HEAT-BALANCE INTEGRAL METHOD FOR HEAT TRANSFER IN SUPERFLUID HELIUM

by

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The heat-balance integral method is used to solve the non-linear heat diffusion equation in static turbulent superfluid helium (He II). Although this is an approximate method, it has proven that it gives solutions with fairly good accuracy in non-linear fluid dynamics and heat transfer. Using this method, it has been possible to develop predictive solutions that reproduce analytical solution and experimental data. We present the solutions of the clamped heat flux case and the clamped temperature case in a semi-infinite using independent variable transformation to take account of temperature dependency of the thermophysical properties. Good accuracy is obtained using the Kirchhoff transform whereas the method fails with the Goodman transform for larger temperature range.

Keyword: *heat-balance integral method, superfluid helium, heat transfer*

Introduction

Analytical treatment of transient heat transfer in He II has received not enough attention, considering the substantial interest as it relates to the cooling and stability of magnet systems design. Analytical treatment is useful for providing a physical description of the phenomenon and scaling laws for engineering to design cooling system of cryogenics device cooled by superfluid helium. Only, Dresner, using similarity solutions method, has developed analytical solutions for the clamped temperature and heat flux cases and the pulsed-source problem [1, 2]. These cases deal with linear boundary conditions and temperature independent properties in a semi-infinite media. These solutions are not predictive since an “adequate” temperature has to be chosen to set the thermal properties in order to fit the experimental data [1, 2]. For the clamped heat flux problem, the Dresner’s solution has even a free parameter that has to be adjusted to fit experimental data [1]. The solutions for the clamped heat flux case [1] and for the pulsed source problem [2] reproduce with a high accuracy experimental results and give good physical description of transient heat transfer in superfluid helium.

An adequate method in the solution of non-linear heat diffusion problems is the heat-balance integral method (HBIM), developed by Goodman [3], because of its capability of solving non-linear problems where the non-linearity can be found either in the differential equation itself or in the boundary conditions. With exact method, the resulting solution satisfies locally the system of equations over the entire range of space and time. Such solutions are rather difficult to obtain when the differential equation is non-linear or if the boundary conditions involved are non-linear. The HBIM in the solution of time-dependent boundary-value problems gives solutions, which satisfy the differential system only on the average over the region considered rather than considering a local solution. It is often sufficient for engineering calculations in which many more approximations are used to model complex systems.

Heat transfer in superfluid helium is non-linear since the Fourier law is replaced by a non-linear law between the heat flux and the temperature gradient as it will be developed in the next paragraph [4]. The heat diffusion equation constructed is non-linear and is a perfect candidate to be solved by the HBIM. More over, the thermal properties of superfluid helium are strongly dependent on temperature, especially the equivalent thermal conductivity [5]. To take account of these two characteristics, Baudouy used the Goodman's method to tackle the non-linear heat diffusion equation with temperature dependent thermal properties in superfluid helium [6, 7]. The development is based on a Kirchhoff or Goodman transform where the HBIM method is solved for a new variable which is the integral of the main thermal properties with respect to temperature. This development takes in account both the non-linearity of the heat diffusion equation of He II and the temperature dependency of the thermal properties. Nevertheless, some approximations have been done, mainly to simplify the development to produce simple solution forms to be used by engineers for the design of system cooled by superfluid helium. Even if these approximations lead to some inaccuracy in the final results, the main characteristics of the transient heat transfer in He II were obtained such as the evolution of the temperature at the cooling surface subjected to a heat flux or the time to reach the critical temperature of phase-change between superfluid helium and normal helium [6]. Experimental results for the pulsed source problem is fitted with good accuracy with no adjusting parameter [7].

We proposed in this paper to cover in details the treatment of the heat transfer in superfluid helium by the HBIM and to present the clamped temperature and heat flux cases with both transforms (Kirchhoff's and Goodman's) with a slight improvement.

Heat transfer in superfluid helium

According to the theory of Landau [8], below a temperature named T_λ ($T_\lambda = 2.172$ K), helium undergoes a phase transition from normal helium (He I) to superfluid helium (He II). It is viewed as a mixture of a normal component, the normal fluid, having a density ρ_n and a velocity field v_n and a superfluid component, the superfluid, having a density ρ_s and a velocity field v_s. The superfluid component is associated with the energy ground state and the normal fluid component is associated with energy excitations such as phonons and rotons. The density of the entire fluid is defined from the densities of normal fluid and superfluid as:

$$\rho = \rho_\mathrm{s} + \rho_n \tag{1}$$

For a given temperature, there is a single ratio ρ_s/ρ and this ratio tends to unity when the temperature approaches the absolute zero and goes to zero when to temperature reaches the transition temperature T_λ. This temperature separates the two liquid phases that exist for helium, the normal helium (He I) above T_λ and the superfluid helium (He II) below T_λ.

In this two-fluid model, the normal fluid is considered as a classical fluid in the Newtonian point of view, and therefore it caries entropy and is associated to a viscosity, μ. Given that the superfluid component is associated with the energy ground state, the motion of superfluid is basically different from any classical fluid since its viscosity is null. The superfluid component does not carry entropy. One can thus connect the flow of entropy to the velocity field of the normal fluid by $\vec{\mathrm{s}} = \rho s \vec{\mathrm{v}}_n$, and the heat flux is expressed as:

$$\vec{\mathrm{q}} = \rho s T \vec{\mathrm{v}}_n \tag{2}$$

where the thermal conduction in the liquid is neglected. In most practical cases, the classical thermal conduction is negligible compared to the specific heat transfer in He II.

The momentum of He II is simply written:

$$\rho\vec{v} \quad \rho_s\vec{v}_s \quad \rho_n\vec{v}_n \tag{3}$$

where $\vec{v}$ is the barycentric velocity of the bulk helium flow. The heat transport process in superfluid helium implies that any difference in temperature or heat flux leads to a movement of normal fluid in the opposite direction of the superfluid. This internal convection is the origin of the remarkable properties of the transport of heat in He II. For large heat flux, what is called the "turbulent regime of superfluid helium", a force appears between the two components, called the mutual friction force. It is the physical description for the creation of quantized vortex, which can be viewed as the circulation of the superfluid component around a core of normal helium and it is correlated to the relative velocity of the two components. This force has been expressed by Gorter *et al.* [4] as:

$$\vec{F}_{ns} \quad A\rho_s\rho_n|\vec{v}_n \quad \vec{v}_s|^2(\vec{v}_n \quad \vec{v}_s) \tag{4}$$

This force dominates the equations of motion of the two components, which are written as:

$$\rho_s\frac{\partial\vec{v}_s}{\partial t} \quad \rho_s\vec{v}_s \quad (\vec{v}_s) \quad \rho_s s \quad T \quad \frac{\rho_s}{\rho} \quad p$$
$$\frac{\rho_s\rho_n}{2\rho} \quad |\vec{v}_n \quad \vec{v}_s|^2 \quad A\rho_n\rho_s|\vec{v}_n \quad \vec{v}_s|^2(\vec{v}_n \quad \vec{v}_s) \tag{5}$$

and

$$\rho_n\frac{\partial\vec{v}_n}{\partial t} \quad \rho_n\vec{v}_n \quad (\vec{v}_n) \quad \rho_s s \quad T \quad \frac{\rho_n}{\rho} \quad p \quad \mu\Delta\vec{v}_n$$
$$\frac{\rho_n\rho_s}{2\rho} \quad |\vec{v}_n \quad \vec{v}_s|^2 \quad A\rho_s\rho_n|\vec{v}_n \quad \vec{v}_s|^2(\vec{v}_n \quad \vec{v}_s) \tag{6}$$

In steady-state regime and for static helium, *i. e.* $\rho\vec{v} \quad \vec{0}$, by combining eqs. (5) and (6) and using eqs. (1), (2), and (3), we obtain the expression of the temperature gradient:

$$T \quad \frac{\eta}{\rho_s s}\Delta\vec{v}_n \quad \frac{A\rho_n}{s}|\vec{v}_n \quad \vec{v}_s|^2(\vec{v}_n \quad \vec{v}_s) \tag{7}$$

In most of practical cases, *i. e.* for large heat flux, the first term is negligible compared to the second term [9]. In that case, the temperature gradient can be expressed in eq. (7) in term of the heat flux using eqs. (1), (2), and (3):

$$\frac{\rho_s^3 s^4 T^3}{A\rho_n} \quad T \quad f(T) \quad T \quad q^3 \tag{8}$$

where $f(T)$ is thermal conductivity function in the turbulent regime of He II. So according to eq. (8), the heat flux is proportional to the cube root of the temperature gradient, which is usually named the "Gorter-Mellink law". In steady-state regime, this law has been compared successfully to experimental data numerous time (see reference [9] for example) but to solve eq. (8), integration of $f(T)$ is necessary since it varies largely with temperature as the fig. 1 shows. He II has a thermal conductivity function about two orders of magnitude larger than for high-purity metals at superfluid helium temperatures. To illustrate the high heat transfer rate in He II, the

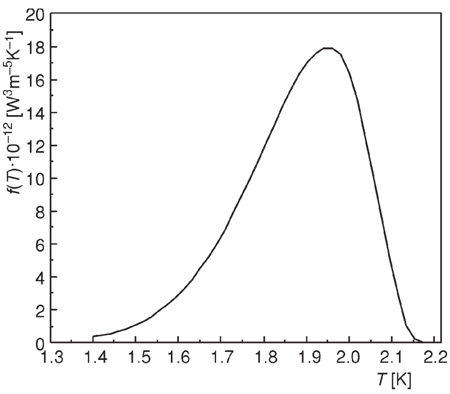

Figure 1. Thermal conductivity function of He II in saturated superfluid helium [10]

calculation of the equivalent thermal conductivity is 100 kW/mK at 1.8 K and for a heat flux density of 10 kW/m^2.

In transient heat transfer, several energy inputs have to be considered to construct a model. Examining eqs. (5) and (6), kinetic energy is the first to come to mind, associated with the acceleration of the component's velocity but it is rather small. The second is the energy to create the superfluid turbulence but for large space, in the order of 1 m in length, this energy is small too. In fact, the principal energy dominating transient heat transfer in He II is related the enthalpy of the helium [9] and this transient phenomenon is controlled by heat transport and enthalpy variation, therefore eq. (8) is used in an energy conservation equation and yields to a non-linear heat diffusion equation:

$$\rho C_{\mathrm{p}} \frac{\partial T}{\partial t} \qquad q \qquad \sqrt[3]{f(T) \quad T} \tag{9}$$

Obviously, such partial differential equation is difficult to solve because of the large variation of the thermal conductivity function with temperature and its non linearity.

Heat transfer in a semi-infinite media with temperature dependent properties using a Kirchhoff transformation

For the turbulent regime of He II, the heat flux is given by the Gorter-Mellink law (8), neglecting the dissipation effects in He II, the partial differential equation modeling our system for one space dimension is:

$$\rho C_{\mathrm{p}} \frac{\partial T}{\partial t} \quad \frac{\partial}{\partial x} \sqrt[3]{f(T) \frac{\partial T}{\partial x}} \ \text{in} \ 0 \quad x \quad \infty \ \text{and for} \ t \quad 0 \tag{10}$$

where ρ is the density, C_{p} – the specific heat at constant pressure, and $f(T)$ – the He II thermal conductivity function.

For a prescribed temperature the boundary condition is:

$$T = T_0 \ \text{at} \ x = 0 \ \text{and for} \ t > 0 \tag{11}$$

and for the clamped heat flux case, the boundary condition is written as:

$$\sqrt[3]{f \frac{\partial T}{\partial x}} \quad q_0 \ \text{at} \ x \quad 0 \ \text{and for} \ t \quad 0 \tag{12}$$

where q_0 is the heat flux at $x = 0$.

At the initial time, the entire media is at constant temperature T_{b}, so the initial condition is:

$$T = T_{\mathrm{b}} \ \text{in} \ 0 \quad x \quad \infty \ \text{and at} \ t = 0 \tag{13}$$

As it is a semi-infinite media, the necessary second boundary condition is a constant temperature when $x \quad \infty$ or practically for large x, *i. e.* the temperature field is not disturbed for large x. This conditions is expressed by:

$$T = T_b \text{ for } x \quad \infty \text{ and for } t > 0 \tag{14}$$

In the HBIM, we assume that the solution of the disturbed temperature field is limited by a distance $\delta(t)$, called the thermal layer, after which the temperature field is not disturbed, *i. e.* $T = T_b$ for $x \quad \delta(t)$. For a semi-infinite media, the thermal layer is defined as being always inferior to the length of the system. From this definition we can modify the boundary condition (14) as:

$$T = T_b \text{ at } x = \delta(t) \text{ and for } t > 0 \tag{15}$$

To take account of the temperature dependence of the thermal conductivity function of He II, we use a Kirchhoff transformation:

$$\theta \quad \int_{T_b}^{T} f(T)\,\mathrm{d}T \tag{16}$$

and the eqs. (10)-(13) and are transformed into:

$$\begin{aligned}
&\frac{1}{\alpha}\frac{\partial\theta}{\partial t} \quad \frac{\partial}{\partial x}\sqrt[3]{\frac{\partial\theta}{\partial x}} \quad \text{in } 0 \quad x \quad \infty \text{ and for } t \quad 0 && \text{(a)}\\
&\theta \quad \theta_0 \text{ at } x \quad 0 \text{ and for } t \quad 0 && \text{(b-1)}\\
&\text{or} \quad \sqrt[3]{\frac{\partial\theta}{\partial x}} \quad q_0 \text{ at } x \quad 0 \text{ and for } t \quad 0 && \text{(b-2)}\\
&\theta \quad 0 \text{ at } x \quad \delta(t) \text{ and for } t \quad 0 && \text{(c)}\\
&\theta = 0 \text{ in } 0 \quad x \quad \infty \text{ at } t \quad 0 && \text{(d)}
\end{aligned} \tag{17}$$

where $\alpha = f(T)/\rho C_p$.

If (17a) is integrated with respect to space over the thermal layer the resulting equation is called the heat balance integral equation. Noticing that in our system $\theta/\partial x|_\delta$ is null because of the definition of the thermal boundary δ since $T/\partial x|_\delta$ is null, the energy equation is then transformed into:

$$\int_0^\delta \frac{\partial\theta}{\partial t}\mathrm{d}x \quad \int_0^\delta \alpha\frac{\partial}{\partial x}\sqrt[3]{\frac{\partial\theta}{\partial x}}\mathrm{d}x \quad \bar{\alpha}\sqrt[3]{\left.\frac{\partial\theta}{\partial x}\right|_0} \tag{18}$$

where we consider α constant. In previous work, we use the value α at $x = 0$ for simplification. But the thermal properties are far for being independent of temperature however to simplify the analytical treatment of the HBIM, we decide to use the arithmetic average value of α named $\bar{\alpha}$ over the temperature range to take account of its variation.

With the rule of differentiation, the integral on the left hand-side of eq. (18) is transformed into:

$$\frac{d}{\mathrm{d}t}\int_0^\delta \theta\,\mathrm{d}x \quad \theta\frac{\mathrm{d}x}{xt}\Big|_0^\delta \quad \bar{\alpha}\sqrt[3]{\left.\frac{\partial\theta}{\partial x}\right|_0} \tag{19}$$

One can notice that, due to the boundary conditions, the second term of the left hand-side of eq. (19) is null which reduces it to a simpler formulation:

$$\frac{d}{\mathrm{d}t}\int_0^\delta \theta\,\mathrm{d}x \quad \bar{\alpha}\sqrt[3]{\left.\frac{\partial\theta}{\partial x}\right|_0} \tag{20}$$

Equation (20) is the heat-balance integral equation for our problem. The second term of the right hand-side of eq. (20) will be evaluated with the knowledge of the solution θ.

Let assume that θ has a polynomial form as $\theta = a_0(t) + a_1(t)x + a_2(t)x^2 + a_3(t)x^3$ where the coefficients $a_i(t)$ are function of time and therefore of the thermal layer $\delta(t)$. Obviously, θ is an approximate solution of the system and to find the different coefficients, we need to use different boundary conditions: the natural conditions, which ensue from the problem, and derived conditions, which are constructed from either the differential equation or the natural boundary conditions. For this expression of the solution, we need two extra boundary conditions. The first one we choose is straightforward and comes from the definition of the thermal layer:

$$\frac{\partial \theta}{\partial x} = 0 \text{ at } x = \delta \text{ for } t \geq 0 \tag{21}$$

One can notice that this condition has been already used to construct the heat balance integral equation. The second one comes from the differential equation at $x = \delta$ where the derivative of the temperature with respect to space is null because of condition (17c). We have, what it is called a derived condition:

$$\frac{\partial^2 \theta}{\partial x^2} = 0 \text{ at } x = \delta \text{ for } t \geq 0 \tag{22}$$

By the use of the natural boundary conditions (17b's), (17c), and (21), and the derived one (22), we can formulate a solution for θ as a function of δ:

$$\theta = \theta_0 \left(1 - \frac{x}{\delta}\right)^3 \tag{23}$$

where $\theta_0 = \int_{T_b}^{T_0} f(T)\mathrm{d}T$ for the clamped temperature case and $\theta_0 = q_0^3\delta/3$ for the clamped heat flux case.

For the prescribed temperature case, if we substitute eq. (23) in the heat integral eq. (20), a first order differential equation for thermal layer is obtained:

$$\sqrt[3]{\delta}\,\frac{\mathrm{d}\delta}{\mathrm{d}t} = 4\overline{\alpha}\sqrt[3]{\frac{3}{\theta_0^2}} \tag{24}$$

and the solution of eq. (24) with the initial condition (17b-1) is:

$$\delta = \frac{8}{\sqrt{3}}\frac{\sqrt[4]{\overline{\alpha}}}{\sqrt{\theta_0}}\sqrt[4]{t} \tag{25}$$

The solution of our problem is then composed of the eqs. (23) and (25) for the prescribed temperature problem where $\theta_0 = \int_{T_b}^{T_0} f(T)\mathrm{d}T$.

For the clamped heat flux case, the heat balance integral eq. (20) is transformed with the boundary condition (17b-2):

$$\frac{d}{\mathrm{d}t}\int_0^\delta \theta\,\mathrm{d}x = q_0\overline{\alpha} \tag{26}$$

Similarly, the first order ordinary differential equation for the thermal layer is:

$$\frac{d}{\mathrm{d}t}(\delta^2) = \frac{12}{q_0^2}\overline{\alpha} \tag{27}$$

The solution of eq. (27) subjected to the initial condition (17b-2) and excluding the negative solution is:

$$\delta = \frac{2\sqrt{3}}{q_0}\sqrt{\overline{\alpha}\,t} \tag{28}$$

For the clamped heat flux problem, the solution is given by eqs. (23) and (28). Since the boundary surface temperature θ_0 is not yet known, eq. (28) cannot be directly used to evaluate δ but we can eliminate the thermal layer in the expression of θ_0 and have a transcendental equation for θ_0 since $\overline{\alpha}$ is also a function of θ_0:

$$\frac{\theta_0}{\sqrt{\overline{\alpha}}} \quad \frac{2}{\sqrt{3}} q_0^2 \sqrt{t} \tag{29}$$

The evolution of the thermal properties of superfluid helium with temperature used to solve eq. (29) with a simple routine are taken from the specific data base for helium [10].

Heat transfer in a semi-infinite media with temperature dependent properties using the Goodman transform

To take account of the temperature dependency of the thermal properties of the helium in the transient heat balance eq. (10), another transformation can be used, the Goodman transformation [3]:

$$\Theta \quad {}_{T_b}^{T} \rho C_{\mathrm{p}}(T)\mathrm{d}T \tag{30}$$

The system of eqs. (10)-(13) and (15) are transformed into:

$$\begin{array}{ll} \dfrac{\partial \Theta}{\partial t} \quad \dfrac{\partial}{\partial x}\sqrt[3]{\alpha \dfrac{\partial \Theta}{\partial x}} \ \text{in } 0 \quad x \quad \infty \text{ and for } t \quad 0 & \text{(a)} \\ \Theta = \Theta_0 \text{ at } x \quad 0 \text{ and for } t \quad 0 & \text{(b-1)} \\ \sqrt[3]{\alpha \dfrac{\partial \Theta}{\partial x}} \quad q_o \text{ at } x \quad 0 \text{ and for } t \quad 0 & \text{(b-2)} \\ \Theta = 0 \text{ at } x = \delta(t) \text{ and for } t \quad 0 & \text{(c)} \\ \Theta \quad 0 \text{ in } 0 \quad x \quad \infty \text{ at } t \quad 0 & \text{(d)} \end{array} \tag{31}$$

where α is also $f(T)/\rho C_{\mathrm{p}}$.

Following the same procedure than in the previous paragraph, the heat balance integral equation is written as:

$$\frac{d}{\mathrm{d}t} {}_0^{\delta} \Theta \mathrm{d}x \quad \sqrt[3]{\alpha_0 \frac{\partial \Theta}{\partial x}\bigg|_0} \tag{32}$$

As before, we use the temperature profile $\Theta = b_0(t) + b_1(t)x + b_2(t)x^2 + b_3(t)x^3$ and two additional boundary conditions identical to the ones used earlier:

$$\frac{\partial \Theta}{\partial x} \quad 0 \text{ at } x \quad \delta \text{ for } t \quad 0 \tag{33}$$

and

$$\frac{\partial^2 \Theta}{\partial x^2} \quad 0 \text{ at } x \quad \delta \text{ for } t \quad 0 \tag{34}$$

Using the boundary condition (31b's), (31c), (33), and (34) the solution for Θ is given by:

$$\Theta \quad \Theta_0 \ 1 \quad \frac{x}{\delta}^{3} \tag{35}$$

where for the prescribed temperature case $\Theta_0 \quad {}_{T_b}^{T_0} \rho C_{\mathrm{p}}(T)\mathrm{d}T$ and for the clamped heat flux case $\Theta_0 \quad (q_0^3/3\alpha_0)\delta$. Similarly to the previous transform, if we introduce eq. (35) into the heat balance eq. (32), then the thermal layer for the prescribed temperature case is:

$$\delta \quad \frac{8}{\sqrt{3}}\frac{\sqrt[4]{\alpha_0}}{\sqrt{\Theta_0}}\sqrt[4]{t^3} \tag{36}$$

and for the prescribed heat flux:

$$\delta \quad \frac{2\sqrt{3}}{q_0}\sqrt{\alpha_0 t} \tag{37}$$

For the prescribed temperature problem, the solution of our problem is then composed of the eqs. (35) and (36) where $\Theta_0 \quad {}_{T_b}^{T_0}\rho C_p \mathrm{d}T$.

For the clamped heat flux problem, the solution is given by eqs. (35) and (37). As before, the boundary surface temperature Θ_0 is not yet known, therefore eq. (37) cannot be directly used to evaluate δ but we can eliminate the thermal layer in the expression of Θ_0 and have a transcendental equation:

$$\Theta_0\sqrt{\alpha_0} \quad \frac{2}{\sqrt{3}}q_0^2\sqrt{t} \tag{38}$$

Finally, one have to note that the use of the Goodman transform does not lead to any approximation on the contrary of the Kirchhoff transform. But the variation of the thermal property, taking in account with this method, is related to the enthalpy variation whereas the Kirchhoff transform takes account of the thermal properties related to the heat transfer. One expects to have some difference in the comparison with analytical solutions or experimental data since the thermal conductivity function of He II varies by several order of magnitude in the practical temperature range of He II (1.6 K; 2.172 K).

Comparison with existing solution and experimental data

Clamped temperature case

The HBIM solutions, given by eqs. (23) and (25) for the Kirchhoff transform and by the eqs. (35) and (36) for the Goodman transform are compared with the exact solution – eq. (39), developed by Dresner in fig. 2 for the temperature range [1.8; 2.0 K]:

$$\frac{T \quad T_b}{T_0 \quad T_b} \quad 1 \quad \frac{z}{\sqrt{\frac{8}{3\sqrt{3}} \quad z^2}} \quad \text{with} \quad z \quad \sqrt{T_0 \quad T_b}\,\frac{\sqrt[4]{(\rho C_p)^3}}{\sqrt[4]{\mathrm{f}}}\frac{x}{\sqrt[4]{t^3}} \tag{39}$$

Since the Dresner solution does not take account of the temperature dependency of the thermal properties, eq. (39) is plotted as an area with the upper limit corresponding to the solution with the thermal properties at T_b = 1.8 K and the lower limit corresponding to the solution with the thermal properties at T_0 = 2.0 K.

In this temperature range, the HBIM solutions gives similar results since the variation of the thermal properties are "not" to important, f(T) varies by 25 % and $\alpha(T)$ by 50%. The discrepancy between the HBIM solutions and the exact solution can be as high as 25% for $x/t^{3/4} < 1$ and diverges at higher $x/t^{3/4}$. This discrepancy is intrinsic to the HBIM due to the fact that it only satisfies the original partial differential equation averaged over a finite distance. And since the thermal layer is underestimated by the HBIM as fig. 2 shows, the discrepancy is found for large x or small t. Several temperature profiles and boundary conditions has been tried but none of them so far presents a better accuracy.

For small x, the accuracy is found to be within a few percent and it is demonstrated in comparing the heat flux at the axis origin. The heat fluxes at the x-origin are defined, respectively, with the Kirchhoff and Goodman transforms as:

and

$$q_0 \quad \frac{\sqrt{3}}{2}\sqrt{\theta_0}\sqrt[4]{\frac{1}{\bar{\alpha}\,t}}$$
$$q_0 \quad \frac{\sqrt{3}}{2}\sqrt{\Theta_0}\sqrt[4]{\frac{\alpha_0}{t}} \tag{40}$$

At the wall we can consider small variation of thermal properties, hence θ_0 and Θ_0 can be replaced by average value as $f \quad \theta_0/(T_0 - T_\mathrm{b})$ and $\rho C_\mathrm{p} \quad \Theta_0/(T_0 - T_\mathrm{b})$ and eq. (40) is simplified to:

$$q_0 \quad \frac{\sqrt{3}}{2}\sqrt[4]{\rho C_\mathrm{p} f}\sqrt{T_0 \quad T_\mathrm{b}}\sqrt[4]{\frac{1}{t}} \tag{41}$$

which has a identical form than the heat flux given by the Dresner's solution except that the $3^{1/2}/2$ (0.87) coefficient in the HBIM solution is $3^{1/4}/2^{1/2}$ (0.93) in Dresner's solution.

For a larger temperature range such as [1.8 K; 2.172 K] as shown in fig. 2, the two HBIM solutions diverge essentially because the large variation of the equivalent thermal conductivity of He II is not taken in account by the Goodman transform. The HBIM solution with the Goodman transform is not even located in the area of possible solutions given by the Dresner's equation. Obviously, this solution can be only used for small variation of temperature. As expected the HBIM solution with the Kirchhoff transform, which takes account of the variation of the equivalent thermal conductivity of He II, gives acceptable results when compared to the analytical possible solutions area.

This constitutes for that problem the only analytical predictive solution known since the Dresner's model does not take account of the temperature dependency of the thermal properties.

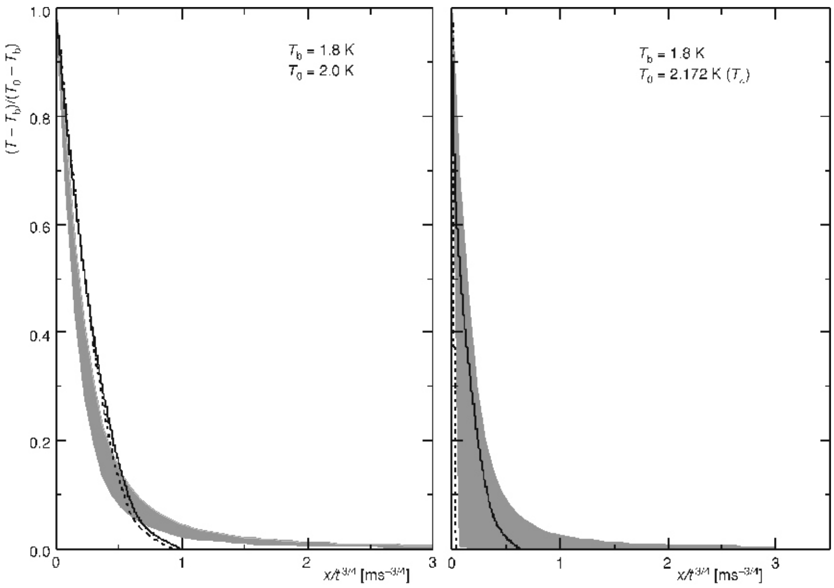

Figure 2. Comparison between the analytical model of Dresner [2] and the HBIM solutions. The Dresner's solution is presented as a area limited by the solutions with the thermal properties at T_b (upper limit) and T_0 (lower limit). The HBIM solution with the Kirchhoff transform – eqs. (23) and (25), is represented with a solid black line and the HBIM solution with the Goodman transform – eqs. (35) and 36), is represented with a dashed black line

Clamped heat flux case

The clamped heat flux problem is the most practical case encountered in cooling with static superfluid helium. It has been studied analytically again by Dresner [1] and experimentally by Van Sciver [11] for example. An important criterion in designing with superfluid helium is to make sure that the temperature of the device to cool does not go over the phase change temperature, named the lambda temperature T_λ, between superfluid helium and normal helium. This information is critical to operation of superconducting magnets, for example, since the heat transfer in normal helium is much lower than in superfluid helium.

The Dresner's analytical solution for the temperature at the heated side (at $x = 0$) is:

$$\frac{T_0 \quad T_\mathrm{b}}{T_\lambda \quad T_\mathrm{b}} \quad a\frac{q_0^2}{\sqrt{\rho C_\mathrm{p} f}\,(T_\lambda \quad T_\mathrm{b})}\sqrt{t} \tag{42}$$

where a was found by identification with experimental data of Van Sciver .The solutions given by the HBIM methods are, respectively:

$$\theta_0 \quad \frac{2}{\sqrt{3}} q_0^2 \sqrt{\alpha t}$$

and

$$\Theta_0 \quad \frac{2}{\sqrt{3}} q_0^2 \sqrt{\frac{t}{\alpha_0}} \tag{43}$$

for the Kirchhoff and Goodman transform. Once more, to compare the form of these solutions with the analytical solution, one can take the assumption that $f \quad \theta_0/(T_0 - T_\mathrm{b})$ and $\rho C_\mathrm{p} \quad \Theta_0/(T_0 - T_\mathrm{b})$ are average values and therefore equations are simplified to:

$$T_0 \quad T_\mathrm{b} \quad \frac{2}{\sqrt{3}} q_0^2 \frac{\sqrt{t}}{\sqrt{\rho C_\mathrm{p} f}} \tag{44}$$

The HBIM solutions have the same form than the analytical solution. Dresner identified the coefficient a to be 0.83 by fitting the experimental data of Van Sciver whereas the HBIM gives straightfully $2\cdot 3^{1/2}$ (1.15).

Another important information to give is the time to reach the critical temperature T_λ. This time is given by replacing T_0 by T_λ, and for the HBIM solution it is:

$$t_\lambda \quad \frac{3}{4}\frac{\rho C_\mathrm{p}(T_\lambda \quad T_\mathrm{b})^2}{q_0^4} \tag{45}$$

This formulation is also similar to Dresner's formulation with the exception of the coefficient 3/4 which is 1.43 in his model but it agrees on the quadric dependence on the heat flux with experimental results reported by Van Sciver [11].

Dresner's coefficients are found by identification with experimental results reported which means that these coefficients are only valid for the thermodynamic conditions of the experimental work they were extracted from. A comparison with experimental data is encouraging, when we look at the proportional function between the time t_λ and q_0^4. The experimental work of Van Sciver gives a value of 110 W^4s/cm^8 for bath temperature 1.802 K whereas eq. (45) gives a value comprised between 52 and 141 W^4s/cm^8 for temperature between 1.8 K and 2.0 K.

The HBIM solutions are compared with the experimental data of Van Sciver on the fig. 3. Only the solutions obtained with the Kirchhoff transform is presented since the solution with the Goodman transform does not fit the data. The Goodman transform HBIM solution does not allow computing a solution for time higher than 0.95 s. This time corresponds to a thermal layer of 0.63 m, which is the cause of the failure of this model, *i. e.* the thermal layer is underestimated. The HBIM solution with the Kirchhoff transform is quite accurate; the difference between the measurement and the HBIM solution is lower than 5%. But it still under estimates the thermal layer. For $t = 3.75$ s, the thermal layer is 1.19 m and at 1.25 s the thermal layer is 0.91 m. The under prediction of the thermal layer is probably due to the approximation of constant $\overline{\alpha}$, further work could be conducted to improve this part of the model.

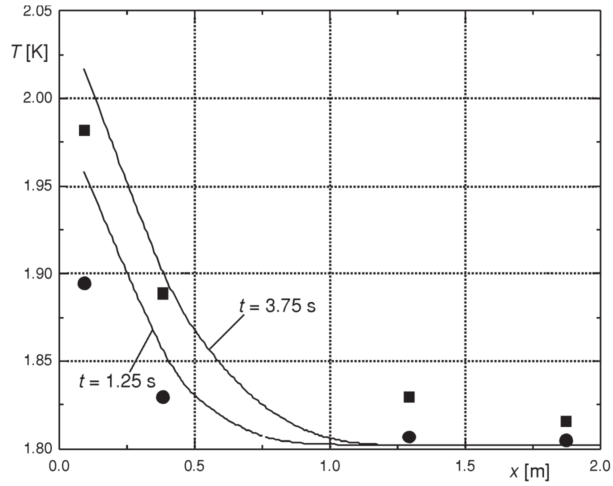

Figure 3. Comparison between experimental data [11] and the HBIM solutions with the Kirchhoff transform, eq. (23) and (28), for the clamped heat flux case. The experimental data were obtained at $T_b = 1.802$ K and $q_0 = 2.22$ kW/m^2. The black square correspond to the measurement at 3.75 s and the black circle for 1.25 s after energizing the heater. The HBIM solution (Kirchhoff transform) is represented with solid black line for t = 3.75 s and $t = 1.25$ s

Finally, one interesting result that can be deduced from the HBIM modeling is that the transient heat transfer in He II is more influenced by transport properties – $f(T)$, than intrinsic properties of helium – ρC_p, for "large" temperature variation, *i. e.* from 1.8 to 2.172 K and for heat flux in the range of kW/m^2, since the HBIM with the Goodman transform fails to reproduce experimental data or exact solution. This remark is not valid for lower temperature range or smaller heat flux, as it can be noticed for the clamped temperature case.

Conclusions

The HBIM method has been applied to solve the non-linear heat diffusion equation for superfluid helium with temperature dependent properties, for the clamped temperature and clamped heat flux.

One of the main contributions of the present work is that the HBIM leads to predictive solutions and do not need any parameter adjustment to fit experimental data. Actually, to the best of our knowledge, these solutions are the only analytical predictive solutions. Moreover, compared to numerical techniques, the present analysis is much simpler and provides analytical forms that can be handled with any spreadsheet program using a helium property data base.

The accuracy obtained with the HBIM using the Kirchhoff transform provides a good accuracy (within few percents) for reproducing experimental results or exact solution.

Nomenclature

A	– Gorter-Mellink coefficient, [mskg^{-1}]
a_i	– polynomial coefficients of the Kirchhoff solution
b_i	– polynomial coefficients of the Goodman solution
C_p	– specific heat at constant pressure, [Jkg^{-1}K^{-1}]
F_{ns}	– mutual friction force per unit volume, [Nm^{-3}]
$f(T)$	– thermal conductivity function in the turbulent regime of He II, [W^{3}m^{-5}K^{-1}]
p	– pressure, [Pa]
q	– heat flux density, [Wm^{-2}]
s	– entropy, [Jkg^{-1}K^{-1}]

T – temperature, [K]
t – time, [s]
v – velocity, [ms^{-1}]
x – space, [m]

Greek letters

α – $=f/\rho C_p$, [J^{2}m^{-2}s^{-1}]
$\overline{\alpha}$ – average value of α over the temperature range, [J^{2}m^{-2}s^{-1}]
δ – thermal layer, [m]
θ – Kirchhoff temperature transform, [W^{3}m^{-5}]
Θ – Goodman temperature transform, [Jm^{-3}]
μ – viscosity, [Pa·s]
ρ – density, [kgm^{-3}]

Subscripts

b – related to the bulk temperature or bath temperature
δ – related to the thermal layer
s – related to the superfluid component
n – related to the normal fluid component
λ – related to the lambda transition
0 – related to the space origin (x = 0)

Superscript

$\rightarrow$ – vector value mark

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