

MELTING AND FREEZING IN A FINITE SLAB DUE TO A LINEARLY DECREASING FREE-STREAM TEMPERATURE OF A CONVECTIVE BOUNDARY CONDITION

by

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One-dimensional melting and freezing problem in a finite slab with time-dependent convective boundary condition is solved using the heat-balance integral method. The temperature, $T_{\infty 1}(t)$, is applied at the left face and decreases linearly with time while the other face of the slab is imposed with a constant convective boundary condition where $T_{\infty 2}$ is held at a fixed temperature. In this study, the initial condition of the solid is subcooled (initial temperature is below the melting point). The temperature, $T_{\infty 1}(t)$ at time $t = 0$ is so chosen such that convective heating takes place and eventually the slab begins to melt (i. e., $T_{\infty 1}(0) > T_f > T_{\infty 2}$). The transient heat conduction problem, until the phase-change starts, is also solved using the heat-balance integral method. Once phase-change process starts, the solid-liquid interface is found to proceed to the right. As time continues, and $T_{\infty 1}(t)$ decreases with time, the phase-change front slows, stops, and may even reverse direction. Hence this problem features sequential melting and freezing of the slab with partial penetration of the solid-liquid front before reversal of the phase-change process. The effect of varying the Biot number at the right face of the slab is investigated to determine its impact on the growth/recession of the solid-liquid interface. Temperature profiles in solid and liquid regions for the different cases are reported in detail. One of the results for Biot number, $Bi_2 = 1.5$ are also compared with those obtained by having a constant value of $T_{\infty 1}(t)$.

Key words: *finite slab, melting, freezing, heat balance integral, time-dependent, convection*

Introduction

Phase-change heat transfer needs to be addressed in various industrial processes. Such solutions provide an improved knowledge of the process under study. But, solutions in explicit form (similarity solutions) exist only for semi-infinite problems with thermophysical properties constant in each phase and constant initial and imposed temperatures [1]. This limitation is on account of the inherent non-linearity of the solid-liquid interface. Thus, for more realistic situations we are forced to seek approximate analytical or numerical solutions [2]. The problems dealing with finite slabs are even more complicated and there is very limited literature available on such phase-change problems in finite domain. Kar *et al.* [3] studied the classical Stefan problem for a finite slab with Dirichlet boundary conditions. There is some literature on starting solutions for phase-change of finite slabs considering both one and two-dimensional heat conduction [4, 5]. Goodman addressed the melting of a finite slab with a constant heat-flux on one side,

the other side of the slab being insulated or at a constant temperature [6]. Zhang *et al.* worked on this same problem but considered the case when the preheating time of the slab was longer than the thermal penetration time [7]. It is well known that there are many phase-change processes that involve convective boundary conditions and such boundary conditions are often time-dependent. Analysis of these phase-change problems is difficult and the literature is scarce. There are a few solutions for phase-change problems involving convective boundary conditions in semi-infinite mediums such as those using quasi stationary approximation [8] and the heat-balance integral technique [9]. A convective cooling problem was studied by Chan *et al.* for a semi-infinite phase change medium [10]. There is a published study involving solidification of a finite slab initially at fusion temperature and convectively cooled from both sides and solved using perturbation technique [11]. More recently, Roday *et al.* examined the problem of one-dimensional melting and freezing with constant *convective* boundary conditions on both faces of a finite slab using the heat balance integral technique [12, 13].

The aim of this paper is to analyze the problem of phase-change in finite slabs using the heat-balance integral method but for a time-dependent convective boundary condition, $[T_{\infty,1}(t)]$, at the left face of the slab. The other face of the slab is imposed with a constant convective boundary condition with $T_{\infty,2}$ being constant and held at the ambient temperature. The temperature, $T_{\infty,1}$, is assumed to decay with time in a linear fashion. The slab is analyzed for the initial condition of the solid being subcooled below the melting temperature. Initially, $T_{\infty,1}$ is so chosen that $T_{\infty,1}(0) > T_f > T_{\infty,2}$. This causes the slab to start melting from the left face. But when $T_{\infty,1}$ decreases with time along with convective cooling at the right face, the slab begins to refreeze. The variation of Bi_2 (Biot number at the right face of the slab) is investigated.

Mathematical analysis

The slab is assumed to be initially subcooled (at the temperature $T_{\infty,2}$). Hence, it starts to change phase only when the left face of the slab reaches the fusion temperature. The problem is solved in two different time periods. The first time period involves transient heat conduction in the solid, during which the left face of the slab reaches the phase-change temperature. During the second time period, the slab undergoes phase-change. The solution for the first time period is discussed in the section *Transient heat conduction in the solid* and forms the initial condition for the phase-change problem which is solved in the section *Phase-change problem*.

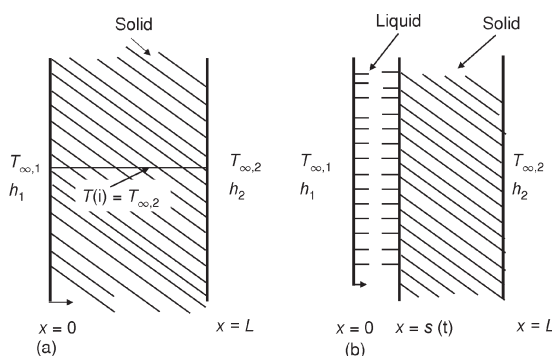


Figure 1. (a) Prior to phase-change, (b) During phase-change

Transient heat conduction in the solid

The heat balance integral method is utilized in this section to determine the temperature distribution in the solid slab before the phase-change process starts. As seen in fig. 1a, the left face of the slab is at $x = 0$ and is of finite length L . Its fusion temperature is T_f . It is assumed that $T_{\infty,1}$ decays linearly with time. The temperature at the other end is $T_{\infty,2}$ which is the ambient temperature. At time $t = 0$, $T_{\infty,1} > T_f > T_{\infty,2}$. The value of Biot numbers on both faces of the

slab are so chosen such that the heat diffusion through the slab ensures that its left face eventually reaches the phase-change temperature.

The heat conduction equation is:

$$\frac{\partial^2 T_s}{\partial x^2} = \frac{1}{\alpha_s} \frac{\partial T_s}{\partial t} \quad 0 \leq x \leq L \quad (1)$$

subject to boundary conditions

$$k_s \frac{\partial T_s}{\partial x} \Big|_{x=0} = h_1 [T_{\infty 1}(t) - T_s \Big|_{x=0}] \quad (2)$$

and

$$k_s \frac{\partial T_s}{\partial x} \Big|_{x=L} = h_2 (T_s \Big|_{x=L} - T_{\infty 2}) \quad (3)$$

The initial condition at time $t = 0$ is:

$$T_s = T_{\infty 2} \quad (4)$$

The temperature $T_{\infty 1}(t)$ is assumed to decay linearly such that:

$$\frac{dT_{\infty 1}(t)}{dt} = \text{const.} \quad (5)$$

at

$$t = 0, T_{\infty 1} = T_{\text{initial}} \quad (6)$$

Non-dimensional variables are introduced using the following definitions:

$$\varphi = \frac{T_s - T_{\infty 2}}{T_{\infty 1} - T_{\infty 2}}, \quad \xi = \frac{x}{L}, \quad \tau_s = \frac{t \alpha_s}{L^2}, \quad \text{Bi}_1 = \frac{h_1 L}{k_s}, \quad \text{Bi}_2 = \frac{h_2 L}{k_s} \quad (7a-e)$$

The non-dimensional heat diffusion thickness is defined by:

$$\eta^* = \frac{\delta}{L} \quad (8)$$

Eqs. (1), (2), and (4) are transformed using the non-dimensional variables as:

$$\frac{\partial^2 \varphi}{\partial \xi^2} = \frac{\partial \varphi}{\partial \tau_s} \quad 0 \leq \xi \leq 1 \quad (9)$$

and the boundary conditions on the left face is:

$$\frac{\partial \varphi}{\partial \xi} \Big|_{\xi=0} = \text{Bi}_1 \frac{T_{\infty 1}(\tau_s) - T_{\infty 2}}{T_{\infty 1} - T_{\infty 2}} - \varphi \Big|_{\xi=0} \quad (10)$$

For $\eta^* \ll 1$, the heat has not yet diffused to the right end of the slab, therefore, the following two conditions exist at $\xi = \eta$

$$\frac{\partial \varphi}{\partial \xi} \Big|_{\xi=\eta} = 0 \quad (11)$$

and since the solid is initially subcooled to $T_{\infty 2}$

$$\varphi \Big|_{\xi=\eta} = 0 \quad (12)$$

The initial condition at $\tau_s = 0$ reduces to:

$$\varphi = 0 \quad (13)$$

The temperature $T_{\infty 1}(\tau_s)$ is assumed to decay linearly with τ such that:

$$\frac{dT_{\infty 1}(\tau_s)}{d\tau_s} = \text{const.} \quad (14)$$

The heat balance integral method is used to solve this problem.

A quadratic expression for the non-dimensional temperature profile is assumed as

$$\varphi = a + b\xi + c\xi^2 \quad (15)$$

At $\tau_s = 0$,

$$\eta^* = 0 \quad \text{and} \quad T_{\infty 1} = T_{\text{initial}} \quad (16a, b)$$

Now, eq. (9) is integrated over the non-dimensional heat diffusion thickness η^* to obtain:

$$\left. \frac{\partial \varphi}{\partial \xi} \right|_{\xi=0} = \frac{d\theta}{d\tau_s} \quad (17)$$

where

$$\theta = \int_0^{\eta^*} \varphi d\xi \quad (18)$$

The following differential equation is then obtained:

$$6 \frac{T_{\infty 1}(\tau)}{T_{\infty 2}} - 1 - (\text{Bi}_1 \eta^* - 2) \frac{T_{\infty 1}(\tau)}{T_{\infty 2}} - 1 - (2\eta^* (\text{Bi}_1 \eta^* - 2) - \eta^{*2} \text{Bi}_1) \frac{d\eta}{d\tau_s} - \frac{(\text{Bi}_1 \eta^* - 2)\eta^{*2}}{T_{\infty 2}} \frac{dT_{\infty 2}}{d\tau_s} \quad (19)$$

Equations (19) and (14) are solved using MATLAB ODE solver to obtain $\eta(\tau_s)$ and $T_{\infty 1}(\tau_s)$. The coefficients a , b , and c of the assumed quadratic profile can be computed from the solution of the differential equations. Using these coefficients, φ can be obtained from eq. (18). If, for a value of $\eta^* < 1$, φ evaluated at $\xi = 0$, $\varphi(0, \tau_{s0})$, equals $\varphi_f = (T_f - T_{\infty 2})/T_{\infty 2}$ it is then concluded that the left end of the slab has reached the fusion temperature at time $\tau_s = \tau_{s0}$. The temperature distribution in the entire thickness of slab at this instant, $\varphi(\xi, \tau_{s0})$, will form the initial condition for the phase change problem.

Note that the condition selected in this paper is limited to those problems of a subcooled solid where the preheating time (to heat one end of the slab to the fusion temperature) is less than the time it takes for the heat to diffuse to the other end to the slab.

Phase-change problem

The section *Transient heat conduction in the solid* provides the temperature distribution in the slab and the value of temperature $T_{\infty 1}$ at the very instant the slab reaches the melting temperature, caused by the initial convective heating at the left face. They form the initial condition for the phase-change problem.

Once melting starts, the slab consists of the liquid and the solid regions containing the melted and the unmelted regions, respectively, as shown in fig. 1b. As the solid melts, this interface moves forward to the right, thus increasing the thickness of the melt. The temperature $T_{\infty 1}(t)$ continuously decreases and at some point in time the interface stops, and reverses direction.

During this re-freezing process, the region from $x = 0$ to $x = s(t)$ is still occupied by the liquid (same as that during melting) and the solid occupies the region from $x = s(t)$ to $x = L$. The direction of the interface movement $s(t)$ from right to the left indicates the growth of the solid portion thus signifying the freezing process. Considering such a coordinate system and geometry, the governing differential equations, boundary conditions and Stefan condition become identical for the melting and freezing process. The phase-change problem can then be mathematically expressed as given below.

The heat conduction equation for the liquid region is:

$$\frac{\partial^2 T_1}{\partial x^2} = \frac{1}{\alpha_1} \frac{\partial T_1}{\partial t} \quad 0 < x < s(t), t > 0 \quad (20)$$

The heat conduction equation for the solid region is:

$$\frac{\partial^2 T_s}{\partial x^2} = \frac{1}{\alpha_s} \frac{\partial T_s}{\partial t} \quad s(t) < x < L, t > 0 \quad (21)$$

The boundary conditions are:

$$k_1 \frac{\partial T_1}{\partial x} \Big|_{x=0} = h_1 [T_{\infty 1}(t) - T|_{x=0}] \quad (22)$$

$$k_s \frac{\partial T_s}{\partial x} \Big|_{x=L} = h_2 (T|_{x=L} - T_{\infty 2}) \quad (23)$$

where k_1 and k_s are the thermal conductivities of the liquid and the solid, respectively.

The conditions at the melt line are given by:

$$T_1 = T_s = T_f \quad \text{at } x = s(t) \quad (24)$$

where T_f is the fusion temperature. This states that the temperature at the melt line is equal to the fusion temperature.

The Stefan condition is given by:

$$k_1 \frac{\partial T_1}{\partial x} \Big|_{x=s(t)} = k_s \frac{\partial T_s}{\partial x} \Big|_{x=s(t)} + \rho \lambda \frac{ds}{dt} \quad (25)$$

where ρ is the density, and λ is the latent heat of fusion.

Initial condition (at $t = t_0$)

$$s(t) = 0 \quad (26)$$

$$T_s(x, t_0) \quad (27)$$

which is obtained by solving the transient heat conduction without phase change to provide the temperature distribution at the instant the slab starts to melt.

The non-dimensionalization is carried out using:

$$\xi = \frac{x}{L}, \quad \eta = \frac{s(t)}{L}, \quad \tau = \frac{t\alpha_1}{L^2}, \quad \text{Bi}_1 = \frac{h_1 L}{k_1}, \quad \text{Bi}_2 = \frac{h_2 L}{k_s}, \quad \varphi_1 = \frac{T_1 - T_f}{T_f - T_{\infty 2}}, \quad \varphi_s = \frac{T_s - T_f}{T_f - T_{\infty 2}}, \quad v = \sqrt{\frac{\alpha_1}{\alpha_s}} \quad (28a-h)$$

Non-dimensionalized governing equations are therefore:

$$\frac{\partial^2 \varphi_1}{\partial \xi^2} - \frac{\partial \varphi_1}{\partial \tau} = 0 \quad 0 < \xi < \eta \quad (\text{liquid}) \quad (29)$$

$$\frac{\partial^2 \varphi_s}{\partial \xi^2} - v^2 \frac{\partial \varphi_s}{\partial \tau} = 0 \quad \eta < \xi < 1 \quad (\text{solid}) \quad (30)$$

Boundary conditions become:

$$\left. \frac{\partial \varphi_s}{\partial \xi} \right|_{\xi=0} = \text{Bi}_1 (M - \varphi_1|_{\xi=0}) \quad (31)$$

where M is given by

$$M(t) = \frac{T_{\infty,1}(\tau) - T_f}{T_f - T_{\infty,2}} \quad (32)$$

The boundary condition at the right face is:

$$\left. \frac{\partial \varphi_s}{\partial \xi} \right|_{\xi=1} = \text{Bi}_2 (1 - \varphi_s|_{\xi=1}) \quad (33)$$

The Stefan condition is obtained as:

$$z \text{St}_s \left. \frac{\partial \varphi_1}{\partial \xi} \right|_{\xi=\eta} = \frac{\text{St}_s}{v^2} \left. \frac{\partial \varphi_s}{\partial \xi} \right|_{\xi=\eta} + \frac{d\eta}{d\tau} \quad (34)$$

$$z = \frac{C_{p1}}{C_{ps}} \quad (35)$$

Also at the melt line

$$\xi = \eta; \varphi_1 = \varphi_s = 0 \quad (36)$$

Initial condition ($\tau = \tau_0$)

$$\eta = 0 \quad (37)$$

$$\varphi_s(x, \tau_0) = \varphi \quad (38)$$

obtained by solving the transient heat conduction without phase change.

The heat balance integral method is used again as in the section *Transient heat conduction in the solid*. The temperature distributions in the solid and liquid region are assumed to be quadratic. The non-dimensional temperature profiles become:

$$\varphi_1 = a_1 + b_1 \xi + c_1 \xi^2 \quad (39)$$

$$\varphi_s = a_2 + b_2 \xi + c_2 \xi^2 \quad (40)$$

If we define θ_1 and θ_2 as:

$$\theta_1 = \int_0^\eta \varphi_1 d\xi \quad (41)$$

$$\theta_2 = \int_\eta^1 \varphi_s d\xi \quad (42)$$

the heat balance integral method leads to three ordinary differential equations:

$$\theta_1 \frac{\text{Bi}_1 M \eta^2}{2(1 - \text{Bi}_1 \eta)} - \frac{\eta^2}{4} \frac{d\theta_1}{d\tau} = \frac{1}{1 - \text{Bi}_1 \eta} - \frac{1}{3} \quad (43a)$$

$$\theta_2 = \frac{Bi_2(1-\eta)^2}{2(1-Bi_2-Bi_2\eta)} - \frac{v^2}{2} \frac{d\theta_2}{d\tau} - \frac{(1-\eta)(2-Bi_2-Bi_2\eta^2)}{2(1-Bi_2-Bi_2\eta)} - \frac{(1-\eta-2\eta^2)}{3} \quad (43b)$$

$$\frac{d\eta}{d\tau} = zSt_s \frac{d\theta}{d\tau} - \frac{2-Bi_1\eta}{2(1-Bi_1\eta)} - \frac{Bi_1M}{1-Bi_1\eta} \quad (43c)$$

$$\frac{St_s}{v^2} - \frac{Bi_2}{1-Bi_2-Bi_2\eta} - v^2 \frac{d\theta_2}{d\tau} - \frac{Bi_2(1-\eta)}{2(1-Bi_2-Bi_2\eta)} = 1$$

with the initial conditions at $\tau = 0$ and $\tau = \tau_0$

$$\eta = 0 \quad (44)$$

$$\theta_1 = \int_0^\eta \varphi_1 d\xi = 0 \quad (45)$$

$$\theta_2 = \int_\eta^1 \varphi_s d\xi = \int_0^1 \varphi_s d\xi \quad (46)$$

Since the solid is initially subcooled then:

$$\theta_2 = \int_0^1 \varphi_s(\xi, \tau_0) d\xi \quad (47)$$

where $\varphi_s(\xi, \tau_0)$ is first obtained by solving the transient heat conduction problem without phase change. The section *Transient heat conduction in the solid* provides the value of $\varphi(\xi, \tau_{s0})$ which is the temperature distribution in the slab at the instant the phase-change starts. This is redefined using eq. (28g) and curve fitted to get an expression for $\varphi_s(\xi, \tau_0)$. The time τ_{s0} can be re-expressed as τ_0 using the relation

$$\tau_0 = v^{2*} \tau_{s0} \quad (48)$$

and the temperature $T_{\infty 1}(\tau)$ can be assumed to decrease in time such that:

$$\frac{dT_{\infty 1}(\tau)}{d\tau} = \text{const.} \quad (49)$$

The value of the constant here is kept exactly the same as in eq. (14), even though the definition of τ is now different. This is done so as to keep the numerical value of the decay rate the same as before.

The differential equations obtained above are coupled and the system needs to be numerically solved. The ODE solver in MATLAB is utilized to obtain the solutions to these differential equations. But before that however, because the solid is subcooled, the transient heat conduction analysis (section *Transient heat conduction in the solid*) first needs to be performed before solving the phase-change.

The next section discusses the results for this phase-change problem. This includes the study of the impact of the change in Biot number (Bi_2) at the right face of the slab.

Results and discussion

The slab of finite thickness is assumed to be a paraffin-based phase-change material with fusion temperature, T_f of 69.3 °C. The thermal conductivity of the slab, for both the solid and liquid portions, is $k = 0.173$ W/mK. The density difference between the solid and liquid phases of this material is neglected. The specific heats of the solid and liquid are not identical. The temperature difference, $T_f - T_{\infty 2}$, imposed on the right face of the slab has a constant value of

49.3 °C. However, the difference in temperature at the left end, $T_{\infty 1}(t)$, continuously decreases with time at a constant assumed rate of -4.5 °C per unit time τ . The initial temperature difference, $T_{\infty 1} - T_f$ at $(t=0)$ is taken as 10.7 °C. The value for the Stefan number of the solid is $St_s = 0.467$. A high value of heat transfer coefficient, h_1 (a constant value of $Bi_1=15$) is assumed to exist on the left end of the slab. The effect of varying Bi_2 ($=1, 1.5,$ and 2) is investigated.

Pre-heating of the slab to the fusion temperature

Transient heat conduction analysis as per the section *Transient heat conduction in the solid* is performed until the left face reaches the fusion temperature. The PCM slab is initially at a temperature equal to $T(\xi, 0) = T_{\infty 2}$. The value of $Bi_1=15$ and the initial temperature difference $T_{\infty 1}(0) - T_f$ of 10.7 °C causes convective heating of the slab with decreasing magnitude as the value of $T_{\infty 1}$ drops down. Melting begins when the value of ϕ evaluated at the left face of the slab – eq. (7a), becomes 2.465.

The solutions for conduction without phase change are obtained by solving the ordinary differential eqs. (14) and (19) for the first stage ($\eta^* = 1$). The temperature distribution in the slab at the very instant the left face reaches the melting temperature is shown in fig. 2. It can be seen from fig. 2 that the heat diffuses to only about 62% of the slab thickness ($\eta^* = 0.6$) during the time the left face attains the fusion temperature.

Figure 3 depicts the depth of heat diffusion (growth of η^*) into the slab in terms of τ_s . It is seen from this figure that it takes $\tau_s = 0.041$ for the fusion temperature to be attained at the left end. This non-dimensional time can be re-written using eq. (48) to be consistent with the definition of τ for the phase-change analysis and it is seen that it takes $\tau = 0.035$ for the slab to reach the fusion temperature in fig. 3.

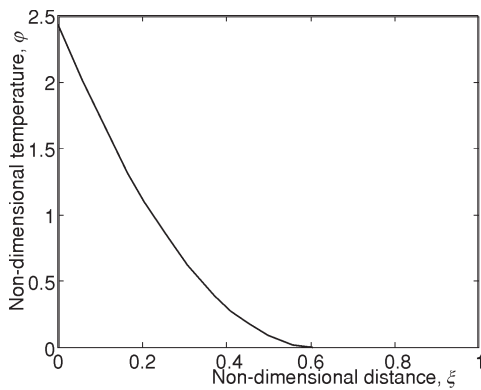


Figure 2. Temperature distribution in the slab when left end reaches T_f ($\tau_s = 0.041$)

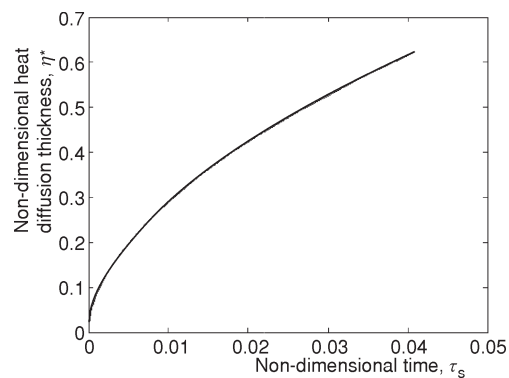


Figure 3. Heat diffusion thickness vs. time

Thus, the diffusion time to the other end of the slab is much more than the pre-heating time it takes for the left end to reach the fusion temperature. Hence, the boundary condition on the other end (the effect of Bi_2) is not felt. So, the same exact temperature profile, as shown by fig. 3, exists for all the cases of $Bi_2 = 1, 1.5,$ and 2 being investigated. This profile is re-constructed using the definition of ϕ as given in eq. (28g) and curved-fit to obtain a mathematical expression in terms of ξ . The temperature $T_{\infty 1}(\tau)$ decreases linearly with time τ_s and drops to 79.8 °C at the start of the phase-change. This particular value of $T_{\infty 1}$, along with the temperature profile obtained from fig. 2 forms the initial condition for the phase-change problem.

Phase-change problem: $Bi_2 = 1.0$

The amount of heat loss from the right face is studied by maintaining the Bi_2 value at 1. The phase-change problem is solved using the heat balance integral method explained in phase-change problem. The slab loses heat (*i. e.* convective cooling) from the right face with $Bi_2 = 1$. The heat transfer coefficient on the left face is maintained at the same value as before, $Bi_1 = 15$. The Stefan number for the solid is 0.467. Using the initial conditions obtained from the earlier conduction solution above, the three ordinary differential equations – eq. (43 a, b and c), are solved. Figure 4 and 4b depicts the rate of decay of temperature $T_{\infty,1}$ and the interface movement with time.

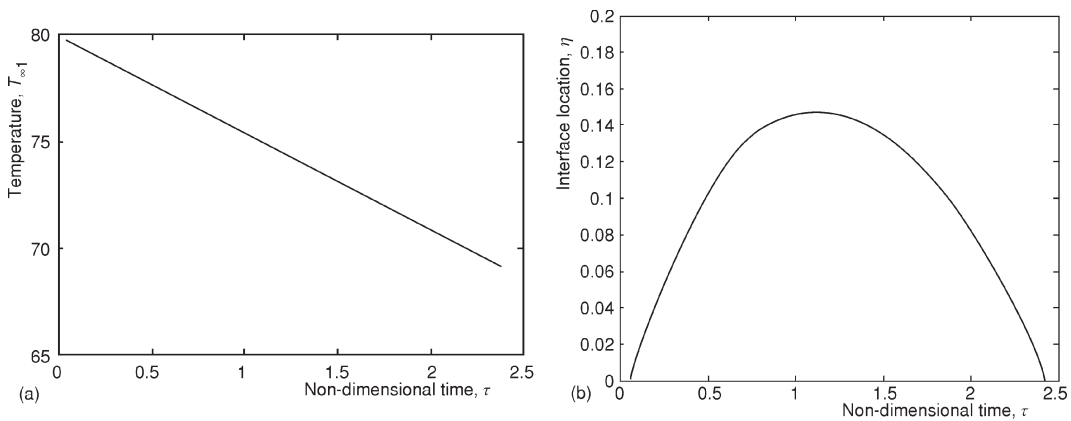


Figure 4. (a) Rate of decay of $T_{\infty,1}$; (b) Non-dimensional interface location vs. time ($Bi_2 = 1.0$)

The slab begins to melt on account of the higher convective heat gain from the left end and as time progresses, $T_{\infty,1}(t)$ decreases, thus causing less heating at the left face. However, simultaneously, the slab is continually losing heat from the right face. After some point in time, the rate of heat loss exceeds the rate of heat gain and the slab begins to freeze back. It is evident, as shown by fig. 4, that some melting takes place ($\eta = 0.147$) before the slab completely freezes back at $\tau = 2.42$. It is seen in fig. 5 that the slab starts melting at time $\tau = 0.035$. However, the temperature $T_{\infty,1}(t)$ drops a little below the fusion temperature at around $\tau = 2.38$, before the slab freezes back completely. Thus, ideally, a second phase-change interface should start developing from $\xi = 0$ (and proceed to the right), in addition to the one already existing traveling left. At the instant, $T_{\infty,1}(t)$ falls below the fusion temperature, the interface, for this case, has already retracted back, more specifically to the point that 99% of the slab is already solid (freezing is almost complete). Figure 5 shows the temperature profiles for the solid and liquid portions of the slab.

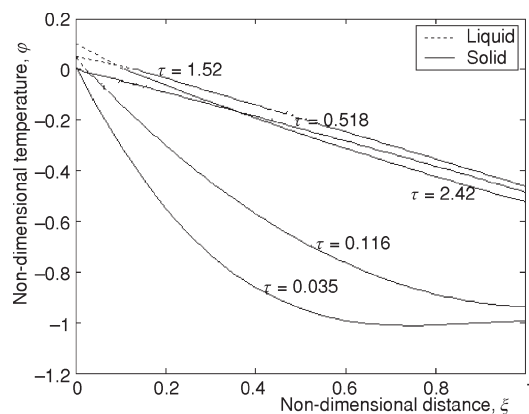


Figure 5. Non-dimensional temperature profiles in the solid and liquid regions for $Bi_2 = 1.0$

Phase-change problem: $Bi_2 = 2.0$

Next, the effect of much higher value of $Bi_2 (= 2.0)$ is considered. Here too, the interface starts moving from the left to the right as the slab melts. As time advances, the interface stops, and then recedes so that the slab continues to freeze (interface moving left) until it becomes all solid. Figure 6 shows that the interface starts advancing forward from $\tau = 0.035$. This indicates melting from the left-end. At time $\tau = 0.9$, the interface movement reverses. The slab melts only to the extent of $\eta = 0.119$ because of a high value of $Bi_2 = 2$ as seen in fig. 6. With further increase of time, $T_{\infty 1}(t)$ decreases and the slab becomes a complete solid again at $\tau = 2.16$. Note that the temperature $T_{\infty 1}(t)$, at this time instant, is 70.3°C , which is still above the fusion temperature of the slab. On account of a very high value of Bi_1 , it is seen that the rate of melting is faster relative to the rate of freezing.

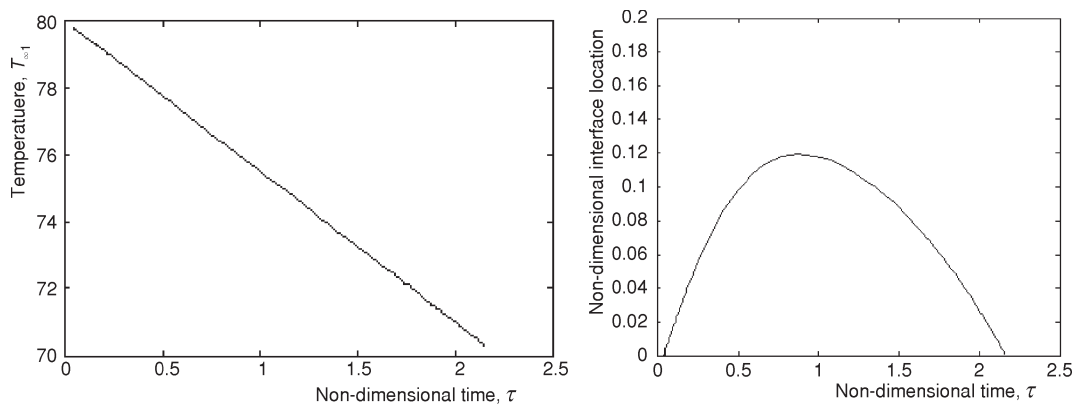


Figure 6. (a) Rate of decay of $T_{\infty 1}$; (b) Non-dimensional interface location vs. time ($Bi_2 = 2.0$)

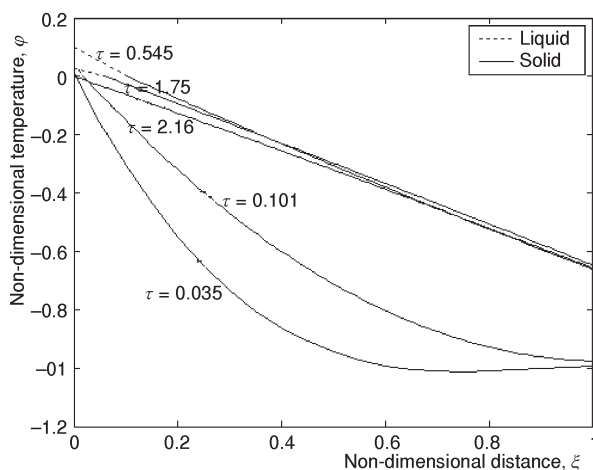


Figure 7. Non-dimensional temperature profiles in the solid and liquid regions for $Bi_2 = 2.0$

Select temperature profiles in the solid and the liquid regions are depicted in fig. 7. Initially, to start with, the slab is all solid at $\tau = 0.035$. With the increase in time the melt front advances forward ($\tau = 0.101, 0.545$). Thus, the slab then consists of both solid (solid line) and liquid (dotted line). Initially, the right face of the slab is very low in temperature at about $T_{\infty 2}$. With passage of time, it gets heated on account of heat diffusion from the left end and temperature peaks but remains solid. With more convective loss now, it can be seen that the temperature of the right face drops down slightly. Finally, observe that the freezing process is completed at $\tau = 2.16$ and the temperature profile in the solid is linear.

Phase-change problem: $Bi_2 = 1.5$

Next, the slab is analyzed when there is less convective heat loss from the right face, $Bi_2 = 1.5$ as compared to the previous case. The heat transfer coefficient on the left face as well as all the other properties are maintained the same as before for the case of $Bi_2 = 2$. Recall, the initial condition for this phase-change problem is the same as that for the previous case. This is because when the left end reaches the fusion temperature, the temperature on the other end is unchanged and the boundary condition on the right hand side is inconsequential. It is seen from fig. 8 that the slab melts a bit more (compared to the previous case), to the extent of about 12.9%. This is because there is a less heat leakage from the right face ($Bi_2 = 1.5$). As shown in fig. 8, as the time increases to about $\tau = 0.98$, the slab starts freezing back even though the temperature $T_{\infty,1} = 75.6^\circ\text{C}$, at this instant of time, is above the fusion temperature similar to the previous case. This is because the rate of heat leakage out of the slab is much higher than the rate of heat entering from the left face. The temperature profiles in the solid and liquid phases are depicted in fig. 9 and are quite similar to those obtained in fig. 7. The final steady linear temperature profile occurs in the complete solid at $\tau = 2.25$.

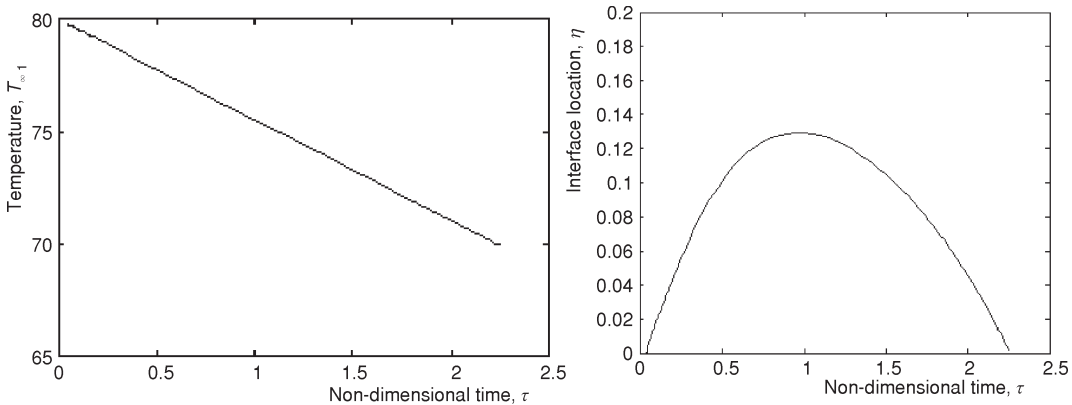


Figure 8. (a) Rate of decay of $T_{\infty,1}$; (b) Non-dimensional interface location vs. time ($Bi_2 = 1.5$)

Comparison of the phase-change problem ($Bi_2 = 1.5$) with that having constant convective boundary conditions

As mentioned above, the slab melts to an extent of about 12.9% in the case of time-dependent convective boundary condition imposed on the left end and with $Bi_2 = 1.5$ before freezing back. It is mentioned again that the initial temperature difference, $T_{\infty,1} - T_f$ at ($t = 0$) is taken as 10.7°C . A high value of heat transfer coefficient, h_1 (a constant value of $Bi_1 = 15$) is assumed to exist on the left end of the slab. A similar problem, but, with constant $T_{\infty,1}$ has been solved earlier by Roday *et al.* [12] using the heat

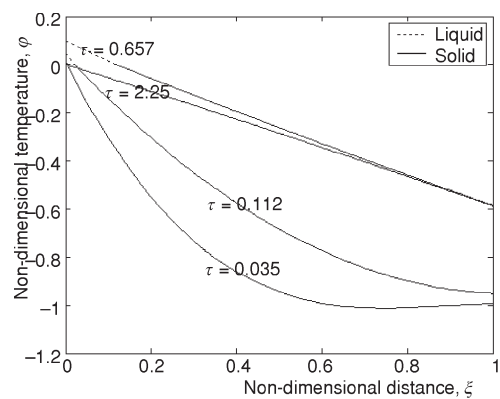


Figure 9. Non-dimensional temperature profiles in the solid and liquid regions for $Bi_2 = 1.5$

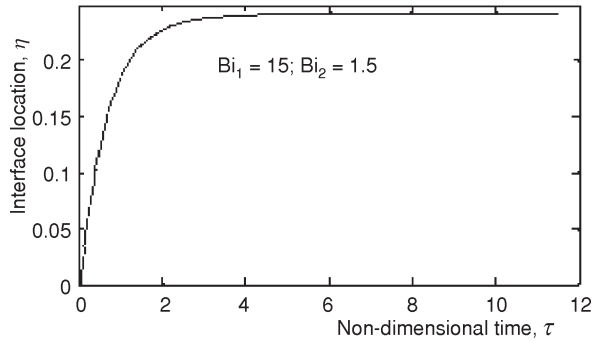


Figure 10. Melting with solid initially subcooled; $Bi_1 = 15$, $Bi_2 = 1.5$ [12]

constant $T_{\infty 1}$ and shown in fig. 10. However, with a decaying rate of $T_{\infty 1}$, melting (up to 12.9%) freezing back for the slab is seen as depicted in fig. 8b.

Conclusions

The heat balance integral approach is an effective tool for solving phase-change problems with convective boundary conditions. With the availability of numerical solvers, the ordinary differential equations, how-so-ever complicated become more manageable to solve. This makes the heat-balance integral approach all the more useful in solving phase-change problems with complex boundary conditions. In this study, this approach has been used to solve the phase change problem (melting and freezing in succession) for a finite slab subjected to a time-dependent convective boundary condition on one side and a constant convective boundary condition on the other side. The slab is initially subcooled. The effect of varying the Biot number on the right face of the slab has been investigated. The heat-balance integral method was also used to determine the temperature distribution in the slab before the phase-change starts. Phase-change analysis revealed that the solid-liquid interface advances forward (melting), stops and then retracts (freezing) when the rate of heat leaking out exceeds the rate of heat input, and finally the slab becomes a complete solid again. Temperature distributions in the solid and liquid phases (during the melting as well as the freezing part) of the phase-change process were also reported for all cases investigated. It was found that the maximum amount of melting that occurred before the slab re-freezes back decreases with an increase in the heat loss (increase in Bi_2) from the right face of the slab. The results are also compared for the case of a constant convective boundary condition.

Nomenclature

Bi	– Biot number ($= hL/k$), [–]	t	– time, [s]
C_p	– specific heat at constant pressure, [$Jkg^{-1}K^{-1}$]	z	– ratio of specific heat of liquid to solid
h	– convective heat transfer coefficient, [$Wm^{-2}K^{-1}$]	<i>Greek letters</i>	
k	– thermal conductivity, [$Wm^{-1}K^{-1}$]	α	– thermal diffusivity, [m^2s^{-1}]
L	– length of slab, [m]	δ	– heat diffusion thickness, [m]
s	– interface location, [m]	η	– non-dimensional interface location
St	– Stefan number [$= C_p(T_f - T_{\infty})/\lambda$], [–]	η^*	– non-dimensional heat diffusion thickness
T	– temperature, [$^{\circ}C$]	θ	– integral
ΔT	– temperature difference, [$^{\circ}C$]		

balance integral method. In that study [12], the temperature differences imposed on the two ends of the slab were

$T_1 = T_{\infty 1} - T_f$ and $T_2 = T_f - T_{\infty 2}$ with values of 10.7 and 49.3, respectively. Choosing the same values of Biot numbers ($Bi_1 = 15$ and $Bi_2 = 1.5$) and the same initial condition, the problem of this study becomes very similar to the previous study [12] except that $T_{\infty 1}$ decays with time. It is interesting to point out that, with similar imposed conditions, the slab underwent partially melting (about 25%) in the case of constant

λ	– latent heat, [Jkg ⁻¹]	2	– right end of the slab
ν	– thermal diffusivity ratio [= (α_l/α_s) ^{1/2}]	∞	– ambient fluid
ξ	– non-dimensional distance	l	– liquid
τ	– non-dimensional time (α/L^2)	s	– solid
φ	– non-dimensional temperature	o	– start of melting/freezing
		f	– fusion temperature (melting or freezing process)
Subscripts			
l	– left end of the slab		

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