CONSTRUCTAL ENTRANSY DISSIPATION MINIMIZATION FOR “VOLUME-POINT” HEAT CONDUCTION BASED ON TRIANGULAR ELEMENT

by

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By taking equivalent thermal resistance, which reflects the average heat conduction effect and is defined based on entransy dissipation, as optimization objective, the “volume to point” constructal problem based on triangular element of how to discharge the heat generated in a fixed volume to a heat sink on the border through relatively high conductive link is re-analyzed and re-optimized in this paper. The constructal shape of the control volume with the best average heat conduction effect is deduced. For the same parameters, the constructs based on minimization of entransy dissipation and the constructs based on minimization of maximum temperature difference are compared, and the results show that the constructs based on entransy dissipation can decrease the mean temperature difference better than the constructs based on minimization of maximum temperature difference. But with the increase of the number of order, the mean temperature difference does not always decrease, and there exists some fluctuations. Because the idea of entransy describes heat transfer ability more suitably, the optimization results of this paper can be put to engineering application of electronic cooling.

Key words: *constructal theory, entransy dissipation, volume-point heat conduction, generalized thermodynamic optimization*

Introduction

The cooling of electronic components can be concluded as “volume-point” heat conduction problem, which can be described as how to determine the optimal distribution of high conductivity material through the given volume such that the heat generated at every point was transferred most effectively to its boundary, has became the focus of attention in the current scientific literature [1-9].

The idea of constructal law was introduced by Bejan [2-9], which was described as: for a finite-size flow system to persist in time (to live), its configuration must change in time such that it provides easier and easier access to its currents (fluid, energy, species, *etc*.). The constructal law was firstly used in the solving of volume-point heat conduction of a rectangular area by taking the minimization of maximum temperature difference in the area as the optimiza-

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tion objective [1]. As shown in fig. 1, firstly, the maximum temperature difference of the element was minimized, and the corresponding optimal elemental shape was obtained. Then, a relatively larger heat generating volume was designed and optimized by introducing a new link of high conductivity material. A number (n_1) of optimized elemental volumes were assembled on the upper and lower sides of the new link. There exists an optimal n_1 corresponding to the minimization of maximum temperature difference of the assembled area (first assembly). The analogy continued until the control area was recovered by the construct. Scholars made many studies on this problem using constructal theory [10-29]. The first aspect is to consider a heat conduction model which is more close to the reality, such as the case that conduction trees with spacing at the tips [10], the case that heat flux is not linear in high conductive link [11], and the three-dimensional model [12]. The second aspect is to relax the constraints of the model, such as adding degrees of freedom to minimize heat resistance [13-18]. The third aspect is to consider various conduction models such as triangle model [19], disc-shaped area model [20], point-circle model, point-line model, and point-plane model [21], *etc*. The fourth aspect is to consider different optimization objectives, such as entropy production minimization [22], exergy destruction minimization [23], and the temperature decreasing time of heat body [24], *etc*.

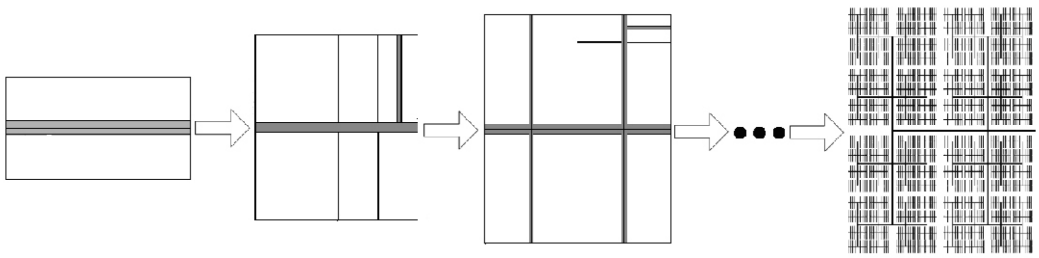

Figure 1. Constructal optimization procedure of rectangular area

References [1, 10-21] set the minimization of the maximum temperature difference as the optimization objective which is to limit maximum temperature difference of the body for safety consideration. Entropy and entropy production are physical quantities defined for transition between power and heat. Minimization of entropy production or exergy destruction can be chosen as an optimization objective when it is intended to decrease the loss of available energy, but the heat transfer mostly focuses on the heat transfer regularity and its transfer speed, not the exergy lost. Therefore, the entropy generation minimization (EGM) is not entirely consistent with the heat transfer optimization objective. To solve this shortage in the current heat transfer theory, based on hypostasis of heat transfer phenomenon, Guo *et al*. [30] defined a heat transfer potential capacity and heat transfer potential capacity dissipation function from heat transfer theory, and pointed out that their physical meanings were heat transfer ability amount and its dissipation rate in the heat transfer process. In terms of the analogy between heat and electrical conduction, Guo *et al*. [31] validated that the heat transfer potential capacity $E_{vh} = Q_{vh}T/2$ was a new physical quantity describing heat transfer ability, which was corresponding to electrical potential energy, and named it as *entransy*. The heat transfer ability lost in heat transfer process was called as entransy dissipation. The extremum principle of entransy dissipation was proposed as follows: for a fixed boundary heat flux, the conduction process is optimized when the entransy dissipation is minimized (minimum temperature difference), while for a fixed boundary temperature, the conduction is optimized when the entransy dissipation is maximized (maximum heat flux).

The volume to point conduction problem is a classical multi-dimensional conduction, equivalent thermal resistance, which is based on entransy, is a good reflection of its average conductive effect. Wei *et al*. [32] took the entransy dissipation minimization as optimization objective for volume-point heat conduction based on rectangular element. In this paper, equivalent thermal resistance is taken as the optimization objective, volume-point heat conduction problem based on triangular element is re-optimized and new structures and equivalent thermal resistance of each element are deduced. The mean temperature difference is deduced based on equivalent thermal resistance and it is compared with the mean temperature difference of the construct from minimization of maximum temperature difference. The results shows that the construct based on minimization of entransy dissipation can improve the conductive ability greatly.

Definition of entransy and equivalent thermal resistance

Entransy, which is a new physical quantity reflecting heat transfer ability of an object, is defined in ref. [31] as:

$$E_{vh} \quad \frac{1}{2} Q_{vh} U_h \quad \frac{1}{2} Q_{vh} T \tag{1}$$

where $Q_{vh} = Mc_v T$ is the thermal energy or the heat stored in an object with constant volume which may be referred to as the thermal charge, U_h or T represents the thermal potential. The entransy dissipation function which represents the entransy dissipation per unit time and per unit volume is deduced as [31]:

$$\dot{E}_{h\phi} \quad \dot{q} \quad T \tag{2}$$

where $\dot{q}$ is the thermal current density vector, and T – the temperature gradient. In steady-state heat conduction, $\dot{E}_{h\phi}$ can be calculated as the difference between the entransy input and the entransy output of the object, *i. e.*:

$$\dot{E}_{h\phi} \quad E_{vh,\mathrm{in}} \quad E_{vh,\mathrm{out}} \tag{3}$$

The entransy dissipation rate of the whole volume in the volume to point conduction is:

$$\dot{E}_{\mathrm{vh}\phi} \quad {}_{v} \dot{E}_{\mathrm{h}\phi} \mathrm{d}v \tag{4}$$

The equivalent thermal resistance for multi-dimensional heat conduction problems with specified heat flux boundary condition is given as [31]:

$$R_h \quad \frac{\dot{E}_{vh\phi}}{\dot{Q}_h^2} \tag{5}$$

where $\dot{Q}_h$ is the heat flow (thermal current). The mean temperature difference for multi-dimensional heat conduction can be expressed as:

$$\Delta\overline{T} \quad R_{\mathrm{h}} \dot{Q}_{\mathrm{h}} \tag{6}$$

Optimization of triangular elemental volume

As shown in fig. 2, a triangular elemental volume ($H_0L_0/2$) generates heat at a constant rate q volumetrically. The heat generation rate per unit volume is constant [$q''' = 2q/\mathrm{H_0L_0}$]. The elemental area size $A_0 = H_0L_0/2$ is constant but the aspect ratio H_0/L_0 is free to vary. The heat generated in the triangular area is first directed to a relatively high conductive link of width D_0, which is located on the longer axes of the triangular elemental area. Then it is channeled to a heat sink located at point M_0 by the D_0 link. The boundary of the triangular elemental area is adiabatic except for the heat sink point M_0. It is assumed that the thermal conductivity of a high conductive link (k_p) is much higher than the thermal conductivity of low conductivity material (k_0) and the area occupied by high conductive material is much smaller than the area by low conductivity material. It is also assumed that the triangular elemental area is slender enough to have one-dimensional (y-direction) heat conduction on the heat generating area.

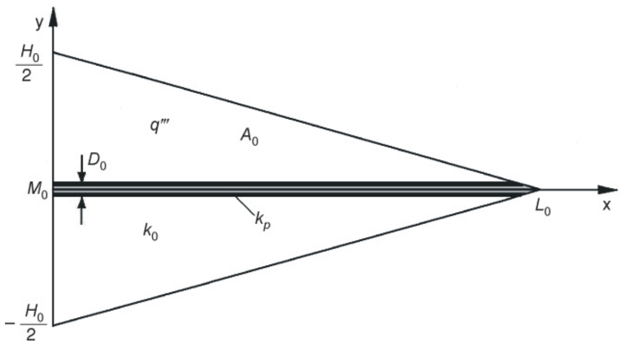

Figure 2. Triangular elemental volume

The temperature difference distribution in fig. 2 where $y > 0$ can be described as [19]:

$$T(x, y) - T_{M_0} = -\frac{q}{2k_0}y^2 + \frac{q H_0}{2k_0}\left(1 - \frac{x}{L_0}\right)y + \frac{q H_0}{k_pD_0}\left(\frac{x^3}{6L_0} - \frac{x^2}{2} + \frac{L_0}{2}x\right) \tag{7}$$

For the case $y < 0$, the temperature difference can be got by replacing H_0 to $-H_0$.
The entransy dissipation rate in the triangular area can be calculated with eq. (4).

$$\dot{E}_{vh\phi 0} = 2\int_0^{L_0}\int_0^{\frac{H_0}{2}\left(1-\frac{x}{L_0}\right)} q\left[-\frac{q}{2k_0}y^2 + \frac{q H_0}{2k_0}\left(1 - \frac{x}{L_0}\right)y + \frac{q H_0}{k_pD_0}\left(\frac{x^3}{6L_0} - \frac{x^2}{2} + \frac{L_0}{2}x\right)\right]\mathrm{d}y\mathrm{d}x = q^2\left(\frac{H_0^3L_0}{48k_0} + \frac{H_0^2L_0^3}{20D_0k_p}\right) \tag{8}$$

For $\widehat{k} = k_p/k_0$, $\phi_0 = 2D_0/H_0$, and considering $A_0 = H_0L_0/2$, eq. (8) can be written as:

$$\dot{E}_{vh\phi 0} = \frac{q^2A_0^2}{k_0}\left(\frac{1}{12}\frac{H_0}{L_0} + \frac{2}{5}\frac{L_0}{H_0}\frac{1}{\widehat{k}\phi_0}\right) \tag{9}$$

Taking the derivative of eq. (9) with respect to H_0/L_0 and setting it to zero yields the optimal H_0/L_0 of the element area and its corresponding optimization results are

$$\left(\frac{H_0}{L_0}\right)_{\mathrm{opt}} = 2\sqrt{\frac{6}{5}}(\widehat{k}\phi_0)^{-1/2} \tag{10}$$

$$\dot{E}_{vh\phi_0,\mathrm{m}} = \frac{q'''^2 A_0^2}{k_0}\sqrt{\frac{2}{15\hat{k}\phi_0}} \tag{11}$$

$$R_{h_0,\mathrm{m}} = \frac{1}{k_0}\sqrt{\frac{2}{15\hat{k}\phi_0}} \tag{12}$$

$$\Delta\overline{T}_{0,\mathrm{m}} = \frac{q''' A_0}{k_0}\sqrt{\frac{2}{15\hat{k}\phi_0}} \tag{13}$$

Optimization of the first assembly

One way to connect the elemental heat currents is shown in fig. 3 [19]. A large number of optimized elemental volumes ($H_{0,\mathrm{opt}}$, $L_{0,\mathrm{opt}}$) are aligned on both sides of a new high conductivity path of width D_1, such that the elemental heat currents are collected by the new path. The outer boundary of this area is adiabatic except for the D_1 patch over the origin, through which the collected heat current is led to the outside. The area size $A_1 = n_1 A_0$ is constant, but the aspect ratio H_1/L_1 or the number of elemental volume n_1 is free to vary. The heat flow of each elemental volume is channeled into D_1 from M_{11}, M_{12}, ... , $M_{1\mathrm{m}}$.

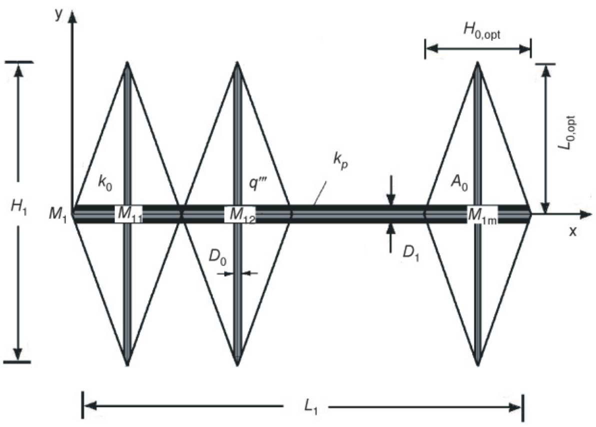

Figure 3. First order assembly construct

The temperature differences between each interval of are [19]:

$$T_{\mathrm{M}_{11}} - T_{\mathrm{M}_1} = \frac{n_1 q''' A_0 H_0}{2k_p D_1} \tag{14}$$

$$T_{\mathrm{M}_{1,j}} - T_{\mathrm{M}_{1,j-1}} = \frac{[n_1 - 2(j-1)]q''' A_0 H_0}{k_p D_1}, \quad 2 \le j \le \frac{n_1}{2} \tag{15}$$

Using eq. (3), the corresponding entransy dissipation rate can be calculated as:

$$\dot{E}_{h\phi,M_{11}M_1} = E_{h,M_{11}} - E_{h,M_1} = \dot{Q}_{h,M_{11}M_1}(T_{M_{11}} - T_{M_1}) = \frac{n_1^2 q'''^2 A_0^2 H_0}{2k_p D_1} \tag{16}$$

$$\dot{E}_{h\phi,M_{1,j}M_{1,j-1}} = E_{h\phi,M_{1,j}} - E_{h\phi,M_{1,j-1}} = \dot{Q}_{h\phi,M_{1,j}M_{1,j-1}}(T_{M_{1,j}} - T_{M_{1,j-1}}) = \frac{[n_1 - 2(j-1)]^2 q'''^2 A_0^2 H_0}{k_p D_1}, \quad 2 \le j \le \frac{n_1}{2} \tag{17}$$

where $E_{h,M_{1,j}}$ $(2 \le j \le n_1/2)$ represents the entransy at point $M_{1,j}$. The entransy dissipation rate along D_1 is equal to the sum of entransy dissipation rates of all intervals on the D_1:

$$\dot{E}_{h\phi,M_{1,n_1/2}} \quad \dot{E}_{h\phi,M_1} \quad (\dot{E}_{h\phi,M_{11}} \quad \dot{E}_{h\phi,M_1}) \quad \sum_{j\ 2}^{n_1/2} (\dot{E}_{h\phi,M_1,j} \quad \dot{E}_{h\phi,M_1,j\ \ 1})$$

$$\frac{n_1^2 q \ ^2 A_0^2 H_0}{2k_p D_1} \quad \sum_{j\ 2}^{n_1/2} \frac{[n_1 \quad 2(j \quad 1)]^2 q \ \ A_0^2 H_0}{k_p D_1}$$

$$= \frac{q \ ^2 A_0^2 H_0}{2k_p D_1} \ \frac{n_1^3}{6} \ \frac{n_1}{3} \qquad n_1 \quad 2, \text{ and } n_1 \text{ is even} \tag{18}$$

The entransy dissipation rate of the first assembly is the sum of the entransy dissipation rate of each elemental area and the entransy dissipation rate along D_1:

$$\dot{E}_{vh\phi 1} \quad n_1 \ \frac{A_0^2 q \ ^2}{k_0} \sqrt{\frac{2}{15\hat{k}\phi_0}} \quad \frac{q \ ^2 A_0^2 H_0}{k_p D_1} \ \frac{n_1^3}{6} \ \frac{n_1}{3}, \quad n_1 \quad 2, \text{ and } n_1 \text{ is even} \tag{19}$$

with $A_1 = n_1 A_0$, becomes:

$$\dot{E}_{vh\phi 1} \quad q \ ^2 A_1^2 \ \frac{n_1 H_0}{6k_p D_1} \ \frac{1}{3n_1} \ \frac{3}{k_0} \sqrt{\frac{2}{15\hat{k}\phi_0}} \ \frac{H_0}{k_p D_1}, \quad n_1 \quad 2, \text{ and } n_1 \text{ is even} \tag{20}$$

Substituting eq. (10) into eq. (20) yields:

$$\dot{E}_{vh\phi 1} \quad \frac{q \ ^2 A_1^2 H_0}{3k_p} \ \frac{n_1}{2D_1} \ \frac{1}{n_1} \ \frac{\hat{k}}{2L_0} \ \frac{1}{D_1}, \quad n_1 \quad 2, \text{ and } n_1 \text{ is even} \tag{21}$$

The derivation of eq. (21) with respect to n_1 is:

$$\frac{\partial \dot{E}_{vh\phi 1}}{\partial n_1} \quad \frac{q \ ^2 A_1^2 H_0}{3k_p} \ \frac{1}{2D_1} \ \frac{1}{n_1^2} \ \frac{\hat{k}}{2L_0} \ \frac{1}{D_1}, \quad n_1 \quad 2, \text{ and } n_1 \text{ is even} \tag{22}$$

Setting eq. (22) equal to zero and solving this equation yields the optimal number of the elements and the corresponding minimum entransy dissipation rate:

$$n_{1,\mathrm{opt}} \quad \sqrt{\frac{\hat{k}D_1}{L_0} \quad 2} \tag{23}$$

$$\dot{E}_{vh\phi 1,\mathrm{m}} \quad \frac{1}{3} \frac{q \ ^2 A_1^2 H_0}{k_p D} \sqrt{\frac{\hat{k}D_1}{L_0} \quad 2} \tag{24}$$

High conductive material can also be optimized. Letting $\phi_1 \ = A_{p,0}/\mathrm{A}_1 = (n_1 A_{p,0} + D_1 L_1)/(n_1 A_0)$, one has $D_1 = L_0(\phi_1 - \phi_0)$. Substituting $D_1 = L_0(\phi_1 - \phi_0)$ and eq. (10) into eq. (24) yields:

$$\dot{E}_{vh\phi 1,\mathrm{m}} \quad \frac{1}{3} \frac{q \ ^2 A_1^2}{k_p (\phi_1 \quad \phi_0)} \sqrt{\frac{24}{5\hat{k}\phi_0}} \sqrt{\hat{k}(\phi_1 \quad \phi_0 \quad 2} \tag{25}$$

The derivation of eq. (25) with respect to n_1 is:

$$\frac{\partial \dot{E}_{vh\phi 1,\mathrm{m}}}{\partial \phi_0} \quad \frac{1}{3}\sqrt{\frac{24}{5}}\frac{q \;\; ^2 A_1^2}{k_p} \frac{\sqrt{\frac{2 \quad \hat{k}(\phi_1 \quad \phi_0)}{\hat{k}\phi_0}}}{(\phi_1 \quad \phi_0)^2} \quad \frac{\frac{1}{\phi_0} \quad \frac{2 \quad \hat{k}(\phi_1 \quad \phi_0)}{\hat{k}\phi_0^2}}{2(\phi_1 \quad \phi_0)\sqrt{\frac{2 \quad \hat{k}(\phi_1 \quad \phi_0)}{\hat{k}\phi_0}}} \tag{26}$$

Setting eq. (26) equal to zero and solving this equation yields the optimal ϕ_0:

$$\phi_{0,\mathrm{opt}} \quad \frac{3}{2}\frac{1}{\hat{k}} \quad \frac{3}{4}\phi_1 \quad \frac{1}{4}\sqrt{\frac{36}{\hat{k}^2} \quad \frac{20\phi_1}{\hat{k}} \quad \phi_1^2} \tag{27}$$

Substituting eq. (27) into eq. (28) yields:

$$n_{1,\mathrm{opt}} \quad \frac{1}{2}\sqrt{\hat{k}\phi_1 \quad 2 \quad \sqrt{36 \quad 20\hat{k}\phi_1 \quad \phi_1^2\hat{k}^2}} \tag{28}$$

The optimized results of eqs. (27) and (28) are too complicated to put into the next optimization step. For simplicity, when $\hat{k}\phi_1 \gg 1$, $n_{1,\mathrm{opt}} \gg 1$ holds, it is reasonable to assume that the temperature gradient distribution along D_1 is linear, which is just the same as the elemental area [18]. For simplicity in this case, the entransy dissipation rate along D_1 is:

$$\dot{E}_{h\phi,M_{1,n_1}/2} \quad \dot{E}_{h\phi,M_1} = \int_0^{L_1} \frac{q \;\; ^2 H_1^2}{k_p D_1} \quad L_1 x \quad \frac{x^2}{2} \quad \mathrm{d}x \atop \frac{q \;\; ^2 A_0^2 H_0}{k_p D_1}\frac{n_1^3}{6} \quad , \quad n_1 \quad 2, n_1 \text{ is even} \tag{29}$$

Comparing eq. (18) with eq. (29), the difference between eqs. (18) and (29) is only $2/n_1^2$ of eq. (29). When $n_{1,\mathrm{opt}} \gg 1$, the difference is very tinny. In this case, eq. (20) becomes:

$$\dot{E}_{vh\phi 1} \quad q \;\; ^2 A_1^2 \quad \frac{1}{n_1 k_0}\sqrt{\frac{2}{15\hat{k}\phi_0}} \quad \frac{n_1 H_0}{6k_p D_1} \quad , \quad n_1 \quad 2, n_1 \text{ is even} \tag{30}$$

Taking the derivative of eq. (30) with respect to n_1 and setting it equal to zero yields the optimal number of the elements and its corresponding minimum entransy dissipation rate:

$$n_{1,\mathrm{opt}} \quad \sqrt{\hat{k}(\phi_1 \quad \phi_0)} \tag{31}$$

$$\dot{E}_{vh\phi 1,\mathrm{m}} \quad \sqrt{\frac{2}{15}}\frac{q \;\; ^2 A_1^2}{\hat{k}}\frac{1}{\sqrt{\phi_0(\phi_1 \quad \phi_0)}} \tag{32}$$

Taking the derivative of eq. (32) with respect to ϕ_0 and setting it to zero yields the optimal ϕ_0:

$$\phi_{0,\mathrm{opt}} \quad \frac{1}{2}\phi_1 \tag{33}$$

The optimization results of the first assembly are:

$$n_{1,\mathrm{opt}} \quad \frac{\sqrt{2}}{2}\sqrt{\hat{k}\phi_1} \tag{34}$$

$$\frac{D_1}{D_0}\;{}_{\text{opt}} \qquad \frac{\sqrt{\frac{5}{3}}}{4}\sqrt{\hat{k}\phi_0} \tag{35}$$

$$\frac{H_1}{L_1}\;{}_{\text{opt}} \qquad 2\sqrt{\frac{5}{6}} \tag{36}$$

$$\dot{E}_{vh\phi 1,\,\text{mm}} \quad 2\sqrt{\frac{2}{15}}\,\frac{q \quad {}^2 A_1^2}{k_p \phi_1} \tag{37}$$

$$R_{1,\,\text{mm}} \quad 2\sqrt{\frac{2}{15}}\,\frac{1}{k_0}\,\frac{1}{\hat{k}\phi_1} \tag{38}$$

$$\Delta\overline{T}_{1,\,\text{m}} \quad 2\sqrt{\frac{2}{15}}\,\frac{q \quad A_1}{k_0 \hat{k}\phi_1} \tag{39}$$

Equation (34) agrees with $n_{1,\text{opt}} \gg 1$, the assumption that the temperature gradient distribution along D_1 is linear is reasonable.

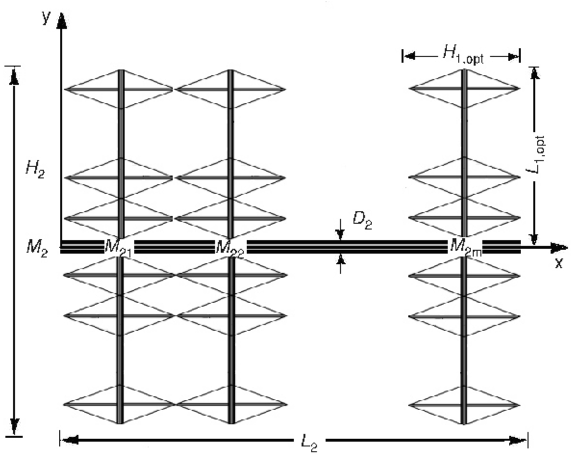

Figure 4. Second order assembly construct [19]

Optimization of second assembly

The new second order assembly construct, as shown in fig. 4 [19], is composed of a number (n_2) of optimized first order assembly constructs. It is assembled just as the first order assembly constructs. The area of A_2 is fixed, but H_2/L_2, the aspect ratio of the new construct, or n_2, the number of constituents inside the construct is free to vary. The boundary of the construct is adiabatic except for the heat sink point located at M_2.

Just like eq. (20), one can derive the entransy dissipation rate at the second assembly:

$$\dot{E}_{vh\phi 2} \quad n_2^2\sqrt{\frac{2}{5}}\,\frac{q \quad {}^2 A_1^2}{k_p \phi_1} \quad \frac{q \quad {}^2 A_1^2 H_1}{k_p D_2} \quad \frac{n_2^3}{6} \quad \frac{n_2}{3}$$
$$\frac{q \quad {}^2 A_2^2}{k_p} \quad \frac{1}{n_2} \quad \sqrt{\frac{8}{15}}\,\frac{1}{\phi_1} \quad \frac{1}{3(\phi_2 \quad \phi_1)}\sqrt{\frac{10}{3}} \quad \frac{n_2}{6(\phi_2 \quad \phi_1)}\sqrt{\frac{10}{3}} \tag{40}$$

Optimizing eq. (40) with respect to n_2 and ϕ_1 gives the optimization results:

$$\phi_{1,\text{opt}} \quad \frac{9 \quad \sqrt{69}}{2}\,\phi_2 \tag{41}$$

$$n_{2,\mathrm{opt}} = 2.55395 \tag{42}$$

The optimal number of the optimized first order assembly constructs is an even number, and it may be 2 or 4.

The optimization results with respect to $n_2 = 2$ are:

$$\phi_{1,\mathrm{opt}} \quad 2(\sqrt{5} \quad 2)\phi_2 \tag{43}$$

$$\dot{E}_{vh\phi 2,\mathrm{n}_2 \ 2,\mathrm{mm}} \quad 1.63808 q \ ^2 \frac{A_2^2}{k_p \phi_2} \tag{44}$$

The optimization results with respect to $n_2 = 4$ are

$$\phi_{1,\mathrm{opt}} \quad \frac{2(3\sqrt{10} \quad 4)\phi_2}{37} \tag{45}$$

$$\dot{E}_{vh\phi 2,\mathrm{n}_2 \ 2,\mathrm{mm}} \quad \frac{2.07558 q \ ^2 A_2^2}{k_p \phi_2} \tag{46}$$

Comparing eq. (44) with eq. (46), one can see that $n_{2,\mathrm{opt}} = 2$ is the best result. The optimal results of the second assembly are:

$$\frac{D_2}{D_1}_{\mathrm{opt}} \quad 4.89898 \tag{47}$$

$$\frac{H_2}{L_2}_{\mathrm{opt}} \quad \sqrt{\frac{5}{6}} \tag{48}$$

$$R_{h,\,2,\mathrm{mm}} \quad \frac{1.63808}{k_0 \widehat{k} \phi_2} \tag{49}$$

$$\Delta \overline{T}_{2,\mathrm{m}} \quad \frac{1.63808 q \ A_2}{k_0 \widehat{k} \phi_2} \tag{50}$$

Optimization results and analyses

Using the same steps mentioned, one can obtain the optimization results of the third assembly and even higher order assemblies. The major results are listed in tab. 1, in which represents the non-dimensional mean temperature difference of the volume and it is the reflection of optimal heat transfer ability of each assembly.

Supposing the volume of each assembly A_i is the same that the volume fraction of high conductivity material in each assembly ϕ_i is the same ($i = 1, 2, 3...$), that is $\phi_1 = \phi_2 = ... = \phi_\mathrm{i} = \phi$ and $A_1 = A_2 = ... = A_\mathrm{i} = A$. Two mean temperature difference curves are obtained according to eqs. (13) and (39) when elemental design and first order design are studied, as shown in fig. 5. Figure 5 shows that the mean temperature difference of the elemental design is smaller than that of the first order design when $\widehat{k}\phi < 4$. While when $\widehat{k}\phi > 4$, the conclusion is reversed. That is to say, 4 is a critical value for $\widehat{k}\phi$. When $\widehat{k}\phi < 4$, the optimal design should be the elemental one.

Table 1. Constructal optimization results based on minimization of entransy dissipation rate

Order of assemby construct	$n_{i,\text{opt}}$	$\frac{D_i}{D_{i\ 1}}$ opt	$\frac{H_i}{L_i}$ opt	$\frac{\Delta\overline{T}_i k_p \phi_i}{q\ \ A_i}$
0			$2\sqrt{\frac{6}{5}}(\widehat{k\phi}_i)^{1/2}$	$\sqrt{\frac{2}{15}}\sqrt{\widehat{k\phi}_i}$
1	$\frac{\sqrt{2}\sqrt{\widehat{k\phi}_1}}{2}$	$\frac{\sqrt{\frac{5}{3}}}{4}\sqrt{\widehat{k\phi}_0}$	$2\sqrt{\frac{5}{6}}$	$2\sqrt{\frac{2}{15}}$
2	2	4.89898	$\sqrt{\frac{5}{6}}$	1.63808
3	8	2.50713	$\frac{1}{2}\sqrt{\frac{6}{5}}$	2.56169
4	6	6.22064	$\frac{4}{3}\sqrt{\frac{6}{5}}$	1.65649

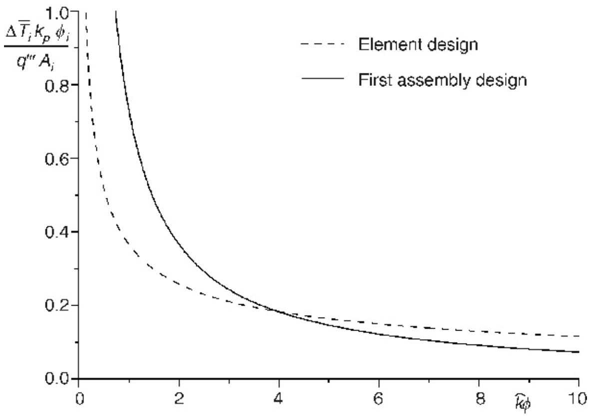

Figure 5. Mean temperature difference of elemental design *vs.* first order design

When $\widehat{k\phi} > 4$, the internal complexity should increase, *i. e.*, the first order design should be adopted.

The major optimization results based on the minimization of the maximum temperature difference obtained in ref. [19] are listed in tab. 2, in which $(\Delta T_i\ \ k_p\phi_i)/(q\ \ A_i)$ represents the non-dimensional mean temperature difference of the volume when each assembly achieves its minimum of the maximum temperature difference. Comparing tab. 1 with tab. 2, the optimal constructs of each assembly based on the entransy dissipation minimization are different from those based on the maximum temperature difference minimization. The reason has been explained in ref. [32]: when the thermal current densities in the high conductive link are linear with the length, the optimized shapes of assemble based on the minimization of the entransy dissipation are the same as those based on the minimization of the maximum temperature difference. When the thermal current densities in the high conductive link are not linear with the length, and the optimized shapes of assemble based on the minimization of entransy dissipation are different from those based on minimization of the maximum temperature difference. The thermal current densities in high conductive link based on triangular elements are all not linear with the length, so the optimized shapes of assemble based on the minimization of entransy dissipation are different from those based on the minimization of the maximum temperature difference.

Table 2. Constructal optimization results based on minimization of maximum temperature difference (ref. [19])

Order of assemby construct	$n_{i,\text{opt}}$	$\frac{D_i}{D_{i\ 1}}$ opt	$\frac{H_i}{L_i}$ opt	$\frac{\Delta T_{i,\max} k_p \phi_i}{q\ A_i}$	$\frac{\Delta \overline{T}_i\ k_p \phi_i}{q\ A_i}$ (*)
0			$\frac{2^{7/6}}{5^{1/6}}\sqrt{\hat{k}\phi_0}$	$\frac{2^{5/6}}{5^{5/6}}\sqrt{\hat{k}\phi_\text{i}}$	$0.376061\sqrt{\hat{k}\phi_\text{i}}$
1	$\frac{2^{1/3}}{5^{1/3}}\sqrt{\hat{k}\phi_\text{i}}$	$\frac{2^{1/3}}{5^{1/3}}\sqrt{\hat{k}\phi_\text{i}}$	$\sqrt{5}$	$\frac{4}{\sqrt{5}}$	1.38847
2	2	$\sqrt[2]{2^3}$	$\frac{2}{\sqrt{5}}$	2.8677	2.50920
3	2	2	$\sqrt{5}$	2.7899	2.53988
4	2	2	$\frac{2}{\sqrt{5}}$	3.7201	3.51595

(*) This column was deduced from ref. [19] by the authors of this paper

A comparison of mean temperature difference between constructs based on minimization of entransy dissipation and those based on minimization of the maximum temperature difference is shown in fig 6. The mean temperature difference based on minimization of entransy dissipation is smaller than that based on minimization of the maximum temperature difference. It means that constructal optimization based on entransy dissipation minimization can improve the heat transfer ability of the whole volume more greatly than that based on the maximum temperature difference minimization. The effect of order on the mean temperature difference based on entransy dissipation minimization and that based on the maximum temperature minimization are also shown in fig. 6. It can be seen that the mean temperature difference based on the entransy dissipation minimization or the maximum temperature difference minimization does not always decrease when the order increases, and there exist some fluctuations. It means that more complex constructs of the high conductivity distribution do not always mean better heat transfer ability and there existing a optimal assembly order to achieve the minimum mean temperature difference.

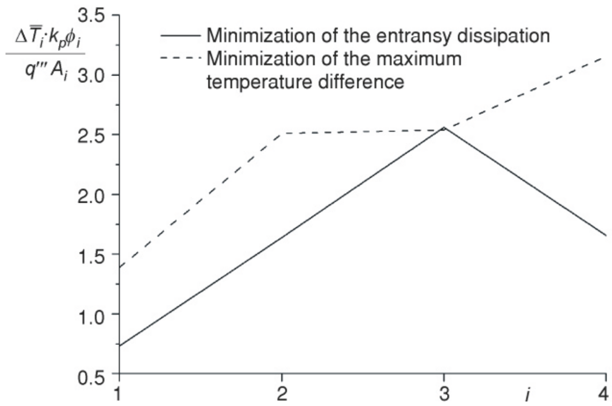

Figure 6. Mean temperature difference based on the minimization of the entransy dissipation *vs.* mean temperature difference based on the minimization of the maximum temperature difference

Conclusions

In the process of energy transfer, the heat transfer is always accompanied by the entransy transfer. The entransy is not in conservation due to dissipation. The equivalent thermal resistance defined based on the entransy dissipation rate reflects the conduction ability in the heat transfer process. The smaller the equivalent thermal resistance, the better the conductive effect, and the lower the mean temperature in the volume. The "volume to point" conduction problem is a classical multi-dimensional conduction with fixed boundary heat flux. The optimized constructs based on the minimization of entransy dissipation rate are results of the optimization of the average conductive effect, which is different from the optimized constructs based on the minimization of the maximum temperature difference in refs. [1, 19]. The optimized results show that the constructs based on the minimization of the entransy dissipation rate can decrease mean temperature difference better than the constructs based on the minimization of the maximum temperature difference. It is seen that both the minimum of the maximum temperature difference and the minimum of the mean temperature difference increase fluctuantly as the order increases. It means that high complex construct dose not mean low thermal resistance. The equivalent thermal resistance reflects the average conductive effect in the volume, and it represents the efficiency. The minimum thermal resistance reflects the maximum temperature difference in the volume, and it represents the maximum temperature limitation in the volume. Both the mean temperature difference and the maximum temperature difference should be combined when considering the efficiency and the temperature limitation simultaneously. Because the idea of entransy describes heat transfer ability more suitably [31], the optimization results of this paper can be put to engineering application of electronic cooling.

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Nomenclature

- A_p – area of high conductive material, [m^2]
- c_v – specific heat at constant volume, [Jkg^{-1}K^{-1}]
- D – width of the conducting path, [m]
- E_{vh} – entransy, [JK]
- $\dot{E}_{h\phi}$ – entransy dissipation rate, [WK]
- $\dot{E}_{vh\phi}$ – entransy dissipation rate of the whole volume, [WK]
- H – height, [m]
- $\widehat{k}$ – thermal conductivity ratio of high conductive material to low conductive material, [–]
- k_p – thermal conductivity of the high conductive material, [Wm^{-1}K^{-1}]
- k_0 – thermal conductivity of the low conductive material, [Wm^{-1}K^{-1}]
- L – length, [m]
- M – weight, [kg]
- n – number
- $\dot{Q}_h$ – thermal current, [W]
- Q_{vh} – heat capacity at constant volume, [J]
- q – heat generation rate, [W]
- $\dot{q}$ – thermal current density vector, [W]
- q – heat generation rate per unit volume, [Wm^{-3}]
- R_h – equivalent thermal resistance, [KW^{-1}]
- T – thermal potential, [K]
- U_h – thermal potential, [K]

v – volume, [m^3]

Greek symbols

ϕ – area size ratio, [–]
T – temperature gradient, [K]
$\overline{T}$ – mean temperature difference, [K]
$\Delta\overline{T}$ – mean temperature difference corresponding to maximum temperature minimization, [K]

Subscripts

in – input
m – minimum value
mm – double minimum value
opt – optimal value
out – output
0, 1, 2, ..., i – element, first order assembly, second order assembly,…i^{th} order assembly order

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