A NOTE ON THE HOMOTOPY PERTURBATION METHOD

by

Ji-Huan HE

Modern Textile Institute, Donghua University, Shanghai, China,
Interdisciplinary Institute for Nonlinear Science, National Engineering
Laboratory of Modern Silk, Soochow University, Suzhou, China

The homotopy perturbation method admits some unknown parameters in the obtained series solutions, which can be identified after few iteration steps using the method of least squares. The solution procedure of the so-called optimal homotopy asymptotic method follows the same way.

Key words: *homotopy equation, parameter-expansion method, method of least squares*

Babaelahi, Ganji, and Joneidi applied the so called Optimal Homotopy Asymptotic Method [1] to solve the problem of steady incompressible mixed convection flow past vertical flat plate [2]. The method is very effective to solve non-linear equations, but this method can not be considered as a new method, it is actually a coupled method of the homotopy perturbation method [3-5] and the method of least squares. The basic idea was systematically presented in 2007 International Symposium of Nonlinear Dynamics, October 27-30, 2007, Shanghai, China, and was then published in Topological Methods in Non-Linear Analysis [6]. I also gave this basic idea in ScienceWatch, see http://sciencewatch.com/inter/aut/2008/08- apr/08aprHe, especially the fifth solution procedure "Optimal identification of the unknown parameter in the trial function".

Hereby I will illustrate the general solution procedure of the homotopy perturbation method [3-5]. Consider a non-linear equation in the form:

$$Lu + Nu = 0 \tag{1}$$

where L and N are linear operator and non-linear operator, respectively. In order to use the homotopy perturbation method, a suitable construction of a homotopy equation is of vital importance. Generally a homotopy can be constructed in the form:

$$\tilde{L}u \quad p(Lu \quad Nu \quad \tilde{L}u) \quad 0 \tag{2}$$

where $\tilde{L}$ can be a linear operator or a simple non-linear operator, and the solution of $\tilde{L}u = 0$ with possible some unknown parameters can basically describe the original non-linear system. For example for a non-linear oscillator we can choose $\tilde{L}u \quad u \quad \omega^2 u$, where ω is the frequency of the non-linear oscillator. We use a simple example to illustrate the solution procedure.

As an example, we consider a nonlinear oscillator:

$$u \quad u^3 \quad 0, \quad u(0) \quad A, \quad u\,(0) \quad 0 \tag{3}$$

We can construct the following homotopy eq. (4):

* Author's e-mail: jhhe@dhu.edu.cn

$$u \quad \omega^2 u \quad p(u^3 \quad \omega^2 u) \quad 0 \tag{4}$$

where ω is the frequency of the oscillator which is to be solved.

When $p = 0$, we have $u'' + \omega^2 u = 0$, and we have the solution $u = A\cos\omega t$. If ω can be optimally identified, then even the zeroth order solution $u = A\cos\omega t$ can excellently describe the oscillation property of eq. (3). When $p = 1$, eq. (4) turns out to be the original one, eq. (3). We assume that the solution can be expressed in the form:

$$u = u_0 + pu_1 + p^2u_2 + ... \tag{5}$$

We write down the linear equations for u_0 and u_1

$$u_0 \quad \omega^2 u_0 \quad 0 \tag{6}$$

$$u_1 \quad \omega^2 u_1 \quad u_0^3 \quad \omega^2 u_0 \quad 0 \tag{7}$$

The standard homotopy perturbation method always stops before the second iteration.

For a beginner, the homotopy equation can be constructed in a simple way, that is:

$$Lu + pNu = 0 \tag{8}$$

For eq. (3), the homotopy equation reads:

$$u'' + pu^3 = 0 \tag{9}$$

This will lead to an infinite series solution which converges to the exact solution. In order to minimize the solution procedure, we re-write eq. (9) in the form [7]:

$$u'' + 0 \cdot u + pu^3 = 0 \tag{10}$$

Using the parameter-expansion method [8-14], the coefficient, 0, of the linear term is also expanded into a series in p, that is:

$$0 = \omega^2 + pa_1 + p^2a_2 + ... \tag{11}$$

where ω and a_i are unknown constants to be further determined.

Substituting eqs.(5) and (11) into eq.(10), we have:

$$\begin{gathered}(u_0 \quad pu_1 \quad p^2u_2 \ \cdots) \quad (\omega^2 \quad pa_1 \quad p^2a_2 \ \cdots)(u_0 \quad pu_1 \quad p^2u_2 \ \cdots) \\ p(u_0 \quad pu_1 \quad p^2u_2 \ \cdots)^3 \quad 0\end{gathered} \tag{12}$$

We write down the linear equations for u_0 and u_1:

$$u_0 \quad \omega^2 u_0 \quad 0 \tag{13}$$

$$u_1 \quad \omega^2 u_1 \quad a_1 u_0 \quad u_0^3 \quad \omega^2 u_0 \quad 0 \tag{14}$$

If the first-order approximate solution is solved, then from eq. (11), by setting $p = 1$, we have:

$$\omega^2 \quad a_1 \quad 0 \tag{15}$$

It is obvious that eqs. (13) and (14) are exact same with eqs. (6) and (7).

If a higher order approximate solution is solved, the unknown can be solved using the method of least squares [6]:

$$\,_0^{T/4} R \frac{\partial R}{\partial a_i} \mathrm{d}t \quad 0 \tag{16}$$

where $T = 2\pi/\omega$, R is the residual of eq. (1) when the iteration procedure stop, $R = Lu_n + Nu_n$.

Now we turn back to the solution procedure of the optimal homotopy asymptotic method [1, 2], where a homotopy equation is constructed in the form:

$$Lu + H(p)Nu = 0 \tag{17}$$

where $H(p)$ is expanded in the form

$$H(p) = pc_1 + p^2c_2 + ... \tag{18}$$

and the unknown constants c_i is identified similar to eq. (16), see ref. [1, 2]. This solution procedure is effective and only few iteration steps are needed.

As I said before, this can not be considered as a new method, the main difference lies in construction of the homotopy equation.

We can write the homotopy equation, eq. (8), in the form:

$$Lu + 1 \cdot pNu = 0 \tag{19}$$

Expanding the constant,1, into a series in p:

$$1 = pc_1 + p^2c_2 + ... \tag{20}$$

The first terms of the obtained series are exactly same with those by the optimal homotopy asymptotic method.

In my previous publication [3], I suggested that a homotopy equation can be also constructed in the form:

$$(1 - p)(Lv - Lu_0) + pNv = 0 \tag{21}$$

where u_0 is the initial solution which must outline the basic solution property and it can involve some unknown parameters which can be identified after few iterations by the method of least squares.

For the problem in [2]:

$$f \qquad \frac{ff}{2} \quad Ri_x T \quad 0, \quad f(0) \quad 0, \quad f\ (0) \quad 0, \quad f\ (\infty) \quad 1 \tag{22}$$

$$T \qquad \frac{\Pr}{2} fT \quad 0, \quad T(0) \quad 1, \quad T(\infty) \quad 0 \tag{23}$$

we can construct the following homotopy equations:

$$(1 \quad p)(u \qquad f_0) \quad p(u \qquad \frac{uu}{2} \quad Ri_x v) \quad 0 \tag{24}$$

$$(1 \quad p)(v \qquad T_0) \quad p(v \qquad \frac{\Pr}{2} uv\ ) \quad 0 \tag{25}$$

There are alternative approaches to construction of homotopy equations for eqs. (22) and (33), the beginners are recommended to read [15, 16].

The initial solutions can be chosen as follows:

$$f_0 \quad \frac{1}{a}(\mathrm{e}^{\ a\eta} \quad 1) \quad \eta \tag{26}$$

$$T_0 \quad \mathrm{e}^{\ \eta}(1 \quad b\eta) \tag{27}$$

where a and b are unknown constants to be further determined.

It is obvious that eqs. (26) and (27) satisfy the boundary/initial conditions, the unknown parameters a and b in eqs. (26) and (27) can be identified using the method of least squares after few iterations. The solution procedure is of course much simpler than that in [2].

Generally the method of least squares can be used for optimal identification of unknown parameters involved in a series solution. Recently Herisanu *et al*. suggested an optimal variational iteration algorithm based on the variational iteration method and the method of the least squares [17].

References

[1] Marinca, V., Herisanu, N., Nemes, I., An Optimal Homotopy Asymptotic Method with Application to Thin Film Flow, *Central European Journal of Physics, 6* (2008), 3, pp. 648-653
[2] Babaelahi, M., Ganji, D. D., Joneidi, A. A., Analytical Treatment of Mixed Convection Flow Past Vertical Flat Plate, *Thermal Science, 14* (2010), 2, pp. 409-416
[3] He, J.-H., A Coupling Method of a Homotopy Technique and a Perturbation Technique for Non-Linear Problems, *Int. J. Nonlinear Mech., 35* (2000), 1, pp. 37-43
[4] He, J.-H., Some Asymptotic Methods for Strongly Nonlinear Equations, *Int. J. Mod. Phys. B, 20* (2006), 10, pp. 1141-1199
[5] He, J.-H., An Elementary Introduction to Recently Developed Asymptotic Methods and Nanomechanics in Textile Engineering, *Int. J. of Mod. Phys. B, 22* (2008), 21, pp. 3487-3578
[6] He, J.-H., Recent Development of the Homotopy Perturbation Method, *Topological Methods in Non-Linear Analysis, 31* (2008), 2, pp. 205-209
[7] He, J.-H., New Interpretation of Homotopy Perturbation Method, *Int. J. Mod. Phys. B, 20* (2006), 18, pp. 2561-2568
[8] He, J.-H., Modified Lindstedt-Poincare Methods for Some Strongly Non-Linear Oscillations Part I: Expansion of a Constant, *Int. J. Nonlinear Mech., 37* (2002), 2, pp. 309-314
[9] Xu, L., Application of He's Parameter-Expansion Method to an Oscillation of a Mass Attached to a Stretched Elastic Wire, *Phys. Lett. A, 368* (2007), 3-4, pp. 259-262
[10] Xu, L., Determination of Limit Cycle by He's Parameter-Expanding Method for Strongly Nonlinear Oscillators, *J. Sound Vib., 302* (2007), 1-2, pp. 178-184
[11] Sweilam, N. H., Al-Bar, R. F., Implementation of the Parameter-Expansion Method for the Coupled Van der Pol Oscillators, *Int. J. Nonlin. Sci. Num., 10* (2009), 2, pp. 259-264
[12] Sweilam, N. H., Khader, M. M., Application of He's Parameter-Expansion Method for the Nonlinear Differential Equations, *Int. J. Nonlin. Sci. Num., 10* (2009), 2, pp. 265-272
[13] Shin, B. C., Darvishi, M. T., Karami, A., Application of He's Parameter-Expansion Method to a Nonlinear Self-Excited Oscillator System, *Int. J. Nonlin. Sci. Num., 10* (2009), 2, pp. 137-143
[14] Zengin, F. O., Kaya, M. O., Demirbag, S. A., Application of Parameter-Expansion Method to Nonlinear Oscillators with Discontinuities, *Int. J. Nonlin. Sci. Num., 9* (2008), 1, pp. 267-270
[15] Ariel, P. D., Homotopy Perturbation Method and the Natural Convection Flow of a Third Grade Fluid through a Circular Tube, *Nonlinear Science Letters A, 1* (2010), 1, pp. 43-52
[16] He, J. H., An Elementary Introduction to the Homotopy Perturbation Method, *Comput. & Maths. with Application, 57* (2009), 3, pp. 410-412.
[17] Marinca, V., Herisanu, N., A Modified Variational Iteration Method for Strongly Nonlinear Problems, *Nonlinear Science Letters A, 1* (2010), 2, pp. 183-192

Paper submited: January 30, 2010
Paper accepted: January 30, 2010