NUMERICAL SOLUTION OF STEFAN PROBLEM WITH TIME-DEPENDENT BOUNDARY CONDITIONS BY VARIABLE SPACE GRID METHOD

by

Svetislav SAVOVIĆ and James CALDWELL

Original scientific paper
UDC: 536.246:517.962
DOI: 10.2298/TSCI0904165S

The variable space grid method based on finite differences is applied to the one-dimensional Stefan problem with time-dependent boundary conditions describing the solidification/melting process. The temperature distribution, the position of the moving boundary and its velocity are evaluated in terms of finite differences. It is found that the computational results obtained by the variable space grid method exhibit good agreement with the exact solution. Also the present results for temperature distribution are found to be more accurate compared to those obtained previously by the variable time step method.

Key words: *Stefan problem, variable space grid method, finite differences*

Introduction

Moving boundary problems known as Stefan problems involving heat conduction in conjunction with change of phase are of great interest in numerous important areas of science, engineering, and industry. Such a process covers a wide range of applications in which phase changes from liquid, solid, or vapour states. The moving boundary problems occur in many areas such as the metal, glass, plastic and oil industries, space vehicle design, preservation of foodstuffs, chemical and diffusion processes, *etc*. The material is assumed to undergo a phase change with a moving boundary whose position is unknown and has to be determined as part of the analysis. Across the phase boundary the heat flux is not continuous, and the heat equation is replaced by a flux condition which relates the velocity of the phase boundary and the jump of heat flux across the phase front.

Since moving boundary problems require solving the heat equation in an unknown region which has also to be determined as part of the solution, they are inherently non-linear. Because of non-linearity of moving boundary problems they can be solved analytically for only a limited number of special cases [1]. Due to difficulties in obtaining analytical solution, numerical techniques are far more common [2-9]. Numerical techniques are specially known to have difficulties with time-dependent boundary conditions and very small time steps are often needed for accurate solutions. Solutions of such Stefan problems reported in the literature include linear, exponential, and periodical variation of the surface temperature or the flux with time [8, 10-13]. Comparison of various numerical methods has been made by Furzeland [11] and Caldwell *et al*. [14].

There are two main approaches in the solution of the Stefan problem. One is the front-tracking method, where the position of the phase boundary is continuously tracked. An example is the heat balance integral method [2], which explicitly tracks the motion of isotherms (the phase boundary being one of them). An alternative approach, namely, variable grid methods (variable space grid and variable time step) provide a way to track the phase front explicitly [15].

Another approach is to use a fixed-domain formulation. An example is the isotherm migration method, which uses the temperature as the independent variable [16]. A more common method is the enthalpy method which uses an enthalpy function together with the temperature as dependent variable [8, 17, 18]. Alternatively, using a suitable coordinate transformation, one may immobilise the moving front at the expense of solving a more complicated problem by a numerical scheme described by Kutluay *et al*. [6].

The one-dimensional Stefan problem with time-dependent boundary conditions describing the solidification/melting process is considered in this paper. The variable space grid (VSG) method is employed in order to determine the evolution of the temperature distribution and phase boundary during the process. The computational results are compared with the exact solution and with those obtained earlier by Caldwell *et al*. [19] who used the variable time step finite difference method. Solutions reported in the literature using the VSG method for solving the moving boundary problem include the one-dimensional Stefan problem describing the process of melting of ice [6] and the process of evaporation of droplets [20].

Formulation of the problem

Here we consider the Stefan problem describing a one-dimensional single phase melting process where the temperature is increased exponentially with time at the fixed boundary $x = 0$. The temperature throughout the solid is assumed to remain at the melting point. We are interested in the temperature distribution $u(x, t)$ in the region $0 \le x \le s(t)$ and in the location of the moving boundary. Within the dimensionless mathematical model, the function $u(x, t)$ is governed by the heat equation:

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < s(t), \quad t > 0 \tag{1}$$

subject to the boundary conditions

$$\begin{aligned} u(x,t) &= e^{\alpha t}, \quad x = 0, \quad t > 0 \\ u(x,t) &= 1, \quad x = s(t), \quad t > 0 \end{aligned} \tag{2}$$

where α is a physical parameter combining the density, specific heat, and the thermal conductivity. The location of the moving boundary is given by the heat balance equation known as the Stefan condition:

$$\frac{1}{\alpha}\frac{\mathrm{d}s}{\mathrm{d}t} = -\frac{\partial u}{\partial x}, \quad x = s(t), \quad t > 0 \tag{3}$$

The initial condition is

$$s(0) = 0 \tag{4}$$

The exact solution of this problem is given by:

$$\begin{aligned} u(x,t) &= e^{\alpha t - x} \\ s(t) &= \alpha t \end{aligned} \tag{5}$$

We use the exact solution (5) to initialise our numerical schemes and to compare it with our computational results.

Here we deal with the finite difference solution of the dimensionless model problem given by eqs. (1)-(4). Several numerical techniques based on finite differences and finite elements have been successfully applied to the treatment of the Stefan problem [4, 5, 21-23]. In this paper, in order to determine $s(t)$ for $t > 0$ and $u(x, t)$ for $0 \leq x \leq s(t)$ and $t > 0$, we employ a variable space grid technique.

A variable space grid (VSG) method

The number of space intervals between a fixed boundary $x = 0$ and a moving boundary $x = s(t)$ is kept constant and equal to N, and thus the moving boundary always lies on the N-th grid. Before writing the finite difference form of eq. (1), it is necessary to take into account the continuous change in the nodal positions due to the boundary movement. The following expression applies at the i-th grid point:

$$\left.\frac{\partial u}{\partial t}\right|_i = \left.\frac{\partial u}{\partial x}\right|_t \left.\frac{\partial x}{\partial t}\right|_i + \left.\frac{\partial u}{\partial t}\right|_x \tag{6}$$

and the node x_i is moved according to the expression:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{x_i}{s(t)}\frac{\mathrm{d}s}{\mathrm{d}t} \tag{7}$$

in which the suffices t, i, and x are to be kept constant during the differentiation process and omitted for clarity below. By substituting eqs. (1) and (7) into eq. (6), the following equation is obtained:

$$\frac{\partial u}{\partial t} - \frac{x_i}{s}\frac{\mathrm{d}s}{\mathrm{d}t}\frac{\partial u}{\partial t} = \alpha\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < s(t), \quad t > 0 \tag{8}$$

subject to the boundary conditions (2). Equation (3), subject to initial condition (4), remains unchanged. One should note here that the grid size $\Delta x = s(t)/N$ varies with time t as the interface moves, since the number N of grid points is constant.

The temperature gradient at the moving interface $[x = s(t) = N\Delta x]$ is given by the following three point backward scheme [11]:

$$\left.\frac{\partial u}{\partial x}\right|_{x=s} = \frac{3u_N - 4u_{N-1} + u_{N-2}}{2\Delta x} + \mathrm{O}(\Delta x^2) \tag{9}$$

Using a forward difference approximation for the time derivative and a central difference approximation for the space derivative, the discretization of eq. (8) can be expressed as:

$$u_{i,m+1} = u_{i,m} + \frac{k x_{i,m}\dot{s}_m}{2h_m s_m}(u_{i+1,m} - u_{i-1,m}) + \frac{k\alpha}{h_m^2}(u_{i+1,m} - 2u_{i,m} + u_{i-1,m}) \tag{10}$$

where $u_{i,m} \equiv u(x_{i,m}, t_m)$, $s_m \equiv s(t_m)$, $\dot{s}_m = (s_{m+1} - s_m)/\Delta t$, $x_{i,m} = ih_m$, $t_m = t_0 + mk$, h_m is the space grid size Δx at m^{th} time step, $k(\equiv \Delta t)$ – the time step, and t_0 – the time at which the numerical process is initialised. A truncation error for this scheme is $\mathrm{O}(k) + \mathrm{O}(h_m^2)$. The schematic diagram in fig. 1 demonstrates the construction of the grids for the finite difference solution.

The temperature distribution at the origin is easily obtained using the boundary condition (2) at $x = 0$, which in discretized form is:

$$u_{i,m} = e^{\alpha t_m}, \quad i = 0, \quad m = 0, 1, 2, \ldots \tag{11}$$

For the temperature distribution at $0 < x < s(t)$ ($i = 1, 2, \ldots, N - 1$, $m = 0, 1, 2, \ldots$,) eq. (10) is to be used. The boundary condition (2) at $x = s(t)$ is:

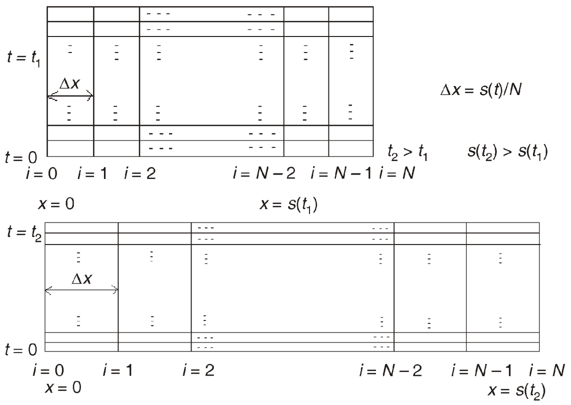

Figure 1. Schematic diagram to illustrate the construction of the grids for the finite difference solution

$$u_{i,m} = 1, \quad i = N, \quad m = 0, 1, 2, \ldots \tag{12}$$

Using eq. (9), the Stefan condition (3) at $x = s(t)$ $(i = N)$ in terms of finite differences is:

$$s_{m+1} \quad s_m \quad \frac{k\alpha}{2h_m}(3u_{N,m} \quad 4u_{N-1,m} \quad u_{N-2,m}), \quad m \quad 0, 1, 2, \ldots, \tag{13}$$

and the initial condition (4) becomes:

$$s_0 = 0 \tag{14}$$

On the basis of the updated interface location s_{m+1}, the updated grid size h_{m+1} is calculated at each time step as $h_{m+1} = s_{m+1}/N$.

Numerical results and discussion

In this section we present the computational results obtained by using the VSG method applied to the one-dimensional Stefan problem describing the melting process of a solid. We compare our computational results with the exact solution for $\alpha = 2$ and 10. Also the present results for temperature distribution are compared with those obtained earlier by Caldwell *et al.* [19] for $\alpha = 10$, who use the variable time step finite difference method. In the VSG method used in the present study, the numerical process is initialised using the exact solution (5) of the Stefan problem defined by eqs. (1)-(4). The initial time $t_0 = 0.01$ which according to eq. (5) corresponds to the initial position of the moving boundary $s(t_0) = 0.02$ and 0.1 for $\alpha =$ $= 2$ and 10, respectively, is used. We investigate the evolution of the temperature distribution, the position of the moving boundary and its velocity in a time interval from $t = t_0 = 0.01$ to 0.5. Applying the VSG method a grid size h_m (Δx) $= s(t)/N$ ($N = 10$ is also adopted) varies between 0.002 and 0.1 for $\alpha = 2$ and between 0.01 and 0.5 for $\alpha = 10$, since we are analyzing the movement of the phase boundary position $s(t)$ between 0.02 and 1 for $\alpha = 2$ and between 0.1 and 5 for $\alpha = 10$. The time steps k (Δt)=0.000001 and 0.000002 are used for $\alpha = 2$ and 10, respectively. Such a choice of time step and grid size guarantees stability of our difference schemes applied within the VSG method.

We first present the results obtained for $\alpha = 10$. The present computational results for the temperature distribution $u(x, t)$ together with the exact solution are shown in tab. 1. Good agreement between the present results and exact solution is seen. Furthermore, the accuracy of the present results for the temperature distribution $u(x, t)$ is about one order of magnitude better than the accuracy of the results obtained earlier in [19] (shown in tab. 2) using the variable time

Table 1. Temperature distribution *u*(*x, t*) obtained using the VSG method compared with the exact solution for α = 10

t	*x/s*	*u*(*x, t*)		Error [%]
		VSG	Exact solution	
0.1	0.0	2.71828183	2.71828183	0.0
	0.1	2.45974218	2.45977508	0.00133752
	0.2	2.22577016	2.22585214	0.00368308
	0.3	2.01403256	2.01417512	0.00707784
	0.4	1.82241728	1.82262844	0.01158547
	0.5	1.64901245	1.64929771	0.01729585
	0.6	1.49208743	1.49245062	0.02433514
	0.7	1.35007570	1.35051958	0.03286735
	0.8	1.22155926	1.22208609	0.04310907
	0.9	1.10525462	1.10586654	0.05533398
	1.0	1.0	1.0	0.0
0.3	0.0	20.08553692	20.08553692	0.0
	0.1	14.91021931	14.90341793	0.04563658
	0.2	11.06523546	11.05829865	0.06272945
	0.3	8.20937495	8.20522981	0.05051827
	0.4	6.08867860	6.08825990	0.00687717
	0.5	4.51419571	4.51747355	0.07255914
	0.6	3.34539139	3.35195402	0.19578520
	0.7	2.47777425	2.48714146	0.37662554
	0.8	1.83367515	1.84545271	0.63819354
	0.9	1.35537724	1.36932128	1.01831760
	1.0	1.0	1.0	0.0
0.5	0.0	148.41315910	148.41315910	0.0
	0.1	90.89914687	90.63872426	0.28731937
	0.2	55.60565785	55.35478380	0.45321122
	0.3	33.97452783	33.80621378	0.49787903
	0.4	20.73231873	20.64609438	0.41763032
	0.5	12.63416185	12.60896046	0.19986890
	0.6	7.68612410	7.70053072	0.18708607
	0.7	4.66455051	4.70285980	0.81459562
	0.8	2.81941818	2.87212546	1.83513150
	0.9	1.69152627	1.75406136	3.56515970
	1.0	1.0	1.0	0.0

Table 2. Temperature distribution *u*(*x, t*) as calculated in [19] using the variable time step method and exact solution for α = 10

t	*u*(*x, t*)								
	x = 0.1			*x* = 0.3			*x* = 0.5		
	Computed	Exact	Error [%]	Computed	Exact	Error [%]	Computed	Exact	Error [%]
0.0199957	1.105159	1.104697	0.041821						
0.2555534	11.68015	11.65265	0.235998	9.607639	9.540384	0.704951	7.902224	7.811006	1.167814
0.2946605	17.27035	17.22920	0.238839	14.20695	14.10607	0.715153	11.68634	11.54907	1.188581
0.5095832	148.1550	147.7959	0.242970	121.8889	120.0050	1.569851	100.2791	99.07057	1.219868
0.5486411	218.9491	218.4181	0.243112	180.1327	178.8256	0.730936	148.1975	146.4100	1.220887

step method. In fig. 2 the computational results and exact solution for moving boundary position versus time are shown. In tab. 3 the computational and exact values for boundary position and its

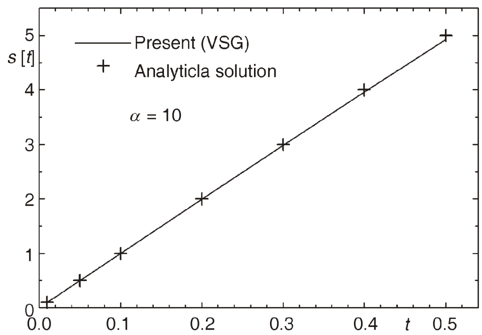

Figure 2. Position of moving boundary *vs*. time for α = 10

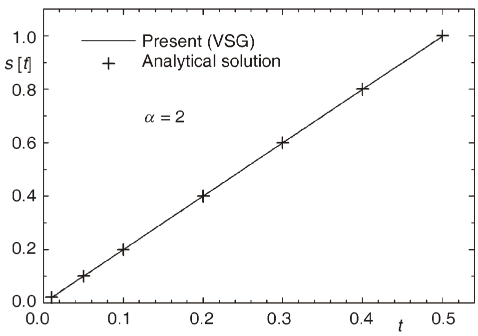

Figure 3. Position of moving boundary *vs*. time for α = 2

velocity are shown together with percentage errors. Reasonably good agreement between the present results and the exact solution is seen.

In tab. 4 is shown a comparison of *u*(*x, t*) determined from the exact solution and from the finite difference calculations for α = 2. Very good agreement between the present results and

Table 3. Position of moving boundary and its velocity obtained using VSG method compared with exact solution for α = 10

t	*s* [t]			*t*	d*x*/d*t*		
	VSG	Error [%]	Exact value		VSG	Error [%]	Exact value
0.01	0.1		0.1	0.01	9.99966996	0.00330040	10.0
0.02	0.19999399	0.00300500	0.2	0.02	9.99903562	0.00964380	
0.05	0.49990592	0.01881600	0.5	0.05	9.99450640	0.05493600	
0.1	0.99930086	0.06991400	1.0	0.1	9.97993878	0.20061220	
0.2	1.99496049	0.25197550	2.0	0.2	9.92856643	0.71433570	
0.3	2.98409422	0.53019267	3.0	0.3	9.84945140	1.50548560	
0.4	3.96387752	0.90306200	4.0	0.4	9.74104545	2.58954550	
0.5	4.93118458	1.37630840	5.0	0.5	9.59898056	4.01019440	

Table 4. Temperature distribution $u(x, t)$ obtained using VSG method compared with exact solution for $\alpha = 2$

t	x/s	$u(x, t)$		Error [%]
		VSG	Exact solution	
0.1	0.0	1.2214027582	1.2214027582	0.0
	0.1	1.1972175448	1.1972181266	0.0000485920
	0.2	1.1735111896	1.1735123676	0.0001003853
	0.3	1.1502742116	1.1502759994	0.0001554206
	0.4	1.1274973175	1.1274997275	0.0002137510
	0.5	1.1051713977	1.1051744418	0.0002754433
	0.6	1.0832875230	1.0832912124	0.0003405778
	0.7	1.0618369408	1.0618412864	0.0004092499
	0.8	1.0408110716	1.0408160839	0.0004815701
	0.9	1.0202015058	1.0202071951	0.0005576653
	1.0	1.0	1.0	0.0
0.3	0.0	1.8221188004	1.8221188004	0.0
	0.1	1.7160228294	1.7160343226	0.0006697538
	0.2	1.6161015667	1.6161261251	0.0015195844
	0.3	1.5219957400	1.5220346224	0.0025546336
	0.4	1.4333669468	1.4334211643	0.0037823844
	0.5	1.3498964432	1.3499668167	0.0052129822
	0.6	1.2712840033	1.2713712143	0.0068596032
	0.7	1.1972468440	1.1973514790	0.0087388738
	0.8	1.1275186122	1.1276412020	0.0108713483
	0.9	1.0618484308	1.0619894849	0.0132820541
	1.0	1.0	1.0	0.0
0.5	0.0	2.7182818285	2.7182818285	0.0
	0.1	2.4597424933	2.4597750458	0.0013233921
	0.2	2.2257706279	2.225852852	0.0036595986
	0.3	2.0140330655	2.0141750415	0.0070488412
	0.4	1.8224177583	1.8226283431	0.0115539052
	0.5	1.6490128544	1.6492976074	0.0172651086
	0.6	1.4920877489	1.4924505088	0.0243063260
	0.7	1.3500759228	1.3505194642	0.0328422797
	0.8	1.2215594026	1.2220859670	0.0430873484
	0.9	1.1052546863	1.1058664094	0.0553161914
	1.0	1.0	1.0	0.0

Table 5. Position of moving boundary and its velocity obtained using the VSG method compared with the exact solution for α = 2

t	s [t]		
	VSG	Error [%]	Exact value
0.01	0.02	–	0.02
0.02	0.0399999484	0.0001291125	0.04
0.05	0.0999991859	0.0008140748	0.1
0.1	0.1999936232	0.0031884082	0.2
0.2	0.3999509301	0.0122674694	0.4
0.3	0.5998399761	0.0266706493	0.6
0.4	0.7996321205	0.0459849380	0.8
0.5	0.9993009904	0.0699009641	1.0
t	ds/dt		
	VSG	Error [%]	Exact value
0.01	1.9999977517	0.0001124500	2.0
0.02	1.9999920331	0.0003983431	
0.05	1.9999514832	0.0024258386	
0.1	1.9998113373	0.0094331333	
0.2	1.9992818233	0.0359088341	
0.3	1.9984515198	0.0774240088	
0.4	1.997347122	0.1326243919	
0.5	1.9959886398	0.2005680093	

the exact solution is seen. The computational results and exact solution for moving boundary position *vs*. time are plotted in fig. 3. In tab. 5 the boundary position and its velocity determined using finite differences are compared with the exact solution. Again, good agreement between the present results and exact solution is evident.

Clearly, our computational results for the worse case tabulated (corresponding to t = 0.5) with α = 10 are approximately within 4% or less of the exact values. Since the values of α for almost all the materials of practical interest are less than 5, the VSG method may be assumed sufficiently accurate for most practical applications. Also the VSG method has been earlier successfully applied to the Stefan problem with Neumann boundary condition at x = 0 by Caldwell *et al.* [20] describing the evaporation of droplets and a time-dependent boundary condition at x = 0 by Kutluay *et al.* [6] describing the process of melting of ice.

On the basis of the results obtained we can conclude that the VSG method, which uses constant time step, can be successfully employed to the Stefan problem describing a one-dimensional single phase melting process where the temperature is increased exponentially at the fixed boundary x = 0. Although the exponentially increasing temperature at the fixed boundary x = 0 makes this problem more difficult than the problem with time-independent boundary conditions successfully treated earlier [20] using the VSG method, this method again proves to be very efficient and accurate.

Conclusions

We report on the implementation of the variable space grid method for the solution of the Stefan problem describing the melting process of a solid. Very good agreement between the computational results obtained using the VSG method with the exact solution is evident. We find that the accuracy of the VSG results for α = 2 is much better than the accuracy of the computational results achieved for α = 10. Since the values of α for almost all the materials of practical interest are less than 5, the VSG method may be assumed sufficiently accurate for most practical applications. One benefit of the VSG method is that the computation time is comparatively short and so it is possible to achieve higher accuracy by refining the mesh size. Furthermore, this method is shown to provide more accurate solutions of the Stefan problem treated in the present work compared to those obtained using the variable time step method [19]. The good agreement achieved in comparison with the analytical solution gives us confidence in the use of this variable space grid approach for other Stefan problems with time-dependent boundary conditions. This is important for those cases where analytical solutions are not available.

Acknowledgments

The authors would like to thank City University of Hong Kong for funding this research by a Strategic Research Grant (Project No. 7001242).

References

[1] Hill, J., One-Dimensional Stefan Problem: An Introduction, Longman Scientific and Technical, Harlow, Essex, UK, 1987
[2] Goodman, T. R., The Heat-Balance Integral Method and its Application to Problems Involving a Change of Phase, *Trans. ASME, 80* (1958), 2, pp. 335-342
[3] Crank, J., Free and Moving Boundary Problems, Clarendon Press, Oxford, UK, 1984
[4] Asaithambi, N. S., On a Variable Time Step Method for the One-Dimensional Stefan Problem, *Comput. Meth. Appl. Mech. Engg., 71* (1988), 1, pp. 1-13
[5] Asaithambi, N. S., A Variable Time-Step Galerkin Method for a One-Dimensional Stefan Problem, *Appl. Math. Comput., 81* (1997), 2-3, pp. 189-200
[6] Kutluay, S., Bahadir, A. R. Özdes, A., The Numerical Solution of One-Phase Classical Stefan problem, *J. Comput. Appl. Math., 81* (1997), 1, pp. 135-144
[7] Wood, A. S., A New Look at the Heat Balance Integral Method, *Appl. Math. Modelling, 25* (2001), 10, pp. 815-824
[8] Esen, A., Kutluay, S., A Numerical Solution of the Stefan Problem with a Neumann-Type Boundary Condition by Enthalpy Method, *Appl. Math. Comput., 148* (2004), 2, pp. 321-329
[9] Ang, W.T., A Numerical Method Based on Integro-Differential Formulation for Solving a One-Dimensional Stefan Problem, *Numerical Methods for Partial Differential Equations, 24* (2008), 3, pp. 939-949
[10] Mennig, J., Özisik, M. N., Coupled Integral Equation Approach for Solving Melting or Solidification, *Int. J. Mass Transfer, 28* (1985), 8, pp. 1481-1485
[11] Furzeland, R. M., A Comparative Study of Numerical Methods for Moving Boundary Problems, *J. Inst. Maths. Appl., 59* (1980), 26, pp. 411-429
[12] Rizwann-Uddin, One-Dimensional Phase Change with Periodic Boundary Conditions, *Num. Heat Transfer, Part A, 35* (1999), 4, pp. 361-372
[13] Savović, S., Caldwell, J., Finite Difference Solution of One-Dimensional Stefan Problem with Periodic Boundary Conditions, *Int. J. Heat and Mass Transfer, 46* (2003), 15, pp. 2911-2916
[14] Caldwell, J., Kwan, Y. Y., Numerical Methods for One-Dimensional Stefan Problems, *Commun. Numer. Meth. Engng., 20* (2004), 7, pp. 535-545
[15] Marshall, G., A Front Tracking Method for One-Dimensional Moving Boundary Problems, *SIAM J. Sci. Stat. Comput., 7* (1986), 1, pp. 252-263
[16] Churchill, S. W., Gupta, J. P., Approximations for Conduction with Freezing or Melting, *Int. J. Heat Mass Transfer, 20* (1977), 11, pp. 1251-1253
[17] Voller, V. R., Cross, M., Applications of Control Volume Enthalpy Methods in the Solution of Stefan Problems, in: Computational Techniques in Heat Transfer (Eds. R. W. Lewis, K. Morgan, J. A. Johnson, W. R. Smith), Pineridge Press Ltd., Mumbles, Swansea, UK, 1985
[18] Caldwell, J., Chan, C. C., Spherical Solidification by the Enthalpy Method and the Heat Balance Integral Method, *Appl. Math. Model., 24* (2000), 1, pp. 45-53
[19] Caldwell, J., So, K. L., Numerical Solution of Stefan Problems Using the Variable Time Step Finite-Difference Method, *J. Math. Sciences, 11* (2000), 2, pp. 127-138
[20] Caldwell, J., Savović, S., Numerical Solution of Stefan Problem by Variable Space Grid Method and Boundary Immobilisation Method, *J. Math. Sciences, 13* (2002), 1, pp. 67-79
[21] Gupta, R. S., Kumar, D., A Modified Variable Time Step Method for the One-Dimensional Stefan Problem, *Comput. Meth. Appl. Mech. Engng., 23* (1980), 1, pp. 101-108

[22] Murray, W. D., Landis, F., Numerical and Machine Solutions of Transient Heat Conduction Involving Melting or Freezing, *J. Heat Transfer, 81* (1959), pp. 106-112
[23] Finn, W. D., Voroglu, E., Finite Element Solution of the Stefan Problem, in: The Mathematics of Finite Elements and Applications, MAFELAP 1978 (Ed. J. R. Whiteman), Academic Press, New York, USA, 1979

Authors´ affiliations:

S. Savović (**corresponding author**)
Department of Mathematics,
Kowloon, City University of Hong Kong,
Hong Kong, People's Republic of China

Faculty of Science,
University of Kragujevac
12, R. Domanovića Str.,
34000 Kragujevac, Serbia
E-mail: savovicŽkg.ac.rs

J. Caldwell
Department of Mathematics,
City University of Hong Kong,
Kowloon, Hong Kong, People's Republic of China

Paper submitted: October 20, 2008
Paper revised: October 25, 2008
Paper accepted October 28, 2008