

## HOW GOOD IS GOODMAN'S HEAT-BALANCE INTEGRAL METHOD FOR ANALYZING THE REWETTING OF HOT SURFACES?

by

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*This paper discusses the application of heat-balance integral method for solving the conduction equation in a variety of rewetting problems. A host of rewetting problems for various geometry, convective boundary conditions and internal heat generation as well as for variable property has been solved by employing the method. Closed form expressions for rewetting velocity and temperature field in the hot solid have been obtained. Further, a unified solution methodology for different geometry and dimension of the problem has been derived. The results obtained agrees well with other analytical techniques namely, Winer-Hopf technique, separation of variables method as well as with the numerical ones. The predicted solutions exhibit a good agreement with experimental data as well. Additionally, an optimal linearization technique has been applied to analyze the effect of temperature dependent properties on the phenomena of rewetting. The results obtained and optimal linearization techniques have been compared and a good agreement has been obtained. All the studies made so far demonstrates the suitability of employing HBIM in the analysis of various rewetting problems.*

Key words: *rewetting, quenching, heat-balance integral method, effective Biot number, analytical methods, optimal linearization*

### Introduction

It is difficult to solve many conduction problems analytically. Such problems include transient heat conduction (one-dimensional or two-dimensional) as well as multidimensional steady or quasi steady conduction. Over the years a number of analytical techniques have been used in order to obtain solutions, some of which include infinite series of complex function which needs to be truncated for practical computation. Others are amenable only through numerical analysis. Among various such methods, the heat-balance integral method (HBIM) formally proposed by Goodman [1] is a landmark development. This technique, in spite of being simple, provides remarkable accuracy of results. Though initially the technique was proposed for a non-linear heat conduction problem involving phase change, subsequently it has been applied to other steady-state two-dimensional conduction problems. In both the cases, this powerful technique renders the partial differential equation into ordinary differential equation which reduces the difficulty of the solution enormously. Recently the method has been adopted in the analysis of rewetting of hot solids. The original transient two-dimensional conduction problem has been converted into a set of two quasi-steady formulations. HBIM has been applied separately to these two problems with proper matching of boundary conditions. The phenomena of

rewetting can be analyzed considering the axial conduction in solid and convective heat transfer from solid to the liquid film.

When a hot surface is brought in contact with the liquid medium a vapour blanket is formed and that prevents the contact between the solid and the liquid phase. As the surface cools off, the vapour blanket collapses and the solid liquid contact is re-established. This phenomenon is known as rewetting. The phenomena of rewetting of hot surfaces is of great interest in various fields namely, cryogenic systems, metallurgical processes, electronics cooling and cooling of nuclear reactors. During a postulated *loss of coolant* accident, the temperature of the fuel elements inside the reactor core increases drastically due to the stored energy inside the fuel. In order to remove the heat, the hot surface essentially undergoes through a rewetting phase. This is achieved either by spraying the coolant from the top of a hot object known as *top flooding* or an upward moving water front can cool down the core and is known as *bottom flooding*. When the coolant is sprayed to the hot surface, a water film forms and it flows down thereby cooling the hot surface. It is observed that the wet front moves with a constant velocity known as *rewetting velocity*. Over the decades various models have been proposed in order to evaluate the rewetting velocity and the temperature field in hot solid.

Initially, efforts were made to analyze rewetting problems based on one-dimensional approximation. These models are reasonably successful in correlating rewetting phenomena at low flow rates. Various models [2-3] have been proposed that demonstrates the transition between one-dimensional and two-dimensional formulations and establishes the limitation of one-dimensional model for high values of Peclet number and Biot number. In view of this, several two-dimensional models have been proposed to solve the conduction equation. Some of the important studies are elaborated below.

Due to mathematical difficulty, mostly the two-dimensional models have been solved either by employing approximate method or numerical ones. A solution to the above problems was reported by Duffey *et al.* [2] by employing separation of variable method. In their solution, a small number of terms of the series solution were retained. Blair [4] presented an approximate solution to the same problem for a cylindrical rod. Among other models that employ either Winer-Hopf technique [3, 5] or numerical techniques [6] in order to solve the two dimensional conduction equation.

HBIM [1] has primarily been used for solving a variety of Stefan problems involving one-dimensional conduction. Later, the method has also been employed for solving some typical two dimensional conduction equations [7-9]. It may be noted that the phenomena of rewetting possesses some similarity with the classical Stefan problem. In both the cases the moving boundary divides the solution space into two distinct regions with a strong temperature gradient at the interface. Recently, HBIM have been extensively used in order to solve a variety of rewetting problems comprising various geometry, convective boundary conditions and internal heat generation. This paper briefly discusses the findings of various rewetting problems analyzed by the HBIM and brings out the strength of this analytical tool. At the same time, it is shown that by employing HBIM highly accurate results are produced without recourse to mathematical complexity and special computation. Further, the analysis of rewetting model with temperature dependent properties, obtained by employing HBIM and optimal linearization technique has been discussed in details.

### **Basic rewetting model**

The basic rewetting models [2-4] consider two different regions (wet and dry) for a hot object (slab or rod) of infinite length with a quasi-steady approximation. The schematic of a two

dimensional object and the variation of temperature and heat transfer coefficient along the axial direction is shown in fig. 1(a)-(c). In these models a constant heat transfer coefficient is assumed in the wet region and an adiabatic condition is assumed in the dry region ahead of the wet front in order to solve one-dimensional/two-dimensional conduction equation. While the water is sprayed to the hot surface the surface temperature of the hot object behind the wet front approaches the liquid saturation temperature  $T_s$ . In general most of the rewetting models assume suitable values of rewetting temperature and heat transfer coefficient in order to solve the conduction equation.

Various analytical models have been proposed to obtain the solution. Some of them compute the solution by truncating the series of complex function by employing either Fourier expansion or Greens function. Thus the truncation of infinite series of the function may lead to errors in the solution. On the contrary, various numerical techniques have also been used to predict the solution. The obtained solution by these methods although powerful, the accuracy of the solution depends on the choice of structural meshing. Therefore, in order to obtain an acceptable accuracy, it is necessary to use dense discretization mesh, which is computationally inefficient and time consuming as well. As an alternative, HBIM although approximate, provide the solution with a good accuracy requiring only a little effort.

Although a number of conduction controlled models have been proposed a few efforts have been made to formulate a unified rewetting model irrespective of geometry and dimension of the problem. Recently, the rewetting model proposed by Sahu *et al.* [10] employs HBIM in order to solve the conduction equation. This provides a unified solution technique to the rewetting of hot object irrespective of geometry and dimension of the problem.

Mostly the rewetting experiments have been performed in rod/slab considering water at atmospheric pressure. The real value of heat transfer coefficients (Biot number) have not been reported precisely. However, experimental data have been used to find an average value of heat transfer coefficient during the analysis. At the same time the rewetting models [2-3] obtained either by employing separation of variable method or Winer-Hopf technique use a suitable value of Biot number to compare the theoretical models with the experimental data. A solution to the above problem [10], obtained by employing HBIM, yields a unique *effective Biot number* which was used to correlate the experimental data comprising of different coolant types, coolant flow rates, heater type, and range of surface temperature.

In this study an attempt has been made to correlate the effective Biot number with the coolant flow rate of all available experimental data. Recently, Duffey [11] have tried to fit the experimental data using the effective Biot number ( $M$ ). A correlation between effective Biot number and flow rate per unit perimeter for the experimental data set obtained by Duffey *et al.* [12] was developed. This exhibits that effective Biot number varies linearly with the flow rate per unit perimeter and is expressed as  $M = 5.0 \text{ flow rate}$ . Taking a cue from his analysis, an attempt has been made to compare the experimental data of Duffey *et al.* [12], and Yamanochi

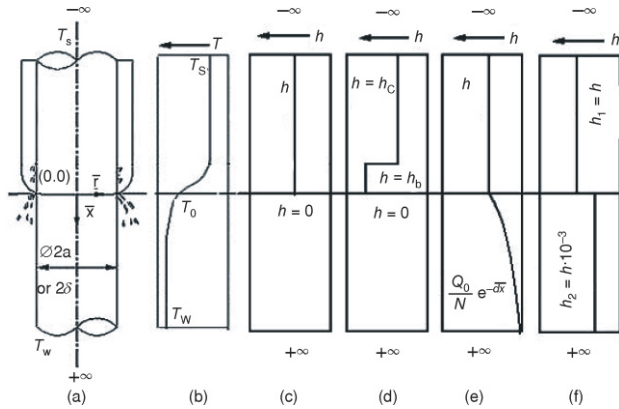
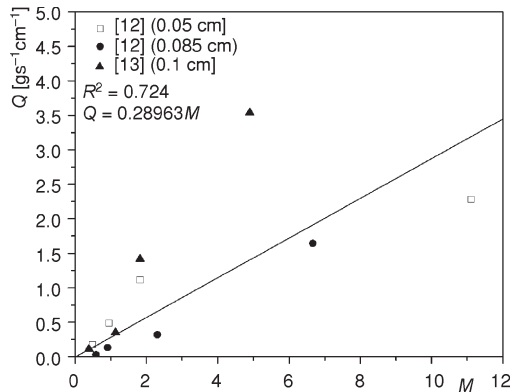


Figure 1. Schematic view of top flooding of a two-dimensional object



**Figure 2.** Variation of flow rate with effective Biot number ( $M$ )

been made in order to improve the capability of predicting the physical phenomena. This has been achieved in various ways, namely, including the effect of heating of the cladding due to decay heat, incorporating the variation in heat transfer by multiple step functions and exponential functions in the hot object. The schematic of various models is shown in fig. 1. The detailed description of various advanced models has been elaborated below. At the same time it is observed that the basic model is obtained as a particular case of these advanced models.

### **Rewetting model with multi region analysis**

The basic rewetting model assumes a constant heat transfer coefficient in the wet region and adiabatic condition in the dry region ahead of the wet front. However, from experiments [12], it has been observed that significant cooling takes place over a very short distance near the quench front. The intense rate of heat removal in this small region is due to the high boiling heat transfer coefficient. This region, known as *sputtering region*, plays an important role during the rewetting process. Therefore, adopting a single heat transfer coefficient in the wet region of hot surface may not be suitable for analyzing rewetting phenomena. In view of this, different profiles of heat transfer coefficient have been considered along the hot object to solve the conduction equation. The schematic of a two dimensional object and the variation of heat transfer coefficient along the axial direction is shown in fig. 1(a) and (d). Rewetting models have been reported [14-17] that propose a three region analysis which divides the wet region into two distinct regions: one liquid region and another sputtering region in a hot solid. Efforts have also been made to include the effect of heat generation, sub cooling in the three region rewetting model [15]. Further, the rewetting velocity predicted by the above models depends on the profile of heat transfer coefficient and sputtering region. Most of these models [15-17] employ either separation of variable method or numerical ones in order to solve the conduction equation for Cartesian geometry as well as in cylindrical geometry.

An attempt has been made to provide a solution methodology [18] valid for both cylindrical as well Cartesian geometries by employing HBIM. This yields a closed form solution for temperature field and rewetting velocity. The HBIM solution incorporates a sputtering length which plays an adjustable parameter and maintains the energy balance between stored and removed heat from the hot object. Thus during a quasi steady-state the propagation of wet front takes place in proportion with the removal of stored heat. It has been seen that by neglecting the sputtering region the three region model reduces to the basic rewetting model. Based on the

[13] with  $M$ . Based on the HBIM analysis [10], a straight-line correlation,  $M = 3.45$  flow rate is suggested and is shown in fig. 2. This shows a good agreement between the HBIM predictions with the experimental data and confirms the validity of HBIM analysis.

### **Refinement over basic rewetting model**

Although the basic conduction controlled rewetting models are successful in correlating the experimental data for moderate flow rates, it could not predict the physical realities of the rewetting phenomena. Therefore a number of refinements over the basic model have

analysis, it may be noted that HBIM can be used as an alternative analytical tool for a three region rewetting model.

**Rewetting model with precursory cooling**

In case of top flooding, when the water is sprayed at lower flow rates, water flows down in the form of a film and cools the hot object. However, at higher flow rates a part of the coolant sputters away from the wet front and cools the dry region ahead of the quench front. This mode of cooling is known as *precursory cooling*. Therefore it is necessary to incorporate the effect of precursory cooling in the model while correlating with the experimental data at higher flow rates. In order to model the precursory cooling, one can consider a constant heat transfer [16, 19-20], variation of heat transfer [21] or variation of heat flux [22-24] ahead of the wet front. If a constant heat transfer coefficient is assumed in the dry region ahead of the wet front, the highest heat flux from the wall occurs at the farthest location in the downstream direction of the quench front. This is physically unrealistic. At the same time by assuming a variation of heat transfer in the dry region ahead of wet front, one may require the knowledge of the variation of temperature of the droplet-vapour mixture  $T_a$  with distance to determine the heat flux in the dry region, which is also difficult to obtain in real situation. In view of this, an exponentially varying heat flux model is generally adopted in most of the models. The schematic of a two dimensional object and the variation of heat transfer coefficient along the axial direction is shown in fig. 1(a) and (c). Thus an exponentially varying heat flux of the form  $q'' = Q_0/N e^{-dx}$  is adopted in the dry region ahead of the wet front, where  $N$  is the magnitude of precursory cooling,  $d$  – the regions of influence for precursory cooling,  $Q_0$  – the heat flux associated at the wet front, and  $q''$  – the heat flux from the surface.

Various models [23-24] have been proposed that employ either separation of variables method or Weiner-Hopf technique to solve the conduction equation with precursory cooling. In most of the models the governing conduction equation is solved either considering a Cartesian geometry or cylindrical geometry. However, a few models have been proposed that reports a unified approach for slab and cylindrical geometry. Recently, Sahu *et al.* [25] presented a solution methodology that is valid for both slab and rod by employing HBIM. Results obtained from HBIM match with the earlier analytical results and exhibit good agreement with experimental results covering a wide range of parametric variation.

From the HBIM analysis a five parameter relationship has been developed and expressed as:

$$\theta_1 = \frac{0.5}{Pe} [\sqrt{Pe^2 + 4M} - Pe - S] \tag{1}$$

$$S = \frac{Bi}{N} \frac{Z_2}{a} \frac{2}{3} (a - Pe) \tag{2}$$

$$j = a - Pe, \quad M = \frac{Bi}{1 - \frac{Bi}{3}}, \quad Z_2 = \frac{2}{3} Z_1, \quad Z_1 = \begin{cases} 3 & \text{for a Cartesian geometry} \\ 6 & \text{for a cylindrical geometry} \end{cases} \tag{3}$$

where  $a$  is the non-dimensional distance controlling the region of influence for precursory cooling and  $Z$  is a parameter associated with geometry of the problem.

From a mathematical standpoint, the precursory cooling effect is due to the parameter  $S$ . The model that accounts for precursory cooling reduces to the basic model if the parameter  $S$  is set to zero. This can be attained in the following ways:

Case 1:  $Bi \rightarrow 0$

This represents the condition that there is no heat transfer from the hot surface ahead of the quench front and obviously it represents an adiabatic condition in the dry region, which is usually adopted in a basic model [2, 10].

Case 2:  $N \rightarrow \infty$

In this case, an exponentially varying heat flux of the form  $q'' = (Q_0/Ne) e^{-ax}$  is adopted in the dry region ahead of the wet front. From the exponential heat flux expression, it is observed that heat flux from hot surface decreases with increase in  $N$ . This leads to the situation when  $N \rightarrow \infty$ , there is no heat transfer from the hot surfaces. This is the same as an adiabatic condition in the dry region ahead of the wet front. On the contrary, one can also define  $N$  as the magnitude of precursory cooling which depends on the flow rate. As the coolant flow rate increases, number of coolant droplets increases in the sputtering region.

This results in an increase in heat transfer from the hot surface, yielding stronger precursory cooling. Thus one can conclude that  $N$  decreases with increasing coolant mass flow rate.

At the same time, with decreasing coolant flow rate the sputtering decreases and heat transfer decreases thus  $N$  increases. Thus when  $N \rightarrow \infty$ , this situation creates no cooling from the dry region ahead of the wet front.

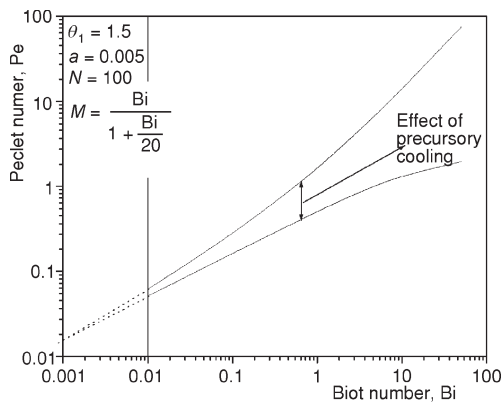


Figure 3. Effect of precursory cooling on the rewetting velocity

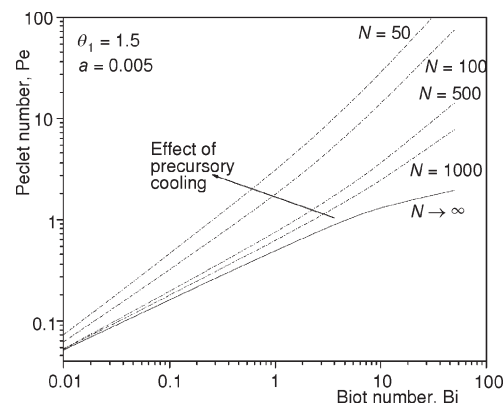


Figure 4. Variation of Pelet number with Biot numbers for various magnitudes of precursory cooling and with no precursory cooling

$$\text{Case 3: } a = \frac{Pe \sqrt{Pe^2 - 6Z_2}}{2}$$

This shows the relation between the region of influence by precursory cooling and Pelet number for a given operating conditions. However, this situation seems physically unrealistic and hence is not considered here for the analysis.

Figure 3 depicts the variation of Pelet number with Biot number for two different cases with precursory cooling and with no precursory cooling. It is observed that in both cases the wet front velocity increases with increasing Biot number. The rate of increase in wet front velocity becomes higher at higher Biot number for the case of precursory cooling. The predicted wet front velocity with precursory cooling significantly differs with that of with no precursory cooling at higher Biot number. However, the difference between the rewetting rates for these two models decreases significantly with decrease in Biot number.

The variation of rewetting rate with Biot number for various values of  $N$  is depicted in fig. 4. It is seen that the rewetting rate varies inversely with  $N$ . With increase in  $N$  the rate

of increase in rewetting rate increases. It is seen that the precursory model approaches the former model [18], when  $N \rightarrow \infty$ . From this analysis it is seen the model with precursory cooling approaches that of without precursory cooling [18] for a higher value of  $N$  and lower Biot number. This essentially proves the equivalence between rewetting models with and without precursory cooling for the same initial and boundary conditions.

### **Rewetting model with heat generation**

In case of a realistic situation, the temperature of the fuel rods increases due to the decay heat generated by fission. This implies that an internal heat source exists inside the fuel rod. Because of the internal heat source, transient conduction may take place across the fuel rod and this affects the rewetting velocity. In this context, the basic model is inadequate to describe the physical phenomena. Depending on the coolant flow rate and internal heat source, the wet front may either propagate in downward direction or cease (dryout condition) at some location. Thus a critical internal heat generation is defined as the minimum heat generation rate at which the wet front ceases at some location. The schematic of a two dimensional object and the variation of heat transfer coefficient along the axial direction is shown in fig. 1(a) and (f). The rewetting velocity obtained by these models depends on the surface temperature and heat removal rate from the surface. In view of this, it becomes essential to include the heat source parameter in the basic rewetting model. Some of the important studies are elaborated below.

Various models [26-29] have been proposed that consider the effect of specified heat flux as well as heat generation in the clad in order to account the decay heating of the fuel element. Chan *et al.* [30] have obtained the solution to the same problem by employing numerical methods as well as by analytical techniques. They have reported a closed form solution for both temperature field and rewetting velocity for both smooth and grooved plates. Satapathy *et al.* [5], Satapathy *et al.* [6] have reported a solution to the above problem by employing a Winer-Hopf technique and finite difference technique, respectively.

In general most of the models consider either a slab or tube with a specified heat flux on one side and flooded on the other side. At the same time one can also modify the governing equation by considering the internal heat generation to account the decay heat during the process of rewetting. However, so far only a single investigation [31] has been proposed to consider the second approach. The model reported by Sahu *et al.* [31] considers both boundary heat flux as well as heat generation applicable for various geometries by employing HBIM. All the different cases have been analyzed by employing HBIM. The HBIM yields a closed form solution for the temperature field and rewetting velocity valid for both boundary heat flux as well as heat generation as applied to different geometry of the problem. Further, it has been shown that a generalization of the analysis is possible through HBIM.

### **Rewetting model with temperature dependent properties**

In general, most of the models do not consider the variation of temperature dependent properties while solving the conduction controlled rewetting. The models based on constant property have been successful in predicting the rewetting velocity for a given range of surface temperature. However, during the cooling of hot surfaces, during metallurgical quenching and rewetting of nuclear fuel rods, the hot objects may undergo a substantial change in surface temperature and hence assumption of constant thermophysical properties can only provide approximate results.

Therefore, it is necessary to incorporate property variation in the model for the rewetting analysis. However only a single investigation [32] has been reported that considers the variation of temperature dependent properties for analyzing rewetting phenomena. Olek *et al.* [32] presented a solution considering the effect of various temperature dependent parameters namely, thermal conductivity, specific heat, density, and thickness of the test object for analyzing the rewetting phenomena. It is revealed that the variation of conductivity with temperature is more significant compared to other parameters. Therefore, in this study, the variation of temperature dependent conductivity has been incorporated in the model in order to analyze the phenomena of rewetting. Two different techniques one HBIM another optimal linearization techniques have been employed for analyzing the variation of properties during a rewetting analysis. The results obtained by both the analytical methods have been presented in closed form expression. The comparisons between the results obtained by both the models have been presented and the effect on conductivity variation on wet front velocity has been discussed.

### Theoretical analysis

#### HBIM analysis

For one-dimensional conduction of heat, the governing equation is given as:

$$\frac{\partial}{\partial \bar{x}} \left[ K \frac{\partial T}{\partial \bar{x}} \right] = \frac{h}{\delta} (T - T_s) - \rho C \frac{\partial T}{\partial t} \quad \infty < \bar{x} < \infty \quad (4)$$

where  $\delta$  is the thickness of the slab and  $\rho$ ,  $C$ , and  $K$ , are density, specific heat, and thermal conductivity of the slab material, respectively. Employing quasi-steady state assumption,  $T/t = -u T/\bar{x}$  yields:

$$\frac{d}{d\bar{x}} \left[ K \frac{dT}{d\bar{x}} \right] = \frac{h}{\delta} (T - T_s) - \rho C u \frac{dT}{d\bar{x}} = 0 \quad \infty < \bar{x} < \infty \quad (5)$$

Considering the variation of thermal conductivity to be a linear function of temperature, one can obtain:

$$K = K_0 [1 + n(T - T_s)] = K_0 [1 + \beta\theta] \quad (6)$$

where  $K_0$  is the thermal conductivity of the solid body at the ambient fluid temperature and  $n$  is the parameter describing the variation of conductivity.

Employing the following dimensionless parameters:

$$x = \frac{\bar{x}}{\delta}, \quad \text{Pe} = \frac{\rho C u a}{K_0}, \quad \text{Bi} = \frac{h a}{K_0}, \quad \theta = \frac{T - T_s}{T_0 - T_s}, \quad \theta_1 = \frac{T_w - T_0}{T_0 - T_s}, \quad \beta = n(T_0 - T_s) \quad (7)$$

and using eqs. (6) and (7), the governing eq. (5) reduces to:

$$[1 + \beta\theta] \frac{d^2\theta}{dx^2} - \frac{\beta}{K_0} \frac{d\theta}{dx} - \text{Pe} \frac{d\theta}{dx} - \text{Bi}\theta = 0 \quad (8)$$

with boundary conditions:

$$\theta(\infty) = 0, \quad \theta(0) = 1 \quad \infty < x < 0 \quad (9a)$$

$$\theta(0) = 1, \quad \theta(\infty) = 1, \quad \theta_1 = \infty < x < 0 \quad (9b)$$

#### Solution procedure by HBIM

At this juncture one needs to choose distribution profiles of temperature along the axial direction. Previously, other researchers have adopted a polynomial guess function in order to



solve the fluid flow and heat transfer problems by employing integral techniques [7, 33]. However, it is seen that in the present configuration the temperature decays (or increases slowly) away from the origin. Recently, an exponential trail function has been suggested by Mosally *et al.* [34] in order to solve decay like spatial variation in temperature for a Stefan problem. It is also reported [34] that if the selection is made properly, exponential profiles give better results compared to polynomial approximation.

In the generalized form, the energy eq. (8) can be integrated as:

$$(1 - \beta\theta) \frac{d^2\theta}{dx^2} - \frac{\beta}{K_0} \frac{d\theta}{dx} - Pe \frac{d\theta}{dx} - Bi\theta = 0 \quad (10)$$

For the present case the following guess profile has been selected:

$$\theta(x) = \begin{cases} D_1 e^{-Ax} + D_2 e^{-Bx} & (-\infty < x < 0) \\ D_3 + D_4 e^{-Ax} & (0 < x < \infty) \end{cases} \quad (11)$$

where  $A$ ,  $B$ , and  $D$  are positive constants. Using boundary condition given in eqs. (9a) and (9b) and eqs. (10) and (11) yields:

$$\theta(x) = \exp\left(\frac{Pe}{\alpha} \sqrt{1 - \frac{2Bi\alpha}{Pe^2}}\right) \begin{cases} 1 - \theta_1 & (-\infty < x < 0) \\ \theta_1 (1 - e^{-\psi x}) & (0 < x < \infty) \end{cases} \quad (12)$$

where

$$\alpha = 2 - \beta \frac{\beta}{K_0}, \quad \psi = \frac{2Pe\theta_1}{2\theta_1[1 - \beta(1 - \theta_1)] - \theta_1^2(\alpha - 2)} \quad (13)$$

It is noted that the dimensionless wet front velocity  $Pe$ , is unknown and can be determined by applying heat balance continuity at origin:

$$\left. \frac{d\theta}{dx} \right|_{x=0} = \left. \frac{d\theta}{dx} \right|_{x=0} \quad (14)$$

Using eq. (14), eq. (12) yields the following expression:

$$\frac{\sqrt{Bi}}{Pe} = \sqrt{\frac{2\theta_1^2 - 2\theta_1[1 - \beta(1 - \theta_1)] - \theta_1^2 - \beta \frac{\beta}{K_0} - \theta_1^2\alpha}{2\theta_1[1 - \beta(1 - \theta_1)] - \theta_1^2 - \beta \frac{\beta}{K_0}}} \quad (15)$$

Equation (15) correlates among Biot number, non-dimensional wet front velocity, non-dimensional temperature and the parameter for conductivity variation. Therefore, the wet front velocity can be evaluated using known values of  $Bi$ ,  $\theta_1$ , and  $\beta$ .

### Linearization techniques

While solving the heat conduction equation, the thermo-physical properties of the test material can either be assumed constant or vary with temperature. When the temperature dependent properties are considered in the model, the governing differential equations become nonlinear. Out of various analytical techniques the optimal linearization methods is one such ap-

proach used to solve non-linear heat transfer problems. This method essentially reduces the non-linear boundary value problem to a linear one and yields the solution in terms of closed form expression. Initially this method was first introduced by West [35] and Blaquiere [36] in order to solve the differential equations in non-linear vibration theory. This has also been applied to solve non-linear conduction problems including property variation by Vujanovic [37]. The solutions obtained by optimal linearization technique, although not exact, are sufficiently accurate for conduction problems. In this study an attempt has been made to employ this technique in order to solve the conduction-controlled rewetting problems with property variation. A one-dimensional conduction model is considered and the conductivity of material is assumed to vary linearly with temperature.

Employing eq. (7), eq. (5) reduces to:

$$\frac{d}{dx} K(\theta) \frac{d\theta}{dx} - \rho C u \delta \frac{d\theta}{dx} = h \delta \theta \quad (16)$$

Together with eq. (16) considering the following equation:

$$\lambda \frac{d^2\theta}{dx^2} - \rho C u \delta \frac{d\theta}{dx} = h \delta \theta \quad (17)$$

where  $\lambda$  is a constant adjustable parameter. At this juncture, one needs to select the best value of  $\lambda$  in order to approximate the linear eq. (17) with the non-linear eq. (16) together with boundary conditions. In this regards, one can evaluate the difference term between eq. (17) and eq. (16) and adopt a suitable minimization criteria to minimize the difference term.

The difference term can be expressed as:

$$\varepsilon = \lambda, \theta, \frac{d\theta}{dx}, \frac{d^2\theta}{dx^2} - \lambda \frac{d^2\theta}{dx^2} - K(\theta) \frac{d\theta}{dx} + K(\theta) \frac{d^2\theta}{dx^2} \quad (18)$$

where

$$K(\theta) = \frac{dK(\theta)}{d\theta} \quad (19)$$

Out of the various methods, the minimization of the mean square of the difference term [38] is mostly used in order to obtain a minimum value of difference term. In this study we adopt the method reported by Iwan *et al.* [38] in order to minimize the difference term as expressed in eq. (18).

Considering the integral:

$$I(\lambda) = \int_{-\infty}^{\infty} \varepsilon^2 \lambda, \theta, \frac{d\theta}{dx}, \frac{d^2\theta}{dx^2} dx \quad (20)$$

In order to evaluate the value of  $I(\lambda)$ , a guess profile for the temperature variation in spatial direction has to be chosen. Considering a known function that satisfies the boundary conditions in eqs. (9a) and (9b) is expressed as:

$$\theta = g(x) \quad (21)$$

The details functional value of  $g(x)$  are discussed later. Employing eq. (21) into eq. (20) one obtain:

$$I(\lambda) = \omega_1 \lambda^2 - 2\lambda(\omega_2 + \omega_3) + \omega_4 \quad (22)$$

where

$$\begin{aligned}
 \omega_1 &= \int_0^\infty \frac{\delta^2 \theta}{dx^2}^2 dx \\
 \omega_2 &= \int_0^\infty K(\theta) \frac{d^2 \theta}{dx^2} \frac{d\theta}{dx}^2 dx \\
 \omega_3 &= \int_0^\infty K(\theta) \frac{d^2 \theta}{dx^2}^2 dx \\
 \omega_4 &= \int_0^\infty K(\theta) \frac{d\theta}{dx}^2 K(\theta) \frac{d^2 \theta}{dx^2} dx
 \end{aligned} \tag{23}$$

The optimal value of  $\lambda$  may be found from the equation:

$$\frac{\partial I(\lambda)}{\partial \lambda} = 0 \tag{24}$$

*i. e.*

$$\lambda = \frac{\omega_2 - \omega_3}{\omega_1} \tag{25}$$

Therefore the optimal value of  $\lambda$  solely depends on the form of chosen function  $\theta = g(x)$  in eq. (21), and the linear differential equation with constant coefficients in eq. (17) should be considered “optimal” subject to  $\theta = g(x)$ . From the prior knowledge, the approximate solution of the problem is known. Based on this, we have selected an exponential trail function for the both wet and dry region of the hot object.

Thus eq. (21) is expressed as:

$$\theta(x) = \begin{cases} e^{Ex} & (-\infty < x < 0) \\ 1 - \theta_1(1 - e^{-Fx}) & (0 < x < \infty) \end{cases} \tag{26}$$

Employing eq. (26) in eq. (23), the constants  $\omega_1$ ,  $\omega_2$ , and  $\omega_3$  are evaluated and expressed as:

$$\begin{aligned}
 \omega_1 &= \frac{\frac{E^4}{2E}}{F^4 \theta_1^2} \int_0^\infty \frac{2E}{2F} dx && \begin{matrix} (-\infty < x < 0) \\ (0 < x < \infty) \end{matrix} \\
 \omega_2 &= \frac{\frac{K_0 \beta E^4}{3E}}{K_0 \beta F^4 \theta_1^3} \int_0^\infty \frac{3E}{3F} dx && \begin{matrix} (-\infty < x < 0) \\ (0 < x < \infty) \end{matrix} \\
 \omega_3 &= \frac{\frac{K_0 E^4}{2E}}{K_0 \beta \theta_1^2 (1 - \theta_1) F^4} \int_0^\infty \frac{\frac{K_0 \beta E^4}{3E}}{3F} dx && \begin{matrix} (-\infty < x < 0) \\ (0 < x < \infty) \end{matrix}
 \end{aligned} \tag{27}$$

Utilizing eq. (27), eq. (25) is expressed as:

$$\lambda \begin{cases} \frac{4K_0}{3} \beta \frac{3}{4} & (\infty < x < 0) \\ K_0 - 1 - \beta \frac{\beta\theta_1}{3} & (0 < x < \infty) \end{cases} \quad (28)$$

Employing eq. (28), the linear eq. (17) reduces to:

$$\begin{cases} \frac{4K_0}{3} \beta \frac{3}{4} \frac{d^2\theta}{dx^2} - \rho Cu \delta \frac{d\theta}{dx} - h\delta\theta & (\infty < x < 0) \\ K_0 - 1 - \beta \frac{\beta\theta_1}{3} \frac{d^2\theta}{dx^2} - \rho Cu \delta \frac{d\theta}{dx} & (0 < x < \infty) \end{cases} \quad (29)$$

Subject to the boundary conditions in eqs. (9a) and (9b), the solution of eq. (29) is expressed as:

$$\theta(x) = \begin{cases} \exp\left(\frac{Pe_1}{2} \sqrt{1 - \frac{4Bi_1}{Pe_1^2}}\right) (1 - x) & (\infty < x < 0) \\ 1 - \theta_1 (1 - e^{-Pe_2 x}) & (0 < x < \infty) \end{cases} \quad (30)$$

where

$$Pe_1 = \frac{Pe}{\frac{4}{3} \beta \frac{3}{4}}, \quad Pe_2 = \frac{Pe}{1 - \beta \theta_1 \frac{b}{3}}, \quad Bi_1 = \frac{Bi}{\frac{4}{3} \beta \frac{3}{4}} \quad (31)$$

It is noted that the dimensionless wet front velocity  $Pe$ , is unknown and can be determined as has been done in the HBIM analysis and one gets:

$$\frac{\sqrt{Bi}}{Pe} = \sqrt{\frac{\theta_1 \frac{Pe_2}{Pe_1} - \theta_1 \frac{Pe_2}{Pe_1} - 1 - 1 - \beta \theta_1 \frac{\beta}{3}}{\frac{4}{3} \beta \frac{3}{4}}^2} \quad (32)$$

### Results and discussion

Based on the present analysis, two different closed form expressions have been obtained for rewetting velocity by employing HBIM and optimal linearization techniques, respectively. Three independent dimensionless parameters, namely,  $\theta_1$ ,  $Bi$ , and  $\beta$  are used to analyze the rewetting process of the hot surface with property variation. In a similar analysis Olek [32] have reported expressions involving these parameters. However, different functional forms were obtained. From the reported relationship as expressed in eqs. (15) and (32), some important aspects of the rewetting process may be described. It is interesting to examine that for the case of  $\beta = 0$ , both the expressions obtained by HBIM and optimal linearization technique reported in eqs. (15) and (32), reduces to the following expression:

$$\frac{\sqrt{Bi}}{Pe} = \sqrt{\theta_1 (1 - \theta_1)} \quad (33)$$

This is the same as that of one-dimensional model of Yamanouchi [13] obtained for fixed properties.

Figure 5 depicts the variation of wet front velocity for various values of Biot number for a given value of  $\theta_1$ ,  $\beta$ , and  $K_0$ . It is observed that Peclet number increases with increasing Biot number. Further, the results obtained by various models, namely HBIM and optimal linearization have been compared with that of Yamanouchi [13] which were obtained considering a constant thermal conductivity and is shown in fig. 5. It is observed that for a given value of  $\theta_1$ ,  $\beta$ , and  $K_0$  the result obtained by HBIM and optimal linearization technique is same with the model that considers a constant thermal conductivity. It may be noted that for a constant dry wall temperature ( $\theta_1$ ), the conductivity remains constant. Therefore, the prediction is same for all the models.

Figure 6 shows the variation of Peclet number with dry wall temperature ( $\theta_1$ ) for a given value of Bi,  $\beta$ , and  $K_0$ . The results obtained by HBIM, optimal linearization technique, and Yamanouchi [13] model shows the same value of Peclet number for lower value of drywall temperature. However, with increase in surface temperature the Peclet number obtained by employing HBIM and optimal linearization technique varies with that of Yamanouchi model [13] and is shown in tab. 1. It is seen that HBIM shows a

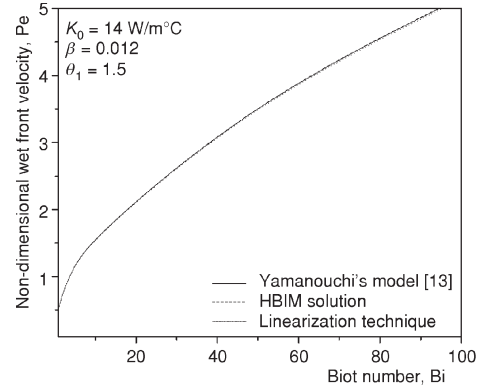


Figure 5. Variation of wet front velocity with Bi

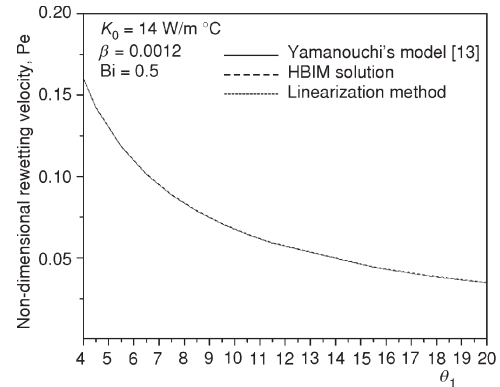


Figure 6. Variation of wet front velocity with  $\theta_1$

Table 1. A comparison between the non-dimensional wet front velocity (Pe) obtained by Yamanouchi<sup>a</sup> [13], HBIM<sup>b</sup>, and OL<sup>c</sup> for  $\beta = 0.0012$ , Bi = 0.5, and  $K_0 = 14 \text{ W/mK}$

$\theta_1$	Yamanouchi [13] (I)	HBIM (II)	Error [%] (I vs. (II))	OL (III)	Error [%] (I vs. (III))
1.0	0.5000	0.5005	0.10	0.5003	0.06
2.0	0.2886	0.2891	0.17	0.2887	0.03
3.0	0.2041	0.2045	0.1959	0.2041	0.0
4.0	0.1581	0.1585	0.2530	0.1581	0.0
5.0	0.1290	0.1295	0.3875	0.1290	0.0
6.0	0.1091	0.1095	0.3666	0.1090	0.09
7.0	0.0944	0.09491	0.5402	0.0944	0.0
8.0	0.08333	0.08375	0.540	0.08326	0.08
9.0	0.07453	0.07495	0.5635	0.07446	0.09
10.0	0.06741	0.06783	0.6230	0.06733	0.14
15.0	0.04564	0.04605	0.8983	0.04554	0.22
20.0	0.03450	0.03490	1.1594	0.03439	0.318
30.0	0.02318	0.02358	1.725	0.02306	0.51

a: Yamanouchi's [13] one-dimensional model; b: heat-balance integral method; c: optimal linearization technique

higher value of Peclet number where as the optimal linearization technique predicts a lower value of Peclet number compared to Yamanouchi's model. The maximum error in variation in Peclet number with Yamanouchi's model is 1.72% and 0.51% in HBIM and optimal linearization technique, respectively. Therefore from this analysis it is observed that the variation of property have a minimal effect on the rewetting velocity.

## Conclusions

Heat-balance integral method (HBIM) has been employed in order to solve the conduction equation in a variety of rewetting problems. These include the basic rewetting model, three region model, inclusion of precursory cooling, heat generation, and property variation in the hot object. A unified solution methodology for a variety of conduction controlled rewetting problems has been obtained by employing HBIM. In all the cases a closed form solutions have been obtained for the temperature field and rewetting velocity of the hot solid. By the application of this technique one can identify a unique function solely dependant on Biot number. Treating this function as a modified Biot number ( $M$ ) unique parametric relationship for the rewetting velocity is obtained. This eliminates the need for the development of different models for different geometry, unifies 1-D and 2-D analysis and also shows a direction for comparing the experimental data with the analytical results (fig. 2). Next, this technique has been extended to include the variation of temperature dependent properties during rewetting process. Additionally, an optimal linearization technique has been employed for the same problem to facilitate comparison. From both the studies, it is observed that the effect of temperature dependent conductivity have a minimal effect on the rewetting velocity.

It may be noted that HBIM has the ability to deal with a host of rewetting problems without recourse to mathematical complexity and special computation. This definitely shows an advantage of HBIM over other analytical and numerical techniques. This technique is simple to use and can be adopted as an alternative method for the class of rewetting problems. Acceptance of HBIM widely depends on application of this method for solving various heat transfer problems as applied to engineering purposes. This study has shown that HBIM has the potential as an acceptable alternative analytical tool for solving general partial differential equations.

## Nomenclature

$A, B, D$	– constants defined in text	$Pe_1, Pe_2$	– Peclet numbers defined in eq. (31)
$a$	– parameter defined in eq. (3)	$Q_0$	– heat flux at the wet front [ $=h(T_0 - T_S)$ ], [Wm <sup>-2</sup> ]
$Bi$	– Biot number ( $=h\delta/K_0$ ) defined in eq. (7)	$q''$	– wall heat flux, [Wm <sup>-2</sup> ]
$Bi_1$	– Biot number defined in eq. (31)	$S$	– parameter defined in eq. (2)
$C$	– specific heat, [Jkg <sup>-1</sup> °C <sup>-1</sup> ]	$T$	– temperature, [°C]
$d$	– constant in exponent, fig. 1(e), [m <sup>-1</sup> ]	$T_0$	– wet front temperature that corresponds to the temperature at the minimum film boiling heat flux, [°C]
$E, F$	– parameters defined in eq. (27)	$T_S$	– saturation temperature, [°C]
$h$	– heat transfer coefficient, [Wm <sup>-2</sup> °C <sup>-1</sup> ]	$T_w$	– initial temperature of the dry surface, [°C]
$I$	– error integral defined in eq. (20)	$t$	– time, [s]
$j$	– parameter defined in eq. (3)	$u$	– wet front velocity, [ms <sup>-1</sup> ]
$K$	– thermal conductivity, [Wm <sup>-1</sup> °C <sup>-1</sup> ]	$\bar{x}$	– length coordinates, [m]
$K_0$	– thermal conductivity at the ambient fluid temperature, [Wm <sup>-1</sup> °C <sup>-1</sup> ]	$x$	– length coordinates, [–]
$M$	– effective Biot number, [–]	$Z$	– parameter defined in eq. (3)
$N$	– magnitude of precursory cooling, [–]		
$n$	– constant defined in eq. 7		
$Pe$	– dimensionless wet front velocity ( $=u\delta\rho C/K_0$ ) defined in eq. (7)		

*Greek letters*

$\alpha$	– parameter defined in eq. (13)	$\psi$	– parameter defined in eq. (13)
$\beta$	– constant defined in eq. (7)	$\rho$	– density, [kgm <sup>-3</sup> ]
$\delta$	– wall thickness, [m]	$\omega$	– parameter defined in eq. (23)
$\varepsilon$	– difference term defined in eq. (18)		
$\theta$	– non-dimensional temperature defined in eq. (7)	<i>Subscripts</i>	
$\theta_1$	– non-dimensional temperature parameter defined in eq. (7)	0	– quench front
$\lambda$	– parameter defined in eq. (28)	1, 2, 3, 4	– separation constants

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