ANALYSIS OF MARANGONI CONVECTION OF NON-NEWTONIAN POWER LAW FLUIDS WITH LINEAR TEMPERATURE DISTRIBUTION

by

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The problem of steady, laminar, thermal Marangoni convection flow of non-Newtonian power law fluid along a horizontal surface with variable surface temperature is studied. The partial differential equations are transformed into ordinary differential equations by using a suitable similarity transformation and analytical approximate solutions are obtained by an efficient transformation, asymptotic expansion and Padé approximants technique. The effects of power law index and Marangoni number on velocity and temperature profiles are examined and discussed.

Key words: *power law fluid, Marangoni convection, similarity transformation, Padé approximants*

Introduction

When a free liquid surface is present, the surface tension variation resulting from the temperature gradient along the surface can also induce motion within the fluid called thermal Marangoni convection. Most investigations relevant to Marangoni convection have been concentrated on Newtonian fluids [1, 2]. It is well known that the flow and heat transfer characteristics are affected significantly by the physical properties of the fluid. Because most of the fluids in the applications are not strictly Newtonian, during the last decades, the studies on non-Newtonian power law fluids boundary layer flow and heat transfer have attracted considerable attention. Although a number of industrially important fluids such as polymers, molten plastics, foods and slurries exhibit non-Newtonian fluid behavior, only very few works are available in the literature dealing with the heat transfer within non-Newtonian power law fluid Marangoni convection. Naimi [3] researched Marangoni convection of non-Newtonian fluids in a horizontal shallow rectangular cavity, which focused on pseudoplastic and dilatant fluids. Chen [4] studied the effect of Marangoni convection on the flow and heat transfer within a power law liquid film on an unsteady stretching sheet.

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The purpose of this study is to investigate Marangoni convection flow and heat transfer of non-Newtonian power law fluid along a horizontal surface with variable surface temperature. The effects of power law index and Marangoni number on the flow and heat transfer characteristics are analyzed and discussed.

Mathematical formulation

Consider steady, laminar, non-Newtonian power law fluid Marangoni convection induced by the linear temperature variation along a horizontal surface, as shown in fig. 1.

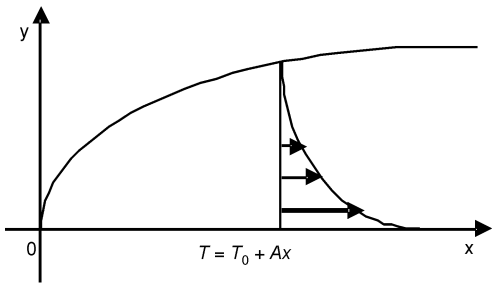

Figure 1. Physics model with interface condition and coordinate system

The physical properties of the fluid are assumed to be constant, except the surface tension which is usually assumed to vary linearly with temperature:

$$\sigma = \sigma_0[1 - \gamma(T - T_0)] \tag{1}$$

where $\gamma = -1/\sigma_0(\partial s/\partial T)$ is the temperature coefficient of surface tension. Since the surface tension decreases with temperature for most liquids, γ is a positive fluid property. The shear stress tensor is defined as:

$$\tau = -K\left|\frac{\partial u}{\partial y}\right|^{n-1}\frac{\partial u}{\partial y} \tag{2}$$

where K is the consistency coefficient and n is the power law index. The fluid is Newtonian for $n = 1$ with $K = \mu$. However, the thermal induced surface tension gradient along the interface, *i. e.*:

$$\frac{\partial \sigma}{\partial x} = \frac{\partial \sigma}{\partial T}\cdot\frac{\partial T}{\partial x} \tag{3}$$

may produce interfacial flow, which is of major concern in this study. This thermally-induced Marangoni convection couples the hydrodynamic and the thermal problems. The boundary layer equations are based on the balance laws of mass, momentum, energy modified to include the effects of Marangoni convection and power law fluid. The resulting equations can be written as [5]:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{4}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{K}{\rho}\frac{\partial}{\partial y}\left(\left|\frac{\partial u}{\partial y}\right|^{n-1}\frac{\partial u}{\partial y}\right) \tag{5}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial}{\partial y}\left(\left|\frac{\partial u}{\partial y}\right|^{n-1}\frac{\partial T}{\partial y}\right) \tag{6}$$

where x and y are Cartesian coordinates measured along the flat surface and normal to it, respectively, u and v being the velocity components along x and y axes, ρ is the density, T is the temperature, α is the thermal diffusivity. The boundary conditions are:

$$K\left|\frac{\partial u}{\partial y}\right|^{n-1}\frac{\partial u}{\partial y}=\frac{\mathrm{d}\sigma}{\mathrm{d}T}\cdot\frac{\partial T}{\partial x},\ v=0,\ T=T_0+Ax \text{ at } y=0 \tag{7}$$

$$u\to 0,\ T\to T_0,\text{ as } y\to\infty \tag{8}$$

Using the standard definition of the stream function such as $u = \partial\psi/\partial y$ and $v = -\partial\psi/\partial x$, introduce the following similarity variables:

$$\psi(x,y)=\left(\frac{K(A\sigma_0\gamma)^{2n-1}}{\rho^{2n}}\right)^{1/3n}x^{2/3}f(\eta),\ \eta=\left(\frac{\rho^n}{K^2(A\sigma_0\gamma)^{n-2}}\right)^{1/3n}x^{-1/3}y,\ T=T_0+Ax\theta(\eta) \tag{9}$$

The governing eqs. (5) and (6) can be rewritten as:

$$[f'(\eta)]^2-2f(\eta)f''(\eta)=3[|f''(\eta)|^{n-1}f''(\eta)]' \tag{10}$$

$$3Mf'(\eta)\theta(\eta)-2Mf(\eta)\theta'(\eta)=3[|f''(\eta)|^{n-1}\theta'(\eta)]' \tag{11}$$

where $M=AK/(\rho\alpha)$ is Marangoni number.

The boundary conditions become

$$f(0)=0,\ f'(\infty)=0,\ f''(0)=-1 \tag{12}$$

$$\theta(0)=1,\ \theta(\infty)=0 \tag{13}$$

Solution of the similarity equations

According to the boundary conditions, it must be $f''(\eta) \leq 0$. So we get:

$$|f''(\eta)|^{n-1}f''(\eta)=\left(-f''(\eta)\right)^{n-1}f''(\eta)=(-1)^{n-1}\left(f''(\eta)\right)^n \tag{14}$$

Substituting eq. (14) into eqs. (10) and (11), we obtain the following ordinary differential equations:

$$f'^2-2ff''=(-1)^{n-1}3n(f'')^{n-1}f''' \tag{15}$$

$$3Mf'\theta-2Mf\theta'=(-1)^{n-1}3(n-1)f''^{n-2}\theta'+(-1)^{n-1}3f''^{n-1}\theta'' \tag{16}$$

The initial condition is assumed as follows

$$f'(0)=c \tag{17}$$

Let

$$f(\eta)=\varepsilon g(\xi)+\varepsilon^{1/3}c\xi-\varepsilon^{2/3}\frac{1}{2}\xi^2,\ \xi=\varepsilon^{-1/3}\eta \tag{18}$$

where ε is an artificial small parameter. Substituting this transformation into eq. (15), because of $f''(\eta) \le 0$, results in the following equation:

$$\left(\varepsilon^{2/3}g' + c - \varepsilon^{1/3}\xi\right)^2 - 2\left(\varepsilon g + \varepsilon^{1/3}c\xi - \varepsilon^{2/3}\frac{1}{2}\xi^2\right)\left(\varepsilon^{1/3}g'' - 1\right) = 3nc^{n-1}\left(1 - \varepsilon^{1/3}g''\right)^{n-1}g''' \tag{19}$$

The boundary conditions become

$$g(0) = 0,\ g'(0) = 0,\ g''(0) = 0 \tag{20}$$

Expanding $|1-\varepsilon^3 g''|$ in a power series development, we can get:

$$\begin{aligned}&\left(\varepsilon^{2/3}g' + c - \varepsilon^{1/3}\xi\right)^2 - 2\left(\varepsilon g + \varepsilon^{1/3}c\xi - \varepsilon^{2/3}\frac{1}{2}\xi^2\right)\left(\varepsilon^{1/3}g'' - 1\right) = \\ &= 3nc^{n-1}\left(1 - \varepsilon^{1/3}(n-1)g'' + \varepsilon^{2/3}\frac{(n-1)(n-2)}{2}g''^2 - \right. \\ &\left. -\varepsilon\frac{(n-1)(n-2)(n-3)}{6}g''^3 + \cdots\right)g'''\end{aligned} \tag{21}$$

The solution of eq. (21) can be obtained by expanding g in a power series development near $\varepsilon = 0$ as follows:

$$g(\xi) = g_0 + g_1\varepsilon^{1/3} + g_2\varepsilon^{2/3} + g_3\varepsilon + g_4\varepsilon^{4/3} + \cdots \tag{22}$$

Substituting eq. (22) into eq. (21) and equating the coefficients of ε, yields:

$$\begin{aligned}f(\eta) &= c\eta - \frac{1}{2}\eta^2 + \frac{c^{3-n}}{18n}\eta^3 + \frac{1}{216n^2}c^{6-2n}(n-1)\eta^4 \\ &+ \frac{1}{3240n^3}[2c^{9-3n}n^2 - 3c^{9-3n}n + c^{9-3n} - 6c^{5-2n}n]\eta^5 \\ &+ \frac{1}{58320n^4}[6c^{12-4n}n^3 - 11c^{12-4n}n^2 + 6c^{12-4n}n - 42c^{8-3n}n^2 - c^{12-4n} + 42c^{8-3n}n + 18c^{4-2n}n^2]\eta^6 \\ &+ \cdots\end{aligned} \tag{23}$$

$$\begin{aligned}f'(\eta) &= c - \eta + \frac{c^{3-n}}{6n}\eta^2 + \frac{1}{54n^2}c^{6-2n}(n-1)\eta^3 \\ &+ \frac{1}{648n^3}[2c^{9-3n}n^2 - 3c^{9-3n}n + c^{9-3n} - 6c^{5-2n}n]\eta^4 \\ &+ \frac{1}{9720n^4}[6c^{12-4n}n^3 - 11c^{12-4n}n^2 + 6c^{12-4n}n - 42c^{8-3n}n^2 - c^{12-4n} + 42c^{8-3n}n + 18c^{4-2n}n^2]\eta^5 \\ &+ \cdots\end{aligned} \tag{24}$$

In order to get the value of c, we use Padé approximants [2/2] on $f'(\eta)$. We choose a_0, $a1$, a_2, a_3, a_4, b_1, and b_2 as the coefficients of $\eta^i (i = 0, 1, 2, 3, 4)$:

$$f'(\eta)(1+b_1\eta+b_2\eta^2) = a_0 + a_1\eta + a_2\eta^2 + (a_3\eta^3 + a_4\eta^4) \tag{25}$$

Substituting eq. (24) into eq. (25), applying the condition $f'(\infty) = 0$ and equating coefficients of η^i, we get:

$$cb_2 - b_1 + \frac{c^{3-n}}{6n} = 0 \tag{26}$$

$$-b_2 + \frac{c^{3-n}}{6n} b_1 + \frac{1}{54n^2} c^{6-2n}(n-1) = 0 \tag{27}$$

$$\frac{c^{3-n}}{6n} b_2 + \frac{1}{54n^2} c^{6-2n}(n-1) b_1 + \frac{1}{648n^3}[2c^{9-3n}n^2 - 3c^{9-3n}n + c^{9-3n} - 6c^{5-2n}n] = 0 \tag{28}$$

Solving equations (26) to (28), the value of c can be obtained.

The initial condition is assumed as follows

$$\theta'(0) = h \tag{29}$$

Let:

$$\theta(\eta) = \varepsilon^{2/3}\varphi(\xi) + \varepsilon^{1/3}h\xi + 1, \quad \xi = \varepsilon^{-1/3}\eta \tag{30}$$

Substituting this transformation into eq. (16), the governing equation can be rewritten as:

$$\begin{aligned}3M(\varepsilon^{2/3}g' + c - \varepsilon^{1/3}\xi)(\varepsilon^{2/3}\varphi + \varepsilon^{1/3}h\xi + 1) - 2M(\varepsilon g + \varepsilon^{1/3}c\xi - \varepsilon^{2/3}\frac{1}{2}\xi^2)(\varepsilon^{1/3}\varphi' + h)\\ = -3(n-1)(1-\varepsilon^{1/3}g'')^{n-2}(\varepsilon^{1/3}\varphi' + h) + 3\varphi''(1-\varepsilon^{1/3}g'')^{n-1}\end{aligned} \tag{31}$$

Expanding $(1 - \varepsilon^{1/3}g'')^{n-2}$ and $(1 - \varepsilon^{1/3}g'')^{n-1}$ in a power series development, we can get:

$$\begin{aligned}&3M(\varepsilon^{2/3}g' + c - \varepsilon^{1/3}\xi)(\varepsilon^{2/3}\varphi + \varepsilon^{1/3}h\xi + 1) - 2M(\varepsilon g + \varepsilon^{1/3}c\xi - \varepsilon^{2/3}\frac{1}{2}\xi^2)(\varepsilon^{1/3}\varphi' + h)\\ &= -3(n-1)\varepsilon^{1/3}\varphi' + h)(1 - \varepsilon^{1/3}(n-2)g'' + \varepsilon^{2/3}\frac{(n-2)(n-3)}{2}g''^2 - \varepsilon\frac{(n-2)(n-3)(n-4)}{6}g''^3\\ &+\varepsilon^{4/3}\frac{(n-2)(n-3)(n-4)(n-5)}{24}g''^4 + \cdots) + 3\varphi''(1 - \varepsilon^{1/3}(n-1)g'' + \varepsilon^{2/3}\frac{(n-1)(n-2)}{2}g''^2\\ &-\varepsilon\frac{(n-1)(n-2)(n-3)}{6}g''^3 + \varepsilon^{4/3}\frac{(n-1)(n-2)(n-3)(n-4)}{24}g''^4 + \cdots)\end{aligned} \tag{32}$$

The boundary conditions become

$$\varphi(0) = 0, \quad \varphi'(0) = 0, \tag{33}$$

The solution of eq. (32) can be obtained by expanding ϕ in a power series development near $\varepsilon = 0$ as follows:

$$\varphi(\xi) = \varphi_0 + \varphi_1\varepsilon^{1/3} + \varphi_2\varepsilon^{2/3} + \varphi_3\varepsilon + \varphi_4\varepsilon^{4/3} + \cdots \tag{34}$$

Substituting eq. (34) into eq. (33) and equating the coefficients of ε, we obtain:

$$\theta(\eta) = 1 + h\eta + \frac{1}{2}(nh - h + Mc)\eta^2 + \frac{1}{18n}(3n^3h - 6n^2h + 3n^2Mc + 3nh - 3nMc + nhc^{3-n} - $$

$$- hc^{3-n} + Mnc^{4-n} - Mc^{4-n} + Mcnh - 3Mn)\eta^3 + \cdots \tag{35}$$

where the constant h may be established by Padé approximants technique as the value of c.

Results and discussion

The values of the wall velocity and the wall temperature gradient are presented in tabs. 1-2. Table 1 presents the effect of power law index on the wall velocity. The approximate results indicate that the wall velocity $f'(0)$ increases with the increasing of n till $n = 1$, then decreases with the increasing of n. The value of $f'(0)$ excellently agrees with the result 1.1447 submitted by Zhang [6] for $n = 1$. Table 2 presents the effect of power law index and Marangoni number on the wall temperature gradient. It is obvious that the wall temperature gradient is a decreasing function of n and an increasing function of M.

Table 1. To different values of *n*, the comparison of $f'(0)$

$n = 0$	$n = 8$	$n = 1$	$n = 1.1$	$n = 1.2$
1.1211644	1.142355	1.144714	1.141509	1.13625

Table 2. To different values of *n* and *M*, the comparison of $-\theta'(0)$

	$M = 2$	$M = 3$	$M = 4$
$n = 0.8$	1.8100513	2.2817272	2.6821645
$n = 1$	1.6230846	2.1100871	2.5214073
$n = 1.1$	1.5408525	2.0262522	2.4377732
$n = 1.2$	1.4628330	1.9432602	2.3527847

Figure 2 shows the comparison of the dimensionless temperature profiles in the different Marangoni number as n fixed. It may be seen that dimensionless temperature and the boundary layer thickness all decrease with the increasing of Marangoni number. Figure 3

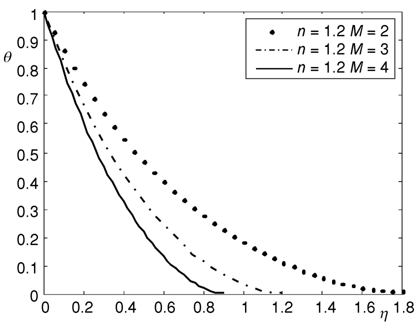

Figure 2. Effects of *M* on temperature profiles for *n* = 1.2

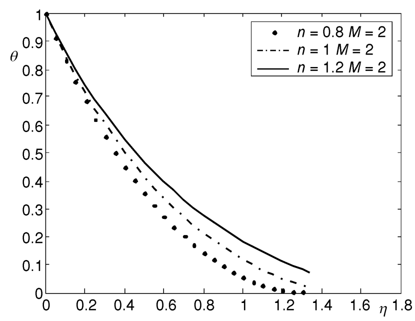

Figure 3. Effects of *n* on temperature profiles for *M* = 2

shows the comparison of the dimensionless temperature profiles with $M = 2$. It is also observed, to the same Marangoni number, the dimensionless temperature and the boundary layer thickness all increase with the increasing of power law index.

Conclusions

The influence of Marangoni convection on the flow and heat transfer characteristics of the power law liquid with an imposed linear temperature on the subsurface has been explored. It is demonstrated that the wall velocity increases with the increasing of power law index till $n = 1$, then decreases with the increasing of power-law index. Both temperature and temperature boundary layer thickness decrease with the increasing of Marangoni number and increase with the increasing of power law index.

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Nomenclature

C_s – surface morphology, [–]
A – surface temperature variation, [–]
c, h – constant, [–]
f – dimensionless velocity, [–]
g – dimensionless velocity defined by eq. (18), [–]
n – power law index, [–]
T – temperature, [K]
T_0 – ambient temperature, [K]
u – velocity component along x, direction, [$\mathrm{ms^{-1}}$]
v – velocity component along y direction, [$\mathrm{ms^{-1}}$]
x – coordinate along the flat surface, [m]
y – coordinate normal to the flat surface, [m]

Greek symbols

α – thermal diffusivity, [$\mathrm{m^2s^{-1}}$]
η – similarity variable, [–]
σ – surface tension, [$\mathrm{Nm^{-1}}$]
σ_0 – surface tension at origin, [$\mathrm{Nm^{-1}}$]
θ – dimensionless temperature, [–]
τ – shear stress tensor (see eq.2), [N]
ε – artificial small parameter, [–]
ρ – density, [$\mathrm{kgm^{-3}}$]
ψ – stream function, [$\Omega^{-1}\mathrm{m}^{-1}$]
ξ – similarity variable defined by eq. (18), [–]

References

[1] Christopher, D. M., Wang, B. X., Prandtl Number Effects for Marangoni Convection over a Flat Surface, *Int. J. Therm. Sci., 40* (2001), 6, pp. 564-570
[2] Zheng, L. C., *et al*., An Approximately Analytical Solution for the Marangoni Convection in an In-Ga-Sb System, *Chin. Phys. Lett., 21* (2004), 10, pp. 1983-1985
[3] Naimi, M., Hasnaoui, M., Platten, J. K., Marangoni Convection of Non-Newtonian Power Law Fluids in a Shallow Rectangular Cavity, *Engineering Computations, 17* (2000), 6-7, pp. 638-668
[4] Chen, C. H., Marangoni Effects on Forced Convection of Power-Law Liquids in a Thin Film over a Stretching Surface, *Phys. Lett. A, 370* (2007), 1, pp. 51-57
[5] Zheng, L. C., Zhang, X. X., Skin Friction and Heat Transfer in Power Law Fluid Laminar Boundary Layer along a Moving Surface, *Int. J. Heat Mass Transfer., 45* (2002), 13, pp. 2667-2672

[6] Zhang, Y., Zheng, L. C., Zhang, X. X., Approximate Analytical Solutions for Boundary Layer Equations of Marangoni Convection Driven by a Power-Law Temperature Gradient, *Proceedings*, 3rd International Conference on Mechanical Engineering and Mechanics, Beijing, China, 2009, pp. 127-132

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