A NUMERICAL STUDY FOR INWARD SOLIDIFICATION OF A LIQUID CONTAINED IN CYLINDRICAL AND SPHERICAL VESSEL

by

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This study presents a numerical solution of inward solidification of phase change material contained in cylinder/sphere. Here, constant thermal property is assumed throughout the analysis for the liquid, which is initially at fusion temperature. The governing dimensionless equations of the above problem and boundary conditions are converted to initial value problem of vector matrix form. The time function is approximated by Chebyshev series and the operational matrix of integration is applied. The solution is utilized iteratively in the interface condition to determine the time taken to attain a fixed interface position.

Key words: *inward solidification in cylindrical/spherical region, moving interface, Chebyshev polynomial, operational matrix*

Introduction

Solidification problems have application in many fields of scientific and technological endeavour. They are interesting both because of diversity of their application and because of their non-linearity, which is associated with the moving interface. Due to presence of moving interface and non-linearity, the exact solution of these problems are limited and restricted only for a few specific cases [1-3]. Very few analytical solutions to the solidification problems are available. Hill [4] summarized some techniques for analytical solution and series solution for solidification problems. Some approximate analytical solutions for inward solidification in cylindrical/spherical region are discussed in [5-8].

Beside analytical methods, numerical solutions are more common and computer intensive. Hence many numerical methods have been developed. In 1967, Tao [9] developed a numerical method for the solidification problem of a saturated liquid contained in cylindrical or a spherical container. Voller *et al*. [10] presented an explicit algorithm to obtained the solidification and melting time in circular regions. They assumed first kind of boundary condition (Dirichlet boundary condition). An implicit finite difference method based on the enthalpy method for the analysis of phase change problem was reported by Voller [11]. Caldwell *et al*. [12] applied a numerical method based on the enthalpy method to spherical solidification. The

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results are compared to the results obtained by the heat-balance method. Ismail *et al.* [13] reported a numerical study for spherical solidification by using finite difference approximation and moving grid approach. They analyzed the effect of the size, thickness and material of the container and the external wall temperature on the solidification rate. Ismail *et al*. [14] presented a numerical study for the spherical solidification under convective boundary conditions. In 2005, Bilir *et al*. [15] have reported the results of numerical study of inward solidification problem of a phase change material encapsulated in cylinder/sphere container. Third kind of boundary condition (Robin boundary condition) is assumed. They have used enthalpy method with control volume approach. Recently, Assis *et al*. [16] presented numerical and experimental study of solidification in a spherical shell.

In the present study, a numerical solution of inward solidification of a liquid contained in cylinder/sphere is reported. The dimensionless differential equations governing the above process are converted into initial value problem by using central difference operator. The time function is approximated by Chebyshev series of the second kind and operational matrix of integration is applied [17] on it. The solution of initial value problem is utilized iteratively in the interface condition to determine the time taken to cover a given interface position. The results are presented through figures.

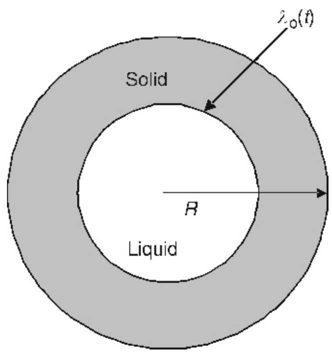

Figure 1. Schematic diagram of freezing process in cylinder//sphere co-ordinates

Mathematical formulation

Consider a cylindrical/spherical vessel filled with a molten material at an initial temperature, which is its freezing temperature. At $t = 0$, the boundary is cooled by imposing a constant temperature T_0 which is lower than T_f. As time proceeds the molten material will eventually solidify. The geometry of the problem is depicted in fig. 1.

The following assumptions are considered here:
– the density change from liquid to solid is ignored,
– thermal properties of solid and liquid are equal,
– the thermal resistance of the vessel is negligible, and
– the heat transfer process inside the vessel is only by conduction in radial direction.

Under this assumption the dynamics of freezing can be described by the following equation:

$$\frac{\partial T}{\partial t} \quad \frac{\alpha}{r^n}\frac{\partial}{\partial r} \quad r^n \frac{\partial T}{\partial r}, \quad \lambda_0(t) \quad r \quad R \tag{1}$$

with, $n = 1$ and 2 for cylindrical and spherical configuration, respectively. The associated initial and boundary conditions are specified as:

$$T = T_f \quad \text{at} \quad t = 0 \tag{2}$$

$$T = T_0 \quad \text{at} \quad r = R \tag{3}$$

The energy balance at the solidification front can be written as:

$$T = T_f \quad \text{at} \quad r = \lambda_0(t) \tag{4}$$

$$k\frac{\partial T}{\partial r} \quad \rho L\frac{\mathrm{d}\lambda_0(t)}{\mathrm{d}t} \quad \text{at} \quad r \quad \lambda_0(t) \tag{5}$$

$$\lambda_0(0) = R \tag{6}$$

where α is the thermal diffusivity, k – the thermal conductivity, ρ – the density, $\lambda_0(t)$ – the location of interface, and L the latent heat of the solidification.

Solution of the problem

Introducing the dimensionless variables [6] and similarity criteria:

$$x \quad \frac{r}{R}, \quad S \quad \frac{c\Delta T}{L}, \quad \lambda \quad \frac{\lambda_0(t)}{R}, \quad \tau \quad \frac{tk\Delta T}{\rho LR^2}, \quad \theta \quad \frac{T \quad T_0}{\Delta T}, \tag{7}$$

where $\Delta T = T_f - T_0$.

The system of the eqs. (1)-(6) reduces to the form:

$$S\frac{\partial\theta}{\partial\tau} \quad \frac{1}{x^n}\frac{\partial}{\partial x} \quad x^n\frac{\partial\theta}{\partial x} \tag{8}$$

$$\theta(1, \tau) = 0 \tag{9}$$

$$\theta(\lambda, \tau) = 1 \tag{10}$$

$$\theta(x, 0) = 1 \tag{11}$$

$$\frac{\partial\lambda}{\partial\tau} \quad \frac{\partial\theta}{\partial x} \quad \text{at} \quad x \quad \lambda(\tau) \tag{12}$$

$$\lambda(0) = 1 \tag{13}$$

Replacing the domain [1, 0] [0, ∞] by a rectangular grid of points (x_i, τ_i) with

$$x_i = ih \qquad i = 0, 1, 2, \ldots, k+1$$
$$\tau_j = i\Delta\tau \qquad j = 0, 1, 2, \ldots$$

where $h = 1/(k+1)$, and τ is the discretization step for the normalized time variable. Taking discretization in the space variable x only. By using central difference, the eqs. (8)-(11) can be written in vector matrix form as:

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} \quad \frac{1}{2\mathrm{Sh}^2}\mathbf{A}\theta + \mathbf{B} \tag{14}$$

$$\theta'(0) = [1, 1, 1, \ldots\ldots, 1]' \tag{15}$$

where

$$\theta \quad \begin{matrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \vdots \\ \vdots \\ \theta_k \end{matrix} \qquad \mathbf{B} \quad \begin{matrix} 0 \\ 0 \\ 0 \\ \vdots \\ \vdots \\ \dfrac{2x_k \quad hn}{2Sh^2 x_k} \end{matrix}$$

$$
\mathbf{A}\quad
\begin{array}{ccccccc}
4 & 2\ \dfrac{hn}{x_1} & 0 & \ldots.0 & 0 & 0 & 0 \\
2\ \dfrac{hn}{x_2} & 4 & 2\ \dfrac{hn}{x_2} & \ldots.0 & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \ldots.0 & 2\ \dfrac{hn}{x_{\mathrm{k}\ 1}} & 4 & 2\ \dfrac{hn}{x_{\mathrm{k}\ 1}} \\
0 & 0 & 0 & \ldots.0 & 0 & 2\ \dfrac{hn}{x_{\mathrm{k}}} & 4
\end{array}
$$

Integrating eq. (14) and using the eq. (15),we obtain:

$$\theta(\tau)\quad \theta(0)\quad \mathbf{A}_{0}^{\tau}\theta(x)\,\mathrm{d}x\quad \mathbf{B}_{0}^{\tau}1\mathrm{d}x \tag{16}$$

The approximation of $\theta(\tau)$ by Chebyshev series gives:

$$\theta(\tau) = \mathbf{D}F(\tau) \tag{17}$$

$$1 = \mathbf{E}F(\tau) \tag{18}$$

where

$$
\mathbf{D}\quad
\begin{array}{cccc}
d_{11} & d_{12} & \ldots\ldots & d_{1m} \\
d_{21} & d_{22} & \ldots\ldots & d_{2m} \\
.. & .. & \ldots\ldots & .. \\
.. & .. & \ldots\ldots & .. \\
d_{\mathrm{k}1} & d_{\mathrm{k}2} & \ldots\ldots & d_{\mathrm{k}m}
\end{array}
$$

d_{ij} (i = 1, 2, 3, ..., k; j = 1, 2, 3, ..., m) are the Chebyshev coefficients of matrix **D**,

$$\mathbf{E} = [1, 0, 0, 0, \ldots\ldots, 0]_{1m}$$

and

$$F\quad [f_0, f_1, f_2, \ldots\ldots, f_{m\ 1}]'_{1m}$$

f_i is Chebyshev polynomial of second kind such that:

$$f_0\quad 1$$

$$f_1\quad 2\quad 4\ \frac{\tau}{\tau^*}$$

$$f_2\quad 3\quad 16\ \frac{\tau}{\tau^*}\quad 16\ \frac{\tau}{\tau^*}^{\,2}$$

$$\vdots$$

$$\vdots$$

$$f_{j\ 1}\quad 2\,1\quad 2\ \frac{\tau}{\tau^*}\quad f_j\quad f_{j\ 1}\ \frac{\tau}{\tau^*}$$

Moreover, integration of the Chebyshev vector gives:

$$\int_0^\tau F(y)\mathrm{d}y \approx \mathbf{P}F(\tau) \tag{19}$$

where **P** is the operational matrix of integration, τ^* – the generalized time, and

$$\mathbf{P} = \tau^* \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 & \ldots\ldots & 0 & 0 & 0 & 0 \\ \frac{3}{8} & 0 & \frac{1}{8} & 0 & 0 & \ldots\ldots & 0 & 0 & 0 & 0 \\ \frac{1}{6} & \frac{1}{12} & 0 & \frac{1}{12} & 0 & \ldots\ldots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & & \ldots\ldots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & & \ldots\ldots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{2(m-1)} & 0 & 0 & 0 & 0 & \ldots\ldots & 0 & \frac{1}{4(m-1)} & 0 & \frac{1}{4(m-1)} \\ \frac{1}{2m} & 0 & 0 & 0 & 0 & \ldots\ldots & 0 & 0 & \frac{1}{4m} & 0 \end{bmatrix}_{mm}$$

Substituting eqs. (17) and (18) in eq. (16) and using eq. (19), we obtained:

$$\mathbf{DF} = \mathbf{ADP}F + \mathbf{BEP}F$$

Since the Chebyshev polynomial are independent, equating the coefficients of $F(t)$ gives the following set of linear algebraic equations:

$$\mathbf{D} - \mathbf{ADP} = \mathbf{BEP} \tag{20}$$

Now, we look for the normalized time in which the interface moves a distance λ. The region $(0, \lambda)$ is divided into k + 1 equal parts or sub regions. Replacing the space derivative by using backward operator in the interface condition (12) and integrating it with $\lambda(0) = 1$, we obtain:

$$\lambda(\tau) = 1 - \frac{1}{h}\left[\tau - \int_0^\tau \eta_i(y)\mathrm{d}y\right] \tag{21}$$

where $i = 0, 1, \ldots\ldots, k$

By assuming a fixed value of τ^*, the elements of the matrix **D** whose order is k × m are computed from eq. (20). Replacing η_i by $\sum_{j=1}^{m} d_{ij} f_{j-1}$ and taking $\tau/\tau^* = 1$, the eq. (21) becomes:

$$\tau = \frac{h(\lambda - 1)}{1 - d_{i1} - \frac{1}{3}d_{i3}} \tag{22}$$

which gives the required time in which the phase front is at a distance λ.

Numerical results and discussion

In this section, we present the numerical results of the dimensionless time taken to cover a distance $\lambda(\tau)$ and determination of melt fraction with reference to dimensionless time in cylindrical and spherical geometries. The computations have been made for fixed values of $\tau^* = 10^{-2}$ and $m = 3$.

Figures 2 and 3 depict the dependence of interface location on dimensionless time for three values of Stefan numbers S = 0.75, 1.5, 5.0 [8, 12, 18]. It can be seen from the figures that the velocity of interface is slower for higher values of Stefan numbers for both cylindrical and spherical cases. Moreover, the velocity of interface decreases as it approaches the centre. One can also observe that the solidification process is slow in case of sphere than the cylindrical case for particular value of S. It is also seen from the figures that the dimensionless time for the complete solidification in case of the cylinder is more than that of spherical case. This result is in complete agreement with the results of Prud'homme *et al.* [6] and Lin *et al.* [8].

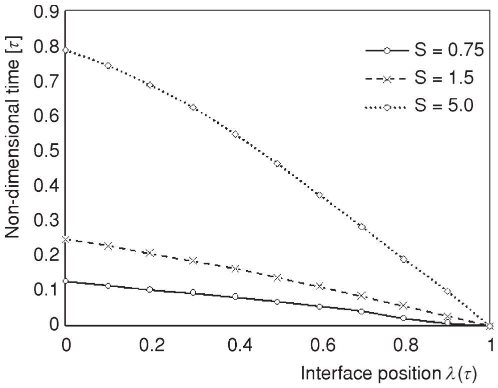

Figure 2. Plot of τ *vs.* $\lambda(\tau)$ for cylindrical solidification

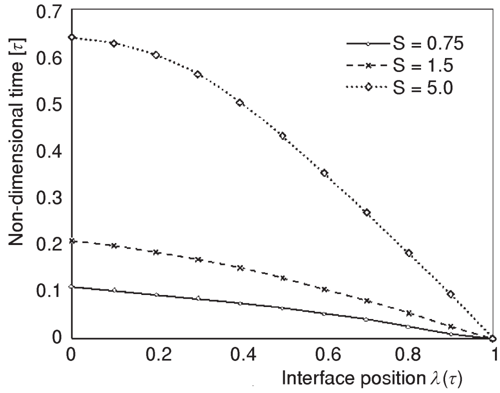

Figure 3. Plot of τ *vs.* $\lambda(\tau)$ for spherical solidification

Figures 4 and 5 represent the dependence of melt fraction on dimensionless time for a fixed value of Stefan number (S = 5.0) for cylindrical and spherical solidification, respectively. Actually, melt fraction is defined as the ratio of melted mass and total mass of the phase change

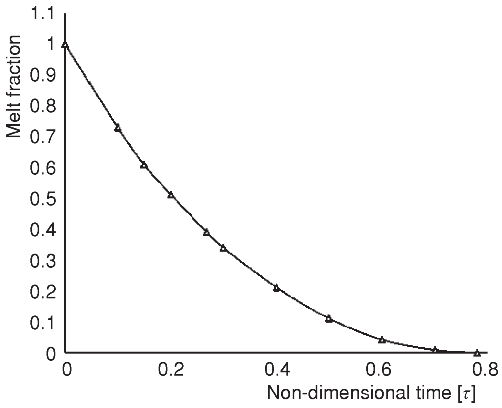

Figure 4. Plot of melt fraction *vs.* τ for cylindrical solidification

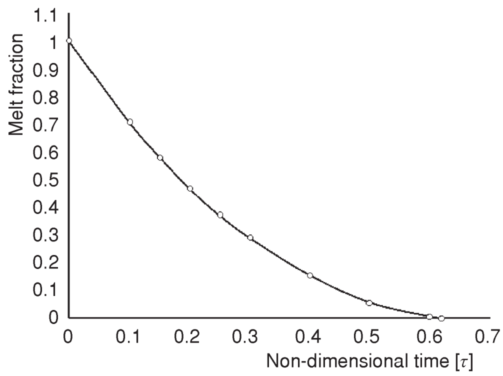

Figure 5. Plot of melt fraction *vs.* τ for spherical solidification

material. Thus, the melt fraction is zero when solidification is complete. It is clear from both figures that total time required to complete solidification is higher for the case of cylinder in comparison with total time for the spherical case. The trend of the result in fig. 5 for spherical case is similar to the result of Assis *et al*. [16].

Conclusions

We have presented a numerical technique to solve inward solidification problem in cylindrical and spherical geometry. It can be seen that the proposed method is efficient, user friendly and accurate to determine the solution of moving boundary problems. In view of rapid convergence of the Chebyshev series of second kind, only a few terms of the series are needed to give satisfactory results. The authors strongly believes that the proposed method will be helpful to the engineers who are working in the area of solidification.

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Nomenclature

c – specific heat, [$\mathrm{Jkg^{-1}K^{-1}}$]
$\mathbf{D}$ – coefficient matrix-vector
d_{ij} – Chebyshev coefficients of matrix $\mathbf{D}$
F – Chebyshev matrix-vector
k – thermal conductivity, [$\mathrm{WK^{-1}m^{-1}}$]
L – latent heat of fusion, [$\mathrm{Jkg^{-1}}$]
R – position of the fixed boundary, [m]
r – position in the solidified region, [m]
S – Stefan number (= $c\Delta T/L$), [–]
T – temperature distribution, [K]
t – time, [s]
x – normalized position (= r/R), [–]

Greek letters

α – thermal diffusivity, [$\mathrm{m^2s^{-1}}$]
θ – normalized temperature distribution [= $(T - T_0)/\Delta T$], [–]
λ_0 – interface position, [m]
ρ – density [$\mathrm{kgm^{-3}}$]
τ – dimensionless time (= $tk\Delta T/\rho LR^2$)

Subscripts

0 – at fixed boundary, $r = R$
f – freezing

Superscript

* – generalized
/ – transpose

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