ANALYTICAL TREATMENT OF MIXED CONVECTION FLOW PAST VERTICAL FLAT PLATE

by

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The problem of steady incompressible mixed convection flow past vertical flat plate has been considered. The velocity and temperature equations for this problem are reduced to set of non-linear ordinary differential equations by appropriate transformation and are solved by optimal homotopy asymptotic method. Results show that this method provides us with a convenient way to control the convergence of approximation series and adjust convergence regions when necessary. It is concluded that increment of the Prandtl number leads to diminishing of the temperature values.

Key words: *mixed convection, flat plate, temperature, optimal homotopy asymptotic method*

Introduction

The boundary layer flow of laminar two-dimensional motion of fluid past a semi-infinite vertical plate, with the free stream velocity and temperature have been considered. As it is shown schematically in fig. 1, for flow past vertical plates, induced body force due to heat transfer is either parallel or anti-parallel to the mean convection direction.

Most of engineering problems cannot be analytically solved using traditional methods. In the old analytical perturbation method, small parameter should be exerted and that was the difficulty of this method. Nayfeh [1] has presented perturbation techniques, pointing out their similarities, differences, and advantages, as well as their limitations. Various powerful mathematical methods have been recently introduced to eliminate the small parameter, such as homotopy perturbation method [2-7], variational iteration method [8], homotopy analysis method [9-11], differential transformation method [12-14], *etc.* The other developed method is optimal homotopy asymptotic method (OHAM) [15-19] which is most applicable in analytical analysis of engineering problems. Marinca *et al*. [16] used this method to solve some non-linear

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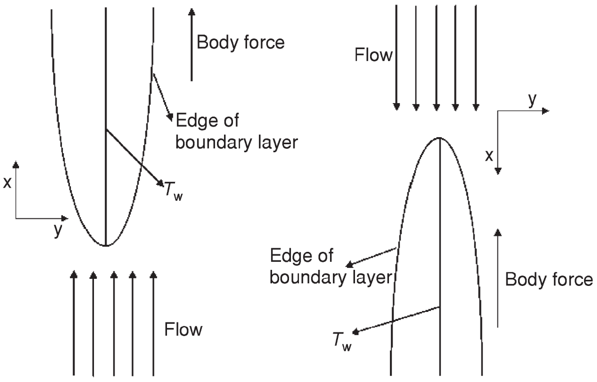

Figure 1. Mixed convection flow over a vertical flat plate

equations arising in heat transfer. Obtained results by OHAM, which did not need small parameters were compared with numerical results and a very good agreement was found. Then they implemented this method to solve other non-linear problems in engineering and have shown that this method provides us with a convenient way to control the convergence of approximation series and adjust convergence regions when necessary [17-18]. After them, Joneidi *et al.* investigated on micropolar flow on a porous channel with high mass transfer using this analytical method [20]. In the present research, mixed convection flow past a vertical flat plate has been studied by OHAM.

Optimal homotopy asymptotic method

We apply this method to the following differential equation:

$$L[u(\tau)] + N[u(\tau)] + g(\tau) = 0, \quad B(u) = 0 \tag{1}$$

where L is a linear operator, τ denotes an independent variable, $u(\tau)$ is an unknown function, $g(\tau)$ – a known function, $N[u(\tau)]$ – a non-linear operator, and B – a boundary operator. By means of OHAM, one first constructs a family of equations:

$$\begin{gathered}(1-p)\{L[\phi(\tau,p)]+g(\tau)\}-H(p)\{L[\phi(\tau,p)]+g(\tau)+N[\phi(\tau,p)]\}=0\\ B[\phi(\tau,p)]=0\end{gathered} \tag{2}$$

where $p \in [0,1]$ is an embedding parameter, $H(p)$ – a non-zero auxiliary function for $p \neq 0$ and $H(0) = 0$, and $\phi(\tau, p)$ – an unknown function. Obviously, when $p = 0$, $p = 1$, it holds that:

$$\phi(\tau,0)=u_0(\tau), \quad \phi(\tau,1)=u(\tau) \tag{3}$$

Thus, as p increases from 0 to 1, the solution $\phi(\tau, p)$ varies from $u_0(\tau)$ to the solution $u(\tau)$, where $u_0(\tau)$ is obtained from eq. (2) for $p = 0$:

$$L[u_0(\tau)]+g(\tau)=0, \quad B(u_0)=0 \tag{4}$$

We choose the auxiliary function in the form:

$$H(p)=pC_1+p_2C_2+\ldots \tag{5}$$

where C_1, C_2, … are constants which can be determined later.

Expanding $\phi(\tau, p)$ in a series with respect to p, one has:

$$\phi(\tau,p,C_i)=u_0(\tau)+\sum_{k=1}u_k(\tau,C_i)p_k, \quad i=1,2,\ldots \tag{6}$$

Substituting eq. (6) into eq. (2), collecting the same powers of p, and equating each coefficient of p to zero, we obtain set of differential equation with boundary conditions. Solving differential equations by boundary conditions $u_0(\tau)$, $u_1(\tau, C_1)$, $u_2(\tau, C_2)$,... are obtained. Generally speaking, the m^{th}-order approximate solution of eq. (1) can be written in the form of:

$$\tilde{u}^{(m)} = u_0(\tau) + \sum_{k=1}^{m} u_k(\tau, C_i) \tag{7}$$

Note that the last coefficient C_m can be function of τ. Substituting eq. (7) into eq. (1), there results the following residual:

$$R(\tau, C_i) = L[\tilde{u}^{(m)}(\tau, C_i)] + g(\tau) + N[\tilde{u}^{(m)} C_i)] \tag{8}$$

If $R(\tau, C_i) = 0$ then $\tilde{u}^{(m)}(\tau, C_i)$ happens to be the exact solution. Generally such a case will not arise for non-linear problems, but we can minimize the functional:

$$J(C_1, C_2, \ldots, C_{\mathrm{n}}) = \int_a^b R^2(\tau, C_1, C_2, \ldots, C_{\mathrm{m}})\mathrm{d}\tau \tag{9}$$

where a and b are two values, depending on the given problem. The unknown constants $C_i (i = 1, 2, \ldots, m)$ can be identified from the conditions:

$$\frac{\partial J}{\partial C_1} = \frac{\partial J}{\partial C_2} = 0 \tag{10}$$

with these constants known, the approximate solution (of order m), eq. (7), is well determined.

Description of the problem

We consider the laminar two-dimensional motion of fluid past a semi-infinite vertical plate, with the free stream velocity and temperature denoted by, U_∞ and T_∞. We consider the flow over isothermal vertical plate, for which the surface temperature (T_{w}) is greater than the free stream temperature (T_∞). Governing equations are presented in non-dimensional form with the buoyancy term modeled by the Boussinesq approximation. Non-dimensionalization of equations are performed by introducing an appropriate length (L), velocity (U_∞), temperature ($\Delta T = T_{\mathrm{w}} - T_\infty$) and pressure scales ($\rho U_\infty^2$). These equations for the velocity and temperature fields are as given in Gebhart *et al.* [21].

$$\nabla \cdot \vec{\mathrm{V}} = 0 \tag{11}$$

$$\frac{D\vec{\mathrm{V}}}{Dt} = \frac{\mathrm{Gr}}{\mathrm{Re}^2} T - \nabla P + \frac{1}{\mathrm{Re}} \nabla^2 \vec{\mathrm{V}} \tag{12}$$

$$\frac{DT}{Dt} = \frac{1}{\mathrm{Re\,Pr}} \nabla^2 T \tag{13}$$

where $T = (T^* - T_\infty)/\Delta T$ and $\mathrm{Gr} = g\beta_t \Delta T L^3/\nu^3$, $\mathrm{Re} = U_\infty L/\nu$, and $\mathrm{Pr} = \nu/\alpha$, where α is the thermal diffusivity of the fluid and T^* – the dimensional temperature in the field. In the momentum conservation equation, the quantity is also known as the Richardson number (Ri). Positive and negative signs of Ri refer to assisting and opposing flows, respectively, that in this work positive sign is considered. The Grashof number weighs the relative importance of buoyancy and viscous diffusion terms and in the mixed convection regime. Ri is of order one.

The mean flow equations are obtained by invoking boundary layer approximation for two-dimensional steady incompressible flow with constant properties and Boussinesq approximation. The mean flow equations are obtained using the following variables $\eta = y(U_\infty/vx)^{1/2}$ for the independent variable $u/U_\infty = F'$ and $(T^* - T_\infty)/(T_\mathrm{w} - T_\infty) = T$ for the dependent variables and are obtained from the solution [22]:

$$f \qquad \frac{ff}{2} \quad \mathrm{Ri}_x T \quad 0 \tag{14}$$

$$f(0) = 0, \;\; f'(0) = 0, \;\; f'(\infty) = 1$$

$$T \qquad \frac{\mathrm{Pr}}{2} f T \quad 0 \tag{15}$$

$$T(0) = 1, \quad T(\infty) = 0$$

where $\mathrm{Ri}_x = \mathrm{Gr}_x/\mathrm{Re}_x^2$ is the buoyancy parameter. In these equations, primes indicate derivatives with respect to η. As Ri_x is a function of x, similarity solution does not exist.

Solution with optimal homotopy asymptotic method

In this section, OHAM has been applied to non-linear ordinary differential eqs. (14) and (15). According to this method, applying of eq. (2) to eqs. (14) and (15) yields:

$$\begin{gathered}(1 \quad p)(f \quad f \quad 1) \quad H_1(p) \; f \quad \frac{ff}{2} \quad \mathrm{Ri}_x T \quad 1 \quad f \quad f \\ (1 \quad p)(T \quad T) \quad H_2(p) \; T \quad \frac{\mathrm{Pr}}{2} f T \quad (T \quad T) \quad 0\end{gathered} \tag{16}$$

where primes denote differentiation with respect to η.

We consider f, T, $H_1(p)$, and $H_2(p)$ as following:

$$\begin{gathered}f = f_0 + pf_1 + p^2 f_2 \\ T = T_0 + pT_1 + p^2 T_2 \\ H_1(p) = pC_{11} + p^2 C_{12} \\ H_2(p) = pC_{21} + p^2 C_{22}\end{gathered} \tag{17}$$

Substituting f, T, $H_1(p)$, and $H_2(p)$ from eq. (17) into eq. (16) and some simplification and rearranging based on powers of p-terms, we have:

$$\begin{aligned}p^0: \;\; & 1 \quad f_0 \quad f_0 \quad 0 \\ & f_0(0) \quad 0, \; f_0(0) \quad 0, f_0(\infty) \quad 1 \\ & T_0 \quad T_0 \quad 0 \\ & T_0(0) \quad 1, \; T_0(\infty) \quad 1\end{aligned} \tag{18}$$

$$p^1: \quad f_0 \quad C_{11} \quad f_0 \quad 1 \quad C_{11} f_0 \quad f_1 \quad 0.5C_{11} f_0 f_0 \quad C_{11} f_0 \quad f_1 \quad f_0 \quad C_{11} \quad C_{11} R T_0 \quad 0 \tag{19}$$

$$f_1(0) \quad 0, \quad f_1(0) \quad 0, \quad f_1(\infty) \quad 0$$

$$T_1 \quad C_{21} T_0 \quad C_{21} T_0 \quad T_0 \quad T_1 \quad C_{21} T_0 \quad T_0 \quad 0.5C_{21} \Pr f_0 T_0 \quad 0$$

$$T_1(0) \quad 0, \quad T_1(\infty) \quad 0$$

$$p^2: C_{12} f_0 \quad 0.5C_{11} f_0 f_1 \quad f_1 \quad 0.5C_{12} f_0 f_0 \quad C_{12} f_0 \quad C_{12} f_0 \quad C_{11} f_1 \quad 0.5C_{11} f_1 f_0 \tag{20}$$

$$C_{12} \quad f_2 \quad f_2 \quad f_1 \quad C_{11} f_1 \quad C_{11} R T_1 \quad C_{12} R T_0 \quad C_{11} f_1 \quad 0$$

$$f_2(0) \quad 0, \quad f_2(0) \quad 0, \quad f_2(\infty) \quad 0$$

$$T_2 \quad C_{21} T_1 \quad T_1 \quad C_{21} T_1 \quad C_{22} T_0 \quad C_{22} T_0 \quad T_2 \quad 0.5C_{21} \Pr f_0 T_1 \quad C_{21} T_1$$

$$0.5C_{22} \Pr f_0 T_0 \quad T_1 \quad C_{22} T_0 \quad 0.5\, C_{21} \Pr f_1 T_0 \quad 0$$

$$T_2(0) \quad 1, \quad T_2(\infty) \quad 0$$

Solving eqs. (18)-(20) under the related boundary conditions, we have:

$$\begin{gathered} f_0 \quad \mathrm{e}^{\ \eta} \quad \eta \quad 1 \\ T_0 \quad \mathrm{e}^{\ \eta} \end{gathered} \tag{21}$$

$$\begin{gathered} f_1(\eta) \quad C_{11}\mathrm{e}^{\ \eta} \quad 0.5C_{11}\mathrm{e}^{\ \eta} \quad 0.25C_{11}\mathrm{e}^{\ 2\eta} \quad 0.25\mathrm{e}^{\ \eta} C_{11}\eta^2 \\ \mathrm{e}^{\ \eta} C_{11} R\eta \quad C_{11} R\mathrm{e}^{\ \eta} \quad 0.75C_{11} \quad CR \end{gathered} \tag{22}$$

$$\begin{aligned} & T_1(\eta) \quad [\ 0.5C_{21}(2\eta \quad \Pr \mathrm{e}^{\ \eta} \quad 0.5\Pr\eta^2 \quad \Pr\eta) \quad 0.5C_{21}\Pr]\mathrm{e}^{\ \eta} \\ & \vdots \\ & f(\eta) \quad f_0(\eta) \quad f_1(\eta) \quad f_2(\eta) \\ & T(\eta) \quad T_0(\eta) \quad T_1(\eta) \quad T_2(\eta) \end{aligned} \tag{23}$$

From eq. (8) by substituting $f(\eta)$ and $T(\eta)$ to eqs. (14) and (15), $R_1(\eta, C_{11}, C_{12})$ and $R_2(\eta, C_{21}, C_{22})$ are obtained and J_1 and J_2 are obtained in the flowing manner:

$$J_1(C_{11}, C_{12}) \quad {}_0^{\infty} R_1^2(\eta, C_{11}, C_{12}) \mathrm{d}\eta$$

$$J_2(C_{21}, C_{22}) \quad {}_0^{\infty} R_2^2(\eta, C_{21}, C_{22}) \mathrm{d}\eta$$

The constants C_{11}, C_{12}, C_{21}, and C_{22} are obtained from the conditions (10). In the particular cases:

$$\Pr = 0.7, \quad R = 0.05$$

$$C_{11} = -0.6431661413, \quad C_{12} = 3.163858714,$$

$$C_{21} = 0.7875530889, \quad C_{22} = 0.9682570954$$

Conclusions

In this present work, the OHAM is successfully applied to obtain analytical solution of laminar two-dimensional motion of fluid past a semi-infinite vertical plate. Validity of this method has been shown in fig. 2 by comparison between OHAM and numerical solution results.

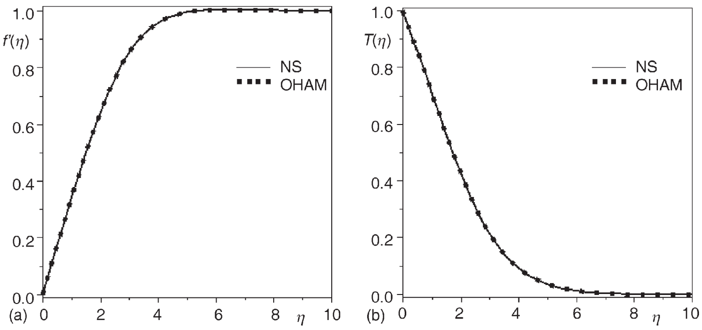

Figure 2. Comparison of the solutions via OHAM and numerical solutions for $f'(\eta)$ and $T(\eta)$
(a) for R = 0.01 and Pr = 0.5;
(b) for R = 0.1 and Pr = 0.7

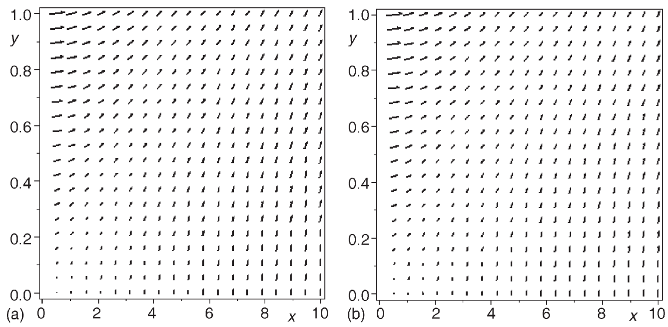

Figure 3. Velocity vector in x-direction;
(a) for R = 0.01 and Pr = 0.5;
(b) for R = 0.1 and Pr = 0.7

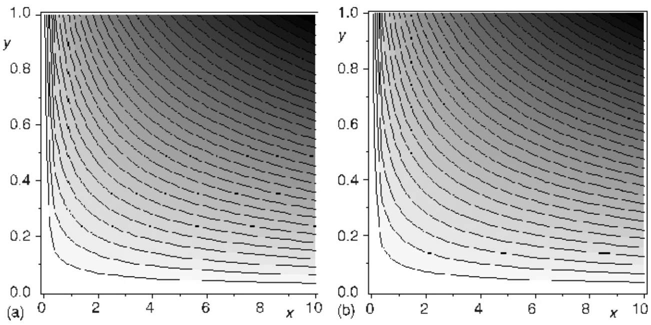

Figure 4. Vector line
(a) for R = 0.01 and Pr = 0.5,
(b) for R = 0.1 and Pr = 0.7

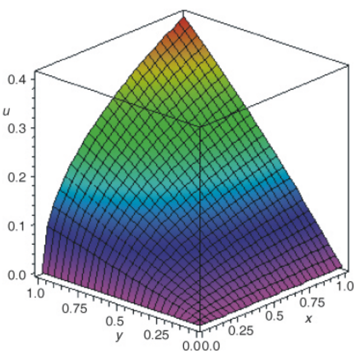

Figure 5. Velocity distribution R = 0.01 and Pr = 0.5

Velocity vectors and lines have been shown in figs. 3 and 4 for two cases. Velocity distribution in x- and y-direction has been also depicted in fig. 5.

In addition, temperature distributions for two cases have been shown in figs. 6-8 by plotting temperature vectors and lines. The effects of buoyancy parameter and Prandtl number on temperature and velocity profile have been shown in figs. 9 and 10. Results show that the velocity increases with increment of the buoyancy parameter and temperature decreases with increasing of Prandtl number.

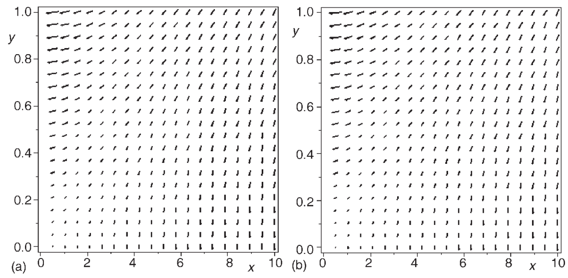

Figure 6. Temperature vector
(a) for $R = 0.01$ and $Pr = 0.5$;
(b) for $R = 0.1$ and $Pr = 0.7$

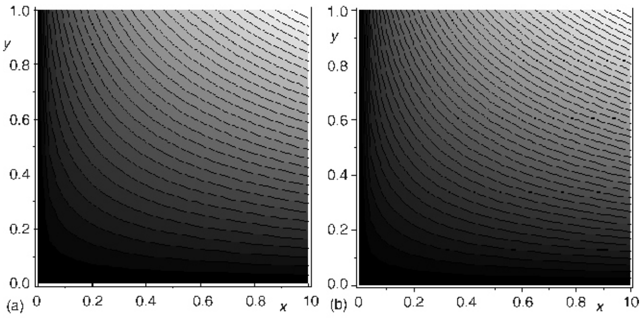

Figure 7. Temperature line
(a) for $R = 0.01$ and $Pr = 0.5$;
(b) for $R = 0.1$, $Pr = 0.7$

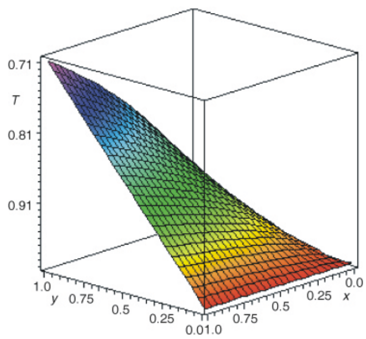

Figure 8. Temperature distribution for $R = 0.1$ and Pr = 0.7

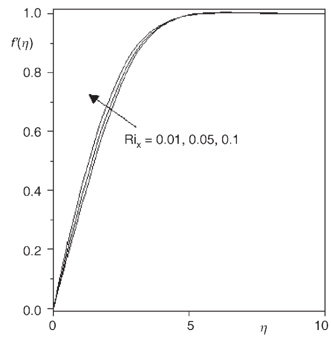

Figure 9. Effect of buoyancy parameter on velocity profile

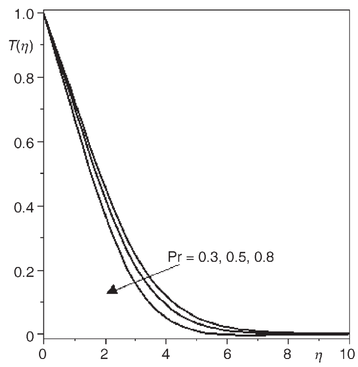

Figure 10. Effect of Prandtl number on temperature profile

References

[1] Nayfeh, A. H., Mook, D. T., Nonlinear Oscillations, John Willey and Sons, New York, USA, 1979
[2] He, J. H., Homotopy Perturbation Method for Bifurcation on Nonlinear Problems, *Int. J. Non-Linear Sci. Numer. Simul., 6* (2005), 2, pp. 207-208
[3] He, J. H., An Elementary Introduction to the Homotopy Perturbation Method, *Computers & Mathematics with Applications, 57* (2009), 3, pp. 410-412
[4] He, J. H., A Review on Some New Recently Developed Nonlinear Analytical Techniques, *Int. J. Non-linear Sci. Numer. Simul., 1* (2000), 1, pp. 51-70
[5] Ariel, P. D., Homotopy Perturbation Method and the Natural Convection Flow of a Third Grade Fluid through a Circular Tube, *Nonlinear Science Letters A, 1* (2010) 1, pp. 43-52
[6] Ganji, D. D., *et al*., A Comparative Comparison of He's Method with Perturbation and Numerical Methods for Nonlinear Vibrations Equations, *International Journal of Nonlinear Dynamics in Engineering and Sciences, 1* (2009), 1, pp. 1-20
[7] Golbabai, A., Ahmadian, D., Homotopy Pade Method for Solving Linear and Nonlinear Integral Equations, *International Journal of Nonlinear Dynamics in Engineering and Sciences, 1* (2009), 1, pp. 59-66
[8] He, J. H., Variational Iteration Method – Some Recent Results and New Interpretations, *Journal of Computational and Applied Mathematics, 207* (2007), 1, pp. 3-17
[9] Liao, S. J., The Proposed Homotopy Analysis Technique for the Solution of Nonlinear Problems, Ph. D. thesis, Shanghai Jiao Tong University, Shanghai, China, 1992
[10] Joneidi, A. A., *et al.*, Analytical Treatment on Magnetohydrodynamic (MHD) Flow and Heat Transfer Due to a Stretching Hollow Cylinder, *Int. J. Numer. Meth. Fluids,* (2009), DOI: 10.1002/fld.2087
[11] Joneidi, A. A., Domairry, G., Babaelahi, M., Analytical Treatment of MHD Free Convective Flow and Mass Transfer over a Stretching Sheet with Chemical Reaction, *Journal of the Taiwan Institute of Chemical Engineers,* (2009), doi:10.1016/j.jtice.2009.05.008
[12] Zhou, J. K., Differential Transformation and Its Applications for Electrical Circuits (in Chinese), Huarjung University Press, Wuuhahn, China, 1986
[13] Babaelahi, M., Ganji, D. D., Joneidi, A. A., Analysis of Velocity Equation of Steady Flow of a Viscous Incompressible Fluid in Channel with Porous Walls, *Int. J. Numer. Meth. Fluids,* (2009), DOI: 10.1002/fld.2114.
[14] Joneidi, A. A., Ganji, D. D., Babaelahi, M., Differential Transformation Method to Determine Fin Efficiency of Convective Straight Fins with Temperature Dependent Thermal Conductivity, *Int. Comm. Heat and Mass Trans., 36* (2009), 7, pp. 757-762
[15] He, J. H., An Elementary Introduction to Recently Developed Asymptotic Methods and Nanomechanics in Textile Engineering, *Int. J. of Modern Phys. B, 221* (2008), 21, pp. 3487-3578
[16] Marinca, V., Herisanu, N., Application of Optimal Homotopy Asymptotic Method for Solving Nonlinear Equations Arising in Heat Transfer, *Int. Comm. Heat and Mass Trans., 35* (2008), 6, pp. 710-715
[17] Marinca, V., *et al*., An Optimal Homotopy Asymptotic Method Applied to the Steady Flow of a Fourth-Grade Fluid Past a Porous Plate, *Applied Mathematics Letters, 22* (2009), 2, pp. 245-251
[18] Marinca, V., Herisanu. N., Optimal Homotopy Asymptotic Method with Application to Thin Film Flow, *Central European Journal of Physics, 6* (2008), 3, pp. 1608-1644
[19] Herisanu, N., *et al*., A New Analytical Approach to Nonlinear Vibration of an Electrical Machine, *Proceedings of the Romanian Academy, Series A, 9* (2008), 3, pp. 229-236
[20] Joneidi, A. A., Ganji, D. D., Babaelahi, M., Micropolar Flow in a Porous Channel with High Mass Transfer, *Int. Comm. Heat and Mass Trans., 36* (2009), 10, pp. 1082-1088
[21] Gebhart, B., *et al*., Buoyancy-Induced Flows and Transport, Hemisphere Publication, Washington DC, USA, 1988
[22] Oosthuizen, P. H., Naylor, D., An Introduction to Convective Heat Transfer Analysis, McGraw Hill, New York, USA, 1999

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