boundary condition is examined in this paper. It is assumed that the left surface of the plate is in contact with a hot fluid while the cold fluid on the right surface of the plate contains a heat source that decays exponentially with the classical similarity variable. The governing non-linear partial differential equations have been transformed by a similarity transformation into a system of ordinary differential equations, which are solved numerically by applying shooting iteration technique together with fourth order Runge-Kutta integration scheme. The effects of physical parameters on the dimensionless velocity and temperature profiles are depicted graphically and analyzed in detail. Finally, numerical values of physical quantities, such as the local skin-friction coefficient and the local Nusselt number are presented in tabular form.*

Key words: *moving vertical plate, internal heat generation, local Grashof number, local Biot number*

## Introduction

Boundary layer flows with internal heat generation past a vertical plate continues to receive considerable attention because of its many practical applications in a broad spectrum of engineering systems like geothermal reservoirs, cooling of nuclear reactors, thermal insulation, combustion chamber, rocket engine, *etc*. Many principal past studies concerning natural convection flows over a semi-infinite vertical plate immersed in an ambient fluid have been found in the literature [1-3]. In many cases, when natural convection takes place from a vertical flat plate, the temperature is not isothermal and these problems may admit similarity solutions. Crepeau *et al*. [4] reported a local similarity solution for a natural convection over a vertical plate with isothermal surface temperature and internal heat generation. Kao [5] investigated free convection from vertical plates with sinusoidal temperature variation and constant transpiration.

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The idea of using a convective boundary condition was recently introduced by Aziz [6] to study the classical boundary layer flow over a flat plate. Makinde *et al*. [7] reported a local similarity solution for the effect of buoyancy forces on thermal boundary layer over a flat plate with a convective boundary condition. Their result also revealed that the buoyancy effects tend to reduce the thermal boundary layer thickness.

In this paper, the similarity method is employed to investigate the effect of an exponentially decaying internal heat generation on a boundary layer flow over a moving vertical plate with a convective boundary condition. The numerical computation for the resulting nonlinear differential equation is performed using a fourth-order Runge-Kutta method with shooting technique. Pertinent results are displayed graphically and discussed quantitatively.

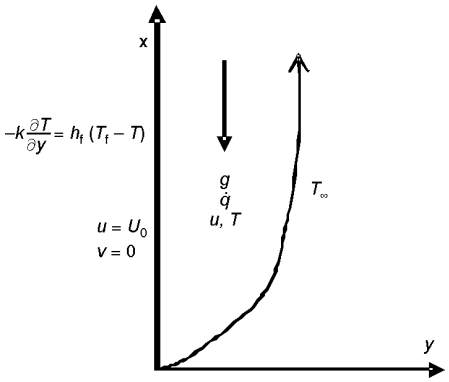


**Figure 1. Flow configuration and coordinate system**

## Mathematical analysis

We consider the steady laminar incompres-sible natural convection boundary layer flows over the right surface of a vertical flat plate moving with uniform velocity $U_0$ in contact with a quiescence cold fluid at temperature $T_\infty$. The cold fluid on the right surface of the plate generates heat internally at the volumetric rate $\dot{q}$. The left surface of the plate is heated by convection from a hot fluid at temperature $T_f$ which provides a heat transfer coefficient $h_f$ as shown in fig. 1. Under the Boussinesq for fluid density variation, the continuity, momentum, and energy equations describing the flow can be written as:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\upsilon\frac{\partial^2 u}{\partial y^2}+g\beta(T-T_\infty) \tag{2}$$

$$\rho c_p\left(u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right)=k\frac{\partial^2 T}{\partial y^2}+\dot{q} \tag{3}$$

where $u$ and $v$ are the $x$ (along the plate) and the $y$ (normal to the plate) components of the velocities, respectively, $T$ is the temperature, $\upsilon$ – the kinematics viscosity of the fluid, $\rho$ – the fluid density, $c_p$ – the specific heat at constant pressure, $k$ – the thermal conductivity of the fluid, and $\beta$ – the thermal expansion coefficient. The boundary conditions at the plate surface and far into the cold fluid may be written as:

$$u(x,0)=U_0, v(x,0)=0,\ \ -k\frac{\partial T}{\partial y}(x,0)=h_f[T_f-T(x,0)] \tag{4}$$

$$u(x,\infty)=0, \qquad T(x,\infty)=T_\infty \tag{5}$$

Introducing a similarity variable $\eta$ and a dimensionless stream function $f(\eta)$ and temperature $\theta(\eta)$ as:

$$\eta=\frac{y}{x}\sqrt{\mathrm{Re}_x},\ \ u=U_0\ f',\ \ v=\frac{\upsilon}{2x}\sqrt{\mathrm{Re}_x}(\eta f'-f),\ \ \theta=\frac{T-T_\infty}{T_f-T_\infty},\ \ \lambda_x=\frac{\dot{q}x^2e^\eta}{k\,\mathrm{Re}_x(T_f-T_\infty)} \tag{6}$$

where the prime symbol denotes differentiation with respect to $\eta$ and $Re_x = U_0x/\upsilon$ is the local Reynolds number. Substituting eq. (6) into the eqs. (1-5), we obtain:

$$f'''+\frac{1}{2}ff''+\mathrm{Gr}_x\theta=0 \tag{7}$$

$$\theta''+\frac{1}{2}\mathrm{Pr}\ f\theta'+\lambda_x e^{-\eta}=0 \tag{8}$$

$$f(0)=0,\ \ f'(0)=1,\ \ \theta'(0)=-\mathrm{Bi}_x[1-\theta(0)],\ \ f'(\infty)=0 \tag{9}$$

where $\mathrm{Bi}_x = h_f(\upsilon x/U_0)^{1/2}/k$ is the local Biot number, $\mathrm{Pr} = rc_p\upsilon/k$ – the Prandtl number and $\mathrm{Gr}_x = g\beta x(T_f - T_\infty)/U_0^2$ is the local Grashof number. The local internal heat generation parameter $\lambda_x$ is defined such that the internal heat generation $\dot{q}$ decays exponentially with the similarity variable $\eta$ as stipulated in highlighted in eq. (8). This type of model can be used in mixtures where a radioactive material is surrounded by inert alloys and in the electromagnetic heating of materials [8, 9]. Moreover, eqs. (7)-(9) will definitely produce a local similarity solution for the problem. In order to have a true similarity solution, the parameters $\mathrm{Gr}_x$, $\mathrm{Bi}_x$, and $\lambda_x$ must be constant. This condition will be satisfied if the heat transfer coefficient $h_f$ is proportional to $x^{-1/2}$, the thermal expansion coefficient $\beta$ and internal heat generation $\dot{q}$ are proportional to $x^{-1}$. Hence, we assume:

$$h_f=cx^{-1/2},\ \ \beta=mx^{-1},\ \ \dot{q}=lx^{-1}e^{-\eta} \tag{10}$$

where $c$, $m$, and $l$ are constants but have the appropriate dimensions. Substituting eq. (10) into the parameters $\mathrm{Gr}_x$, $\mathrm{Bi}_x$, and $\lambda_x$, we obtain:

$$\mathrm{Bi}=\frac{c}{k}\sqrt{\frac{\upsilon}{U_0}},\ \ \mathrm{Gr}=\frac{mg(T_f-T_\infty)}{U_0{}^2},\ \ \lambda=\frac{l\upsilon}{kU_0(T_f-T_\infty)} \tag{11}$$

The coupled nonlinear boundary value problem represented by eqs. (7)-(9) have been solved numerically using the shooting iteration technique together with Runge-Kutta fourth-order integration scheme [10]. From the numerical computation, the plate surface temperature, the local skin-friction coefficient and the local Nusselt number which are, respectively, proportional to $\theta(0)$, $f''(0)$ and $-\theta'(0)$ are worked out and their numerical values are presented in a tabular form.

## Results and discussion

The positive values of local Grashof number $\mathrm{Gr}_x > 0$ is utilised in our computations. This corresponds to the cooling problem with respect to application. The cooling problem is often encountered in engineering applications; for example in the cooling of electronic components and nuclear reactors. Table 1 illustrates the values of local skin friction

coefficient, local Nusselt number and plate surface temperature for different values of parameters embedded in the system. Physically, positive sign of skin friction implies that the fluid exerts a drag force on the right surface of the plate and the negative sign implies the opposite. Both local skin friction and local Nusselt number coefficients increases with an increase in local Biot number, although the presence of back flow of heat into the plate is noticed for Pr = 0.72 (air) since $\theta'(0)$ is positive except for the case of Pr = 7.1 (water) when $\theta'(0)$ is negative and the heat flows out from the left surface of the plate to its right surface. Similar trend is observed as the internal heat generation increases *i. e.* as $\lambda_x$ increases from 1 to 10 with an increase in both local skin friction and local Nusselt number coefficients. As $Gr_x$ increases from 0.1 to 0.5, the local skin friction increases because of the increased strength of the buoyancy forces but the back heat flow into the plate (caused by the internal heat generation) is reduced.

**Table 1. Computation showing $f''(0)$, $\theta(0)$ and $\theta'(0)$ for different parameter values**

| $Bi_x$ | $Gr_x$ | Pr | $\lambda_x$ | $f''(0)$ | $\theta'(0)$ | $\theta(0)$ |
|---|---|---|---|---|---|---|
| 0.1 | 0.1 | 0.72 | 1 | –0.2000518 | 0.076578477 | 1.76578477 |
| 1.0 | 0.1 | 0.72 | 1 | –0.2459676 | 0.281651449 | 1.28165144 |
| 10 | 0.1 | 0.72 | 1 | –0.2695171 | 0.382952717 | 1.03829527 |
| 0.1 | 0.5 | 0.72 | 1 | 0.4221216 | 0.048257030 | 1.48257030 |
| 0.1 | 1.0 | 0.72 | 1 | 0.9895493 | 0.034011263 | 1.34011263 |
| 0.1 | 0.1 | 3.00 | 1 | –0.3748695 | –0.023814576 | 0.76185423 |
| 0.1 | 0.1 | 7.10 | 1 | –0.4138825 | –0.057164001 | 0.42835998 |
| 0.1 | 0.1 | 0.72 | 5 | 0.3741286 | 0.576670381 | 6.76670381 |
| 0.1 | 0.1 | 0.72 | 10 | 0.9010790 | 1.106605802 | 12.0660580 |

Figures 2-5 depict the effects of various thermophysical parameters on the fluid velocity and temperature profiles. Generally, the fluid velocity increases gradually away from the plate, attain its peak value within the boundary layer and then decreases to the free stream zero value satisfying the boundary conditions. The fluid temperature is highest near the plate surface and decreases exponentially to zero value far away from the plate.

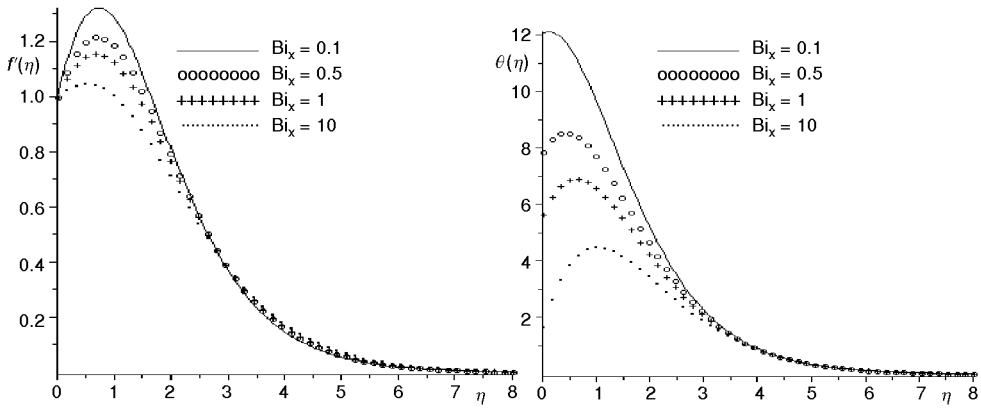


**Figure 2. Effect of local Biot number on the velocity and temperature profiles for Pr = 0.72, $Gr_x$ = 0.1, $\lambda_x$ = 10**

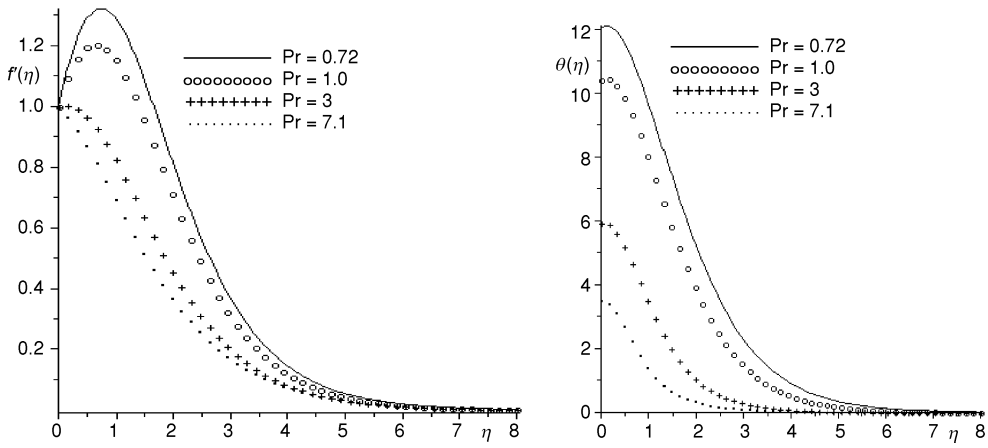


**Figure 3. Effect of Prandtl number on the velocity and temperature profiles for $Gr_x$= $Bi_x$= 0.1, $\lambda_x$ =10**

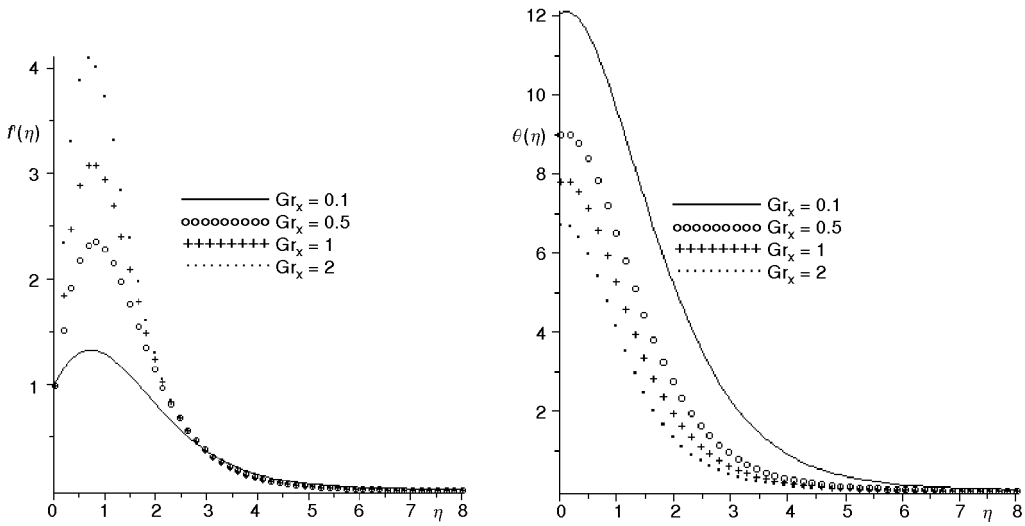


**Figure 4. Effect of local Grashof number of the velocity and temperature profiles for Pr = 0.72, $Bi_x$= 0.1, $\lambda_x$ =10**

It is interesting however to note that both velocity and thermal boundary layer thicknesses decrease with an increase in the Prandtl number (Pr) and intensity of local Biot number ($Bi_x$) due to convective heat transfer the plate surface (see figs. 2 and 3). In fig. 4 we observed that the velocity boundary layer thickness increases while the thermal boundary layer thickness decreases with an increase in the value of local Grashof number ($Gr_x$) due to buoyancy effect. An increase in the exponentially decaying internal heat generation ($\lambda$) causes a further increase in both velocity and thermal boundary layer thicknesses as shown in fig. 5.

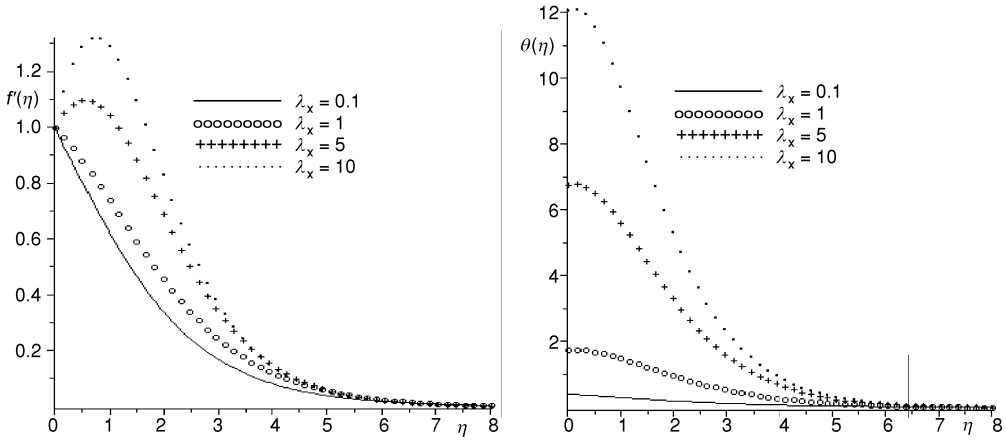


**Figure 5. Effect of internal heat generation on the velocity and temperature profiles for Pr = 0.72, $Gr_x = Bi_x = 0.1$**

## Conclusions

In this paper we have studied numerically the effects of internal heat generation on boundary layer flow over a vertical plate with a convective surface boundary condition. From the present study we have found among others that the internal heat generation prevents the flow of heat from the left surface to the right surface of the plate unless the local Grashof number is strong enough to convert away both the internally generated heat in the fluid.

## Acknowledgements

The authors thank the National Research Foundation and the NRF Thuthuka program for financial support.

## Nomenclature

$Bi_x$ – local Biot number, [–]
$c_p$ – specific heat at constant pressure, [–]
$c$, $m$, $l$ – positive constants, [–]
$f$ – dimensionless stream function, [–]
$Gr_x$ – local Grashof number, [–]
g – gravitational acceleration, [$Ls^{-2}$]
$k$ – thermal conductivity,[$Wm^{-1}K^{-1}$]
Pr – Prandtl number, [–]
$Re_x$ – local Reynolds number, [–]
$T$ – fluid temperature, [K]
$T_\infty$ – free stream temperature, [K]
$T_f$ – hot fluid temperature, [K]
$U_\infty$ – free stream velocity, [$Ls^{-1}$]
$u$, $v$ – velocity components, [$ms^{-1}$]
$x$, y – Cartesian co-ordinates, [m]

*Greek letters*

$\alpha$ – thermal diffusivity, [$m^2s^{-1}$]
$\beta$ – thermal expansion coefficient, [$K^{-1}$]
$\upsilon$ – kinematic viscosity, [$m^2s^{-1}$]
$\lambda$ – internal heat generation parameter, [–]
$\eta$ – similarity variable, [m]
$\rho$ – fluid density, [ $kgm^{-3}$]
$\theta$ – dimensionless temperature, [–]

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Paper submitted: June 7, 2010
Paper revised: July 30, 2010
Paper accepted: November 11, 2010