SOLUTION OF ONE-DIMENSIONAL MOVING BOUNDARY PROBLEM WITH PERIODIC BOUNDARY CONDITIONS BY VARIATIONAL ITERATION METHOD

by

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In this paper, the solution of the one dimensional moving boundary problem with periodic boundary conditions is obtained with the help of variational iterational method. By using initial and boundary values, the explicit solutions of the equations have been derived, which accelerate the rapid convergence of the series solution. The method performs extremely well in terms of efficiency and simplicity. The temperature distribution and the position of moving boundary are evaluated and numerical results are presented graphically.

Key words: *moving boundary problems, variational iteration method, temperature distribution, interface position, Lagrange multiplier, Mittag-Leffler function*

Introduction

Moving boundary (or Stefan) problems involving heat and mass transfer undergoing phase change occur in many physical processes, such as freezing, melting, evaporation, condensation, sublimation, and desublimation. Such problems have wide applications in separation process, food technology, image development in electro photography, medical sciences, heat and mixture migration in solid grounds, *etc*. The history and classical solution to Stefan problems are well covered in the monographs by Crank [1] , Hill [2], *etc*. A lot of work has already been done by many researchers to solve one-dimensional Stefan problems. Asaithambi [3] used time step Galerkin method to solve one-dimensional Stefan problem. Rizwan-Uddin [4] has solved a moving boundary problem with time dependent boundary conditions. In this paper the author has shown the application of nodal integral approach to non-linear Stefan problem which predicts the temperature distribution and position of interface accurately. Rizwan-Uddin [5] also studied the one-dimensional Stefan problem with periodic boundary conditions using a semi analytical numerical scheme. Recently, the similar type of problem has been solved by Savovic *et al*. [6] using finite difference method and also solved by Ahmed [7] using an algorithm based on the boundary integral method.

The analytical approach taken in this literature is the variational iteration method. The variational iteration method was first proposed by He ([8-11]) and was successfully applied to solve non-linear systems of partial differential equations and non-linear differential equations by [12-14]. The main advantage of this method is apart from computational simplicity, the solution obtained by this method is expected to be a better approximation in a straight forward manner.

To the best of authors knowledge solution of Stefan problems with periodic boundary conditions by variational iteration method has not yet been solved. In this paper, the proposed method is used to obtain an approximate analytical solution to solve one-dimensional Stefan problem with periodic boundary conditions. The expressions of the temperature distribution and position of interface are obtained using both the initial and boundary conditions. The numerical results for different particular cases are found and depicted through graphs.

Solution of the problem by He's variational iteration method

In this section, the application of the variational iteration method is discussed for solving the dimensionless set of the Stefan problem [5, 15]:

$$\frac{\partial^2 T(x,t)}{\partial x^2} + xR(t)\frac{\mathrm{d}R(t)}{\mathrm{d}t}\frac{\partial T(x,t)}{\partial x} = R^2(t)\frac{\partial T(x,t)}{\partial x}, \quad 0 < x < 1 \tag{1}$$

$$R(t)\frac{\mathrm{d}R(t)}{\mathrm{d}t} = -\mathrm{Ste}\frac{\partial T(x,\, t)}{\partial x}, \quad x = 1 \tag{2}$$

subjected to the time dependent boundary conditions

$$\begin{aligned} &R = 0, && t = 0 \\ &T(0,t) = f(t), && t > 0 \\ &\frac{\partial T(0,t)}{\partial x} = 0, && T(1,t) = 0 \end{aligned} \tag{3}$$

where Ste is the Stefan number given by $C_1\Delta T_{\mathrm{ref}}/h$, C_1 – the specific heat capacity of liquid, ΔT_{ref} – a reference temperature, h – the latent heat, $T(x, t)$ – the temperature distribution, and $R(t)$ – the position of moving boundary:

$$T_{n+1}(x,t) = T_n(x,t) + \int_0^x \lambda(\xi)\left[\frac{\partial^2 T_n(\xi,t)}{\partial \xi^2} + \xi R(t)\dot{R}(t)\frac{\partial \widetilde{T}_n}{\partial \xi} - R^2(t)\frac{\partial \widetilde{T}_n}{\partial t}\right]\mathrm{d}\xi \tag{4}$$

It is obvious that the successive approximation T_j, $j \geq 0$, can be established by determining Lagrange multiplier λ. The function $\widetilde{T}_n$ is a restricted variation, which means $\delta\widetilde{T}_n = 0$. The successive approximation $T_{n+1}(x, t)$, $n \geq 0$, of the solution $T(x, t)$ will be readily obtained upon using the Lagrange's multiplier and by using any selective function T_0. The initial value $T(0, t)$ and $T_x(0, t)$ are usually used for selecting the zero-th approximation T_0. To find the optimal value of λ, we have:

$$\delta T_{n+1}(x,t) = \delta T_n(x,t) + \delta\int_0^x \lambda(\xi)\frac{\partial^2 T(x,t)}{\partial \xi^2}\mathrm{d}\xi = 0 \tag{5}$$

$$[1 - \lambda'(\xi)]_{\xi=x}\,\delta T_n(\xi,t) + [\lambda(\xi)]_{\xi=x}\,\delta\frac{\partial T_n(\xi,t)}{\partial t} + \int_0^x [\lambda''(\xi)]_{\xi=x}\,\delta T_n(\xi,t)\mathrm{d}\xi = 0 \tag{6}$$

which yields

$$\lambda''(\xi)\big|_{\xi=x} = 0 \tag{7}$$

$$1 - \lambda'(\xi)\big|_{\xi=x} = 0 \tag{8}$$

$$\lambda(\xi)\big|_{\xi = x} = 0 \tag{9}$$

which gives

$$\lambda(\xi) = \xi - x \tag{10}$$

We, therefore, obtain the following iteration formula:

$$T_{n+1}(x,t) = T_n(x,t) + \int_0^x (\xi - x)\left[\frac{\partial^2 T_n}{\partial \xi^2} + \xi R(t)\dot{R}(t)\frac{\partial T_n}{\partial \xi} - R^2(t)\frac{\partial T_n}{\partial t}\right] \mathrm{d}\xi \tag{11}$$

Beginning with an initial approximation:

$$T_0(x, t) = T(0, t) + xT_x(0, t) = f(t) \tag{12}$$

we obtain the following successive approximations:

$$T_1(x,t) = f(t) + R^2(t)f'(t)\frac{x^2}{2!},$$

$$T_2(x,t) = f(t) + R^2(t)f'(t)\frac{x^2}{2!} + R^4(t)f''(t)\frac{x^4}{4!}$$

$$T_3(x,t) = f(t) + R^2(t)f'(t)\frac{x^2}{2!} + R^4(t)f''\frac{x^4}{4!} + R^6(t)f'''(t)\frac{x^6}{6!}$$

and so on.

Using this procedure for sufficiently large values of n we get $T_n(x, t)$ as an approximation of the exact solution.

Thus the exact solution may be obtained by using

$$T(x,t) = \lim_{n \to \infty} T_n(x,t)$$

$$= f(t) + R^2(t)f'(t)\frac{x^2}{2} + R^4(t)f''(t)\frac{x^4}{4!} + R^6(t)f'''(t)\frac{x^6}{6!} + \dots \tag{13}$$

$$= \sum_{k=0}^{\infty} \frac{f^{(k)}(t)[R^2(t)]^k (x^2)^k}{\Gamma(2k+1)} \tag{14}$$

$$= E_2[\psi(t)R^2(t)x^2]$$

Now using boundary condition $T(1, t) = 0$ (eq. 3) and eq. (13), we get:

$$T(x,t) = R^2(t)f'(t)\frac{(1-x^2)}{2!} + R^4(t)f''(t)\frac{(1-x^4)}{4!} + R^6(t)f'''(t)\frac{(1-x^6)}{6!} + \dots, \quad 0 \le x \le 1 \tag{15}$$

Boundary condition (2) with the help of eq. (14) gives rise to:

$$R(t)\frac{\mathrm{d}R(t)}{\mathrm{d}t} = \mathrm{Ste}\sum_{k=0}^{\infty}\frac{R^{2k}(t)f^k(t)}{\Gamma(2k)} \tag{16}$$

where $[\psi(t)]^k = f^k(t)$ and

$$E_\alpha(t) = \sum_{k=0}^{\infty}\frac{t^k}{\Gamma(\alpha k + 1)}$$

is the generalized Mittag-Leffler function.

Numerical results and discussion

In this section, numerical results of the position of interface *vs*. time and temperature distribution *vs*. space are presented through figs. 1-3 for $f(t) = 1 + \varepsilon\sin\omega t$ where $0 \le t \le 1$. It is

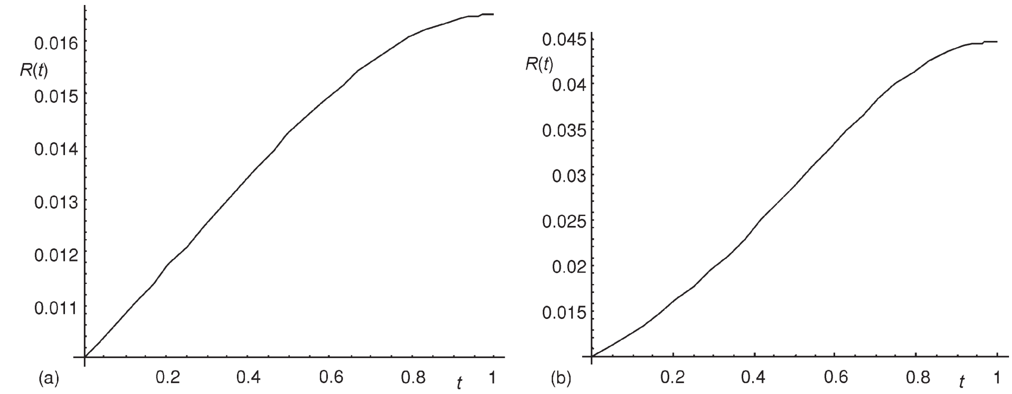

Figure 1. (a) Plot of *R*(*t*) *vs*. *t* for Ste = 1, and ε = 0.5; (b) Plot of R(t) vs. *t* for Ste = 1, and ε = 1.5

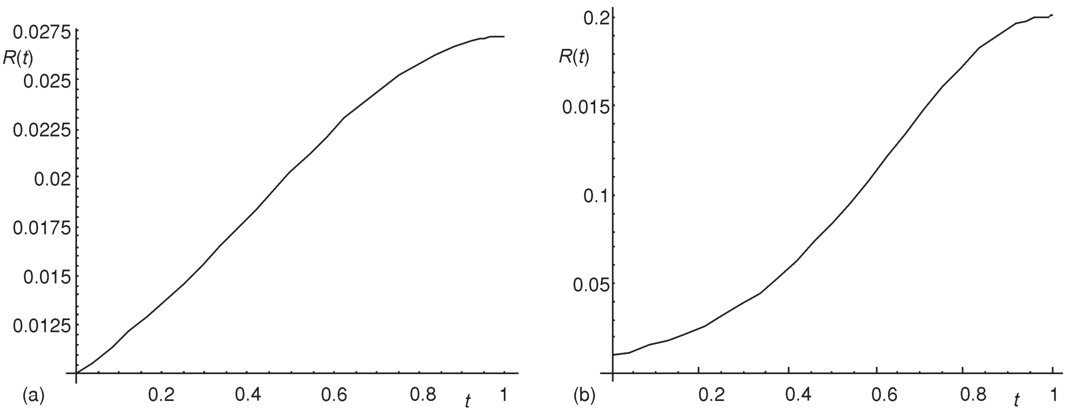

Figure 2. (a) Plot of *R*(*t*) *vs*. *t* for Ste = 2, and ε = 0.5; (b) Plot of *R*(*t*) vs. *t* for Ste = 2, and ε = 1.5;

seen from figs. 1 and 2 that the movement of interface position becomes fast with the increase of both values of oscillation amplitude (ε) and Stefan number (Ste).

Figures 3(a) and 3(b) depict the temperature distribution of a material in the whole domain $0 \quad x \quad 1$ for $\varepsilon = 0.5$, $\varepsilon = 1.5$, and Ste = 1.0. It is clear from figures that increase in the oscil-

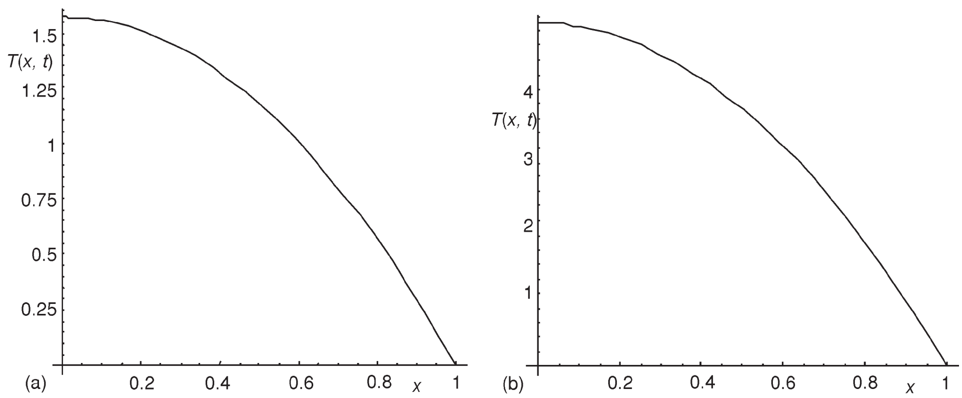

Figure 3. (a) Plot of *T*(*x*, *t*) *vs*. *x* for ε = 0.5; (b) Plot of *T*(*x*, *t*) *vs*. *x* for ε = 1.5

lation amplitude leads to a more pronounced change in the temperature distribution. It is also observed from the figures that the temperature distribution is maximum at $x = 0$ and then decreases continuously to become zero at the tip $x = 1$. This result is in complete agreement with [5], and [6].

Conclusions

The variational iteration method is very powerful in finding the solutions for various scientific and engineering problems. Sharing its application for Stefan problem with periodic boundary conditions, we may conclude that this method will be very much useful for solving many physical problems both analytically and numerically .The proper implementation of proposed method can dramatically minimize the size of the calculations if compared with the existing methods. The advantage of this method comparing with existing methods consists in obtaining the interface position and temperature distribution in the form of continuous function, instead of discrete form. The other advantage of the method is its fast convergence of the series solution. Moreover, no linearization or perturbation is needed and it avoids the accuracy of finding the temperature distribution by the numerical techniques like finite difference and boundary integral methods.

References

[1] Crank, J., Free and Moving Boundary Value Problem, Oxford University Press, Oxford, U. K., 1984

[2] Hill, J. M., One-Dimensional Stefan Problems, an Introduction, Longman Scientific and Technical, New York, USA, 1987

[3] Asaithambi, N. S., A Variable Time Step Galerkin Method for One Dimensional Stefan Problem, *Appl. Math. Comput., 81* (1997), 2-3, pp. 189-200

[4] Rizwan-Uddin, A Nodal Method for Phase Change Moving Boundary Problems, *Int. J. Comp. Fluid Dynamics, 11* (1999), 3, pp. 211-221

[5] Rizwan-Uddin, One Dimensional Phase Change Problem with Periodic Boundary Conditions, *Numerical Heat Transfer A, 35* (1999), 4, pp. 361-372

[6] Savovic, S., Caldwell, J., Finite Difference Solution of One-Dimensional Stefan Problem with Periodic Boundary Conditions, *Int. J. Heat and Mass Transfer, 46* (2003), 15, pp. 2911-2916

[7] Ahmed, S. G., A New Algorithm for Moving Boundary Problem Subject to Periodic Boundary Conditions, *Int. J. of Numerical Method for Heat and Flow, 16* (2006), 1, pp. 18-27

[8] He, J. H., Approximate Solution of Nonlinear Differential Equations with Convolution Product nonlinearities, *Comput. Methods Appl. Mech. Engg., 167* (1998), 1-2, pp. 69-73

[9] He, J. H., Variational Iteration Method – a Kind of Nonlinear Analytical Technique: Some Examples, *Internat. J. Nonlinear Mech., 34* (1999), 4, pp. 699-708

[10] He, J. H., Variational Iteration Method for Autonomous Ordinary Differential Systems, *Appl. Math. Comput., 114* (2000), 2-3, pp.115-123

[11] He, J. H., Some Asymptotic Methods for Strongly Nonlinear Equations, *Internat. J. Modern Phys. B 20* (2006), 10, pp. 1141-1199

[12] Golbabai, A., Javidi, M., A Variational Iteration Method for Solving Parabolic Partial Differential Equations, *Int. J. Computers & Mathematics with Applications, 54* (2007), 7-8, pp. 987-992.

[13] Abassy, T. A., El-Tawil M. A., El-Zoheiry, H., Exact Solution of Some Non Linear Partial Differential Equations Using the Variational Iteration Method Linked with Laplace Transforms and the Pade Technique, *Computers and Mathematics with Applications, 54* (2007), 7-8, pp. 940-954

[14] Soliman, A. A., Abdou, M. A., Numerical Solutions of Nonlinear Evolution Equations Using Variational Iteration Method, *Journal of Computational and Applied Mathematics, 207* (2007), 1, pp. 111-120

[15] Yao, L. S., Prusa, J., Melting and Freezing, in: Advances in Heat Transfer, Academic Press, New York, USA Vol. 19, 1989, pp. 1-95

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