

SOLUTION OF THE TWO-PHASE STEFAN PROBLEM BY USING THE PICARD'S ITERATIVE METHOD

by

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Original scientific paper
UDC: 536.242:517.538
DOI: 10.2298/TSCI11S1021W

In this paper, an application of the Picard's iterative method for finding the solution of two phase Stefan problem is presented. In the proposed method an iterative connection is formulated, which allows to determine the temperature distribution in considered domain. Another unknown function, describing position of the moving interface, is approximated with the aid of linear combination of some base functions. Coefficients of this combination are determined by minimizing a properly constructed functional.

Key words: Stefan problem, solidification, Picard's iterative method

Introduction

In this paper, we consider the two-phase Stefan problem which consists in determining the temperature distribution in the given domain and the function describing position of the moving interface (the freezing front) [1]. Stefan problem is a mathematical model of thermal processes during which the change of phase takes place. The examples of such kind of processes can be solidification of pure metals, melting of ice, freezing of water, deep freezing of foodstuffs and so on.

In some simple cases of the Stefan problem there are chances for finding the analytical solution [2], but in most of cases the approximated methods must be applied [3,4]. In paper [5], the authors have applied the Adomian decomposition method, combined with some minimization procedure, for finding the approximate solution of one-phase Stefan problem. Application of the variational iteration method [6-9] for calculating the approximate solution of direct and inverse Stefan problem is considered in paper [10]. Moreover, in papers [11, 12] some new approach for solving the one-phase Stefan problem is presented. In this approach, the considered problem is at first transformed for the domain of unit square and after that, such transformed problem is solved by using the variational iteration method. Another applications of the variational iteration method for solving problems connected with the heat conductivity are presented in papers [13, 14].

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In the current paper, we propose to apply the Picard's iterative method for solving the two-phase Stefan problem. The Picard's iterative method [15] consists in formulating the iterative procedure, which enables to determine form of the unknown function, describing the temperature distribution in the given domain, on the ground of the equation and initial condition which should be satisfied. Another unknown function, describing position of the moving interface, is approximated in the form of linear combination of some base functions. Coefficients of this linear combination are calculated in the course of minimizing the properly constructed functional. An example illustrating the accuracy of the obtained approximate solution (compared with the known analytical solution of the problem) and speed of convergence of the iterative procedure is also shown. Application of the Picard's iterative method for the solution of one-phase Stefan problem is presented in paper [16].

Two-phase Stefan problem

The two-phase Stefan problem [1] consists in finding position of the moving interface, described by means of function $\xi(t)$, and determining the functions $u(x,t)$ and $v(x,t)$ defined in domains D_1 and D_2 (see fig. 1), respectively, which fulfill the heat conduction equations:

$$\frac{\partial u(x,t)}{\partial t} = \alpha_1 \frac{\partial^2 u(x,t)}{\partial x^2} \quad \text{in } D_1, \quad (1)$$

$$\frac{\partial v(x,t)}{\partial t} = \alpha_2 \frac{\partial^2 v(x,t)}{\partial x^2} \quad \text{in } D_2, \quad (2)$$

where α_p is the thermal diffusivity in liquid phase ($p = 1$) and solid phase ($p = 2$), respectively, u and v denote the temperature in liquid phase and solid phase, while t and x refer to the time and spatial location, respectively.

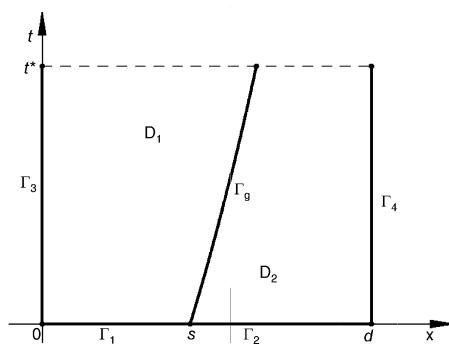


Figure 1. Domain of the two-phase problem
and the Stefan condition:

$$u(\xi(t), t) = v(\xi(t), t) = u^*, \quad (7)$$

$$L \frac{d\xi(t)}{dt} = \kappa_2 \frac{\partial v(x,t)}{\partial x} \Big|_{x=\xi(t)} - \kappa_1 \frac{\partial u(x,t)}{\partial x} \Big|_{x=\xi(t)}, \quad (8)$$

On boundaries Γ_1 and Γ_2 functions u and v fulfill the initial conditions:

$$u(x,0) = \phi_1(x) \quad \text{on } \Gamma_1, \quad (3)$$

$$v(x,0) = \phi_2(x) \quad \text{on } \Gamma_2, \quad (4)$$

on boundaries Γ_3 and Γ_4 they satisfy the Dirichlet boundary conditions:

$$u(x,t) = \vartheta(t) \quad \text{on } \Gamma_3, \quad (5)$$

$$v(x,t) = \eta(t) \quad \text{on } \Gamma_4, \quad (6)$$

and finally, on the moving interface Γ_g they fulfill the condition of temperature continuity

where κ_p denotes the thermal conductivity in liquid phase ($p = 1$) and solid phase ($p = 2$) respectively, u^* – the melting-point temperature, L denotes the latent heat of fusion per unit volume, and $\xi(t)$, – the function describing position of the moving interface Γ_g .

In the discussed Picard's method, we transform the heat conduction eqs. (1) and (2) into the following integral form:

$$u(x, t) = u(x, 0) + \alpha_1 \int_0^t \frac{\partial^2 u(x, \tau)}{\partial x^2} d\tau, \quad t \in (0, t^*), \quad x \in (0, \xi(t)), \quad (9)$$

$$v(x, t) = v(x, 0) + \alpha_2 \int_0^t \frac{\partial^2 v(x, \tau)}{\partial x^2} d\tau, \quad t \in (0, t^*), \quad x \in (\xi(t), d), \quad (10)$$

from which we receive two iterative formulas:

$$u_k(x, t) = \phi_1(x) + \alpha_1 \int_0^t \frac{\partial^2 u_{k-1}(x, \tau)}{\partial x^2} d\tau, \quad (11)$$

$$v_k(x, t) = \phi_2(x) + \alpha_2 \int_0^t \frac{\partial^2 v_{k-1}(x, \tau)}{\partial x^2} d\tau, \quad (12)$$

for $k = 1, 2, \dots$, where $u_0(x, t)$ and $v_0(x, t)$ denote the initial approximations of the unknown functions $u(x, t)$ and $v(x, t)$. The initial approximations are introduced in such way that they satisfy the initial conditions (3) or (4) and the boundary conditions (5) or (6), respectively:

$$u_0(x, t) = e^x[\mathcal{G}(t) - \mathcal{G}(0)] + \phi_1(x), \quad (13)$$

$$v_0(x, t) = e^{d-x}[\eta(t) - \eta(0)] + \phi_2(x) \quad (14)$$

In this way we receive the sequences $\{u_k\}_{k=0}^\infty$ and $\{v_k\}_{k=0}^\infty$, which are convergent (under the proper assumptions – see [15]) to the exact solutions of eqs. (1) and (2). The sufficient conditions for convergence of the Picard's iterative method are formulated in paper [15]. However, checking whether or not the given equation satisfies those conditions is difficult in many cases (for example in case of problem considered in the present paper). That is why the problem of formulating (and proving) the convergence conditions, sufficient and necessary, which would be easy to verify for any given equation is still open.

The function $\xi(t)$, describing position of the moving interface, is defined in the form of linear combination:

$$\xi(t) = \sum_{i=1}^m p_i \psi_i(t), \quad (15)$$

where the coefficients $p_i \in R$ and the base functions $\psi_i(t)$ are linearly independent. For determining the unknown coefficients p_i we use the following functional derived with the aid of the condition of temperature continuity (7) and the Stefan condition (8):

$$J(p_1, \dots, p_m) = \int_0^{t^*} (u_n(\xi(t), t) - u^*)^2 dt + \int_0^{t^*} (v_n(\xi(t), t) - u^*)^2 dt + \int_0^{t^*} \left(\kappa_2 \frac{\partial v_n(x, t)}{\partial x} \Big|_{x=\xi(t)} - \kappa_1 \frac{\partial u_n(x, t)}{\partial x} \Big|_{x=\xi(t)} - L \frac{d\xi(t)}{dt} \right)^2 dt \quad (16)$$

Example

Theoretical consideration introduced in the previous section will be now illustrated with example, in which the calculated approximate solution will be compared with the known exact solution. In the presented calculations, as the base functions of linear combination (15) we will use the monomials:

$$\psi_i(t) = t^{i-1}, \quad i = 1, \dots, m \quad (17)$$

It is assumed in this example that: $d = 3$, $s = 1.5$, $\alpha_1 = 2.5$, $\alpha_2 = 1.25$, $\kappa_1 = 6$, $\kappa_2 = 2$, $L = 0.8$, $u^* = 1$. The initial conditions (3) and (4) are determined by the functions:

$$\phi_1(x) = e^{-0.2x+0.3} \quad (18)$$

$$\phi_2(x) = e^{-0.4x+0.6} \quad (19)$$

whereas the Dirichlet boundary conditions (5) and (6) are described by the functions:

$$g(t) = e^{0.1t+0.3} \quad (20)$$

$$\eta(t) = e^{0.2t-0.6} \quad (21)$$

Exact solution of the considered two-phase Stefan problem takes the form:

$$u(x, t) = e^{-0.2x+0.1t+0.3} \quad (22)$$

$$v(x, t) = e^{-0.4x+0.2t+0.6} \quad (23)$$

$$\xi(t) = 0.5t + 1.5 \quad (24)$$

As it was announced, the initial approximations are get in the form of functions which satisfy the given initial and boundary conditions (see (13) and (14)):

$$u_0(x, t) = e^{0.1t+x+0.3} + e^{0.3-0.2x} - e^{0.3+x}, \quad (25)$$

$$v_0(x, t) = e^{0.2t-x+2.4} + e^{0.6-0.4x} - e^{2.4-x}. \quad (26)$$

After the proper calculations we derive the following sequences $\{u_n\}_{n \in N}$ and $\{v_n\}_{n \in N}$ of approximated functions describing distributions of temperature ($n \geq 1$):

$$u_n(x, t) = e^{\frac{3}{10}+x} \left(25^n e^{\frac{t}{10}} - 25^n \sum_{k=0}^n \frac{(t/10)^k}{k!} \right) + e^{\frac{3}{10}-\frac{x}{5}} \sum_{k=0}^n \frac{(t/10)^k}{k!}, \quad (27)$$

$$v_n(x, t) = e^{\frac{12}{5}-x} \left[\left(\frac{25}{4} \right)^n e^{\frac{t}{5}} - \left(\frac{25}{4} \right)^n \sum_{k=0}^n \frac{(t/5)^k}{k!} \right] + e^{\frac{3}{5}-\frac{2x}{5}} \sum_{k=0}^n \frac{(t/5)^k}{k!}. \quad (28)$$

The above sequences are convergent to the exact solution:

$$u(x, t) = \lim_{n \rightarrow \infty} u_n(x, t) = \exp\left(\frac{3}{10} - \frac{x}{5} + \frac{t}{10}\right) \quad (29)$$

$$v(x, t) = \lim_{n \rightarrow \infty} v_n(x, t) = \exp\left(\frac{3}{5} - \frac{2x}{5} + \frac{t}{5}\right) \quad (30)$$

Function describing position of the moving interface is defined as the linear combination (15) of base functions (17) with coefficients determined in the course of minimizing the functional (16). In tab. 1 there are compiled the errors of approximated solution for $n = 2, 3, 4$ and $m = 2$.

Figure 2 presents the errors with which the approximated solution ($n = 6, m = 2$) reconstructs the exact position of the moving interface and errors with which it fulfills the Stefan condition (8). The other conditions, (5)-(7), are satisfied with the comparably small errors, whereas the initial conditions are satisfied exactly. Presented results show that after only several iterations we can receive the approximated solution with the small error.

Table 1. Values of errors in the reconstructed position of the moving interface and temperature distributions

	$n = 2$	$n = 3$	$n = 4$
Δ_ξ	0.08973	0.00884	0.00153
δ_x [%]	5.51664	0.54321	0.09403
Δ_u	0.01854	0.00553	0.
δ_u [%]	1.56160	0.46610	0.
Δ_v	0.00313	0.00040	0.
δ_v [%]	0.40258	0.05109	0.

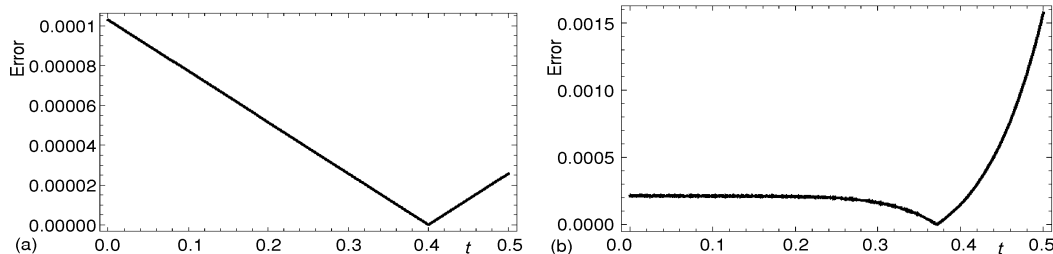


Figure 2. Error in the reconstructed position of the moving interface (a) and error of satisfying the Stefan condition by the approximated solution (b)

Conclusions

The paper presents application of the Picard's iterative method for finding the approximated solution of the two-phase Stefan problem. The proposed approach consists in determining the temperature distribution with the aid of some iterative formulas and calculating the coefficients of the linear combination of some base functions, approximating position of the moving interface, in course of minimizing the properly constructed functional. Presented example shows that the sequence of successive approximations, which we receive in this method, is convergent to the exact solution, if it exists. In [15] there are formulated the sufficient conditions of this convergence, however, they are difficult to check in most of cases (also for example considered in the current paper). That is why the problem of formulating and proving the convergence conditions of the Picard's method, easy to verify for any equation, is still open.

Nomenclature

D_i	– domains of the problem
d	– length of domain, [m]
L	– latent heat of fusion per unit volume, [Jkg^{-1}]
t	– time, [s]
u, v	– temperature, [K]
u^*	– melting-point temperature, [K]
x	– space variable, [m]

Greek letters

α	– thermal diffusivity, [m^2s^{-1}]
Δ	– absolute error
δ	– relative error, [%]
Γ_i	– boundary of domain
κ	– thermal conductivity, [$\text{Wm}^{-1}\text{K}^{-1}$]
ξ	– function describing position of the phase change moving interface

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Paper submitted: April 29, 2010

Paper revised: July 27, 2010

Paper accepted: November 18, 2010