ON THE GOODMAN HEAT-BALANCE INTEGRAL METHOD FOR STEFAN LIKE-PROBLEMS
Further Considerations and Refinements

by

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Since the pioneering studies of Goodman on the application of the integral method to transient non-linear heat diffusion, much attention has been devoted nowadays to what is called heat balance integral method. The present paper considers this technique fifty years later. The genesis and earlier developments, when applied to Stefan like-problems, are reported hereafter. Its simplicity and efficiency are demonstrated. Some numerical results obtained using methods developed on the basis of the heat balance integral are compared. Furthermore, for problems including temperature profile behaviour, such as Stefan problem with forcing term (source or sink) this technique gives highly precise results and may, in some cases, lead to exact solutions.

Key words: *heat balance integral, Stefan problem, analytical solution, source/sink term, phase-change, non-linear heat diffusion, moving boundary*

Introduction

The heat-balance integral method was presented, for the first time in 1957, publicly by Goodman at the Heat Transfer and Fluid Mechanics Institute in Pasadena (USA) before its publication in 1958. The heat-balance integral (HBI) was then adopted in the technical language [1]. This technique, among a great number of numerical and analytical works developed and applied successfully to non-linear transient diffusion problems, has received a considerable attention. Two decades only after being published, the paper was cited over 100 times and was considered by its author as being one of his best efforts [2]. Since that time an increasing interest has been given to such method which is the basis of a great amount of research work. For instance, one should note aerodynamic heating by forced convection [3], heat transfer with transpiration [4], prediction of the response of a positive temperature coefficient thermistor [5, 6], conduction-controlled rewetting problems [7], boilover phenomenon occurring in fuel tanks [8], nuclear reactor safety analysis [9], deicing systems [10], and casting metals and spray forming [11]. The method allows satisfactory results with the least numerical efforts and, in some cases, leads to analytical solutions. This technique shows remarkable flexibility to include cylindrical [12-15] and spherical [12, 14, 16] coordinates with various boundary conditions and trials were carried on for two-dimensional [17] and three-dimensional [18] heat diffusion problems.

The simplicity of the method to generate approximate functional solutions to non-linear diffusion problems motivated many investigations on refining the method or enlarging its application domain. The HBI method allows deriving the moving front velocity, heat fluxes or internal energy variations straightaway from the temperature profile. Furthermore, it provides starting solutions for complex numerical schemes.

The method appears, presently, to be a definite powerful approach to many non-linear diffusion problems since it doesn't need any linearization of the required equations. It should be observed that the non-linearity can either be induced by the temperature dependency of the transport properties [19-21] or by the non-linearity of the boundary conditions [22] such as at a phase change front.

The main advantage is due to the transformation of the governing equation, from a partial differential form to an ordinary differential one. In the case of the heat diffusion in a variable domain, the latter is achieved according to the following procedure:
(1) assuming an adequate function for the temperature distribution,
(2) enforcing the available boundary conditions,
(3) obtaining the heat-balance integral equation by integrating once the diffusion equation, with respect to space, over the domain of interest, and
(4) substituting the assumed profile in the obtained heat integral equation to get an ordinary differential equation for the location of moving boundary as function of time.

From the outlined procedure, it seems clear that the assumed profile constitutes the central part of the original Goodman's technique [1]. It should be pointed out that the method accuracy depends on the chosen profile. Despite a large number of investigations [12, 19, 23-25] there is, unfortunately, no systematic procedure to choose the most appropriate profile. Inspecting the steady-state form of heat transfer problem, as suggested by Goodman himself [3], to identify an appropriate approximant for the transient phase doesn't seem to be a promising way. Then, works were oriented towards a decrease on the accuracy dependence of the method on an arbitrarily profile [20, 26-31].

In the present contribution, the method is developed in the case of the one-phase Stefan problem which was the basis of the Goodman's method derivation. Some refinement procedures, developed in the literature, are reported and compared. Moreover, this work considers application of the method in the case of Stefan problem with a source-sink term (forcing term). In that case, the technique may lead to analytical solution in closed form.

Classical one-phase Stefan problem

The non-linear mathematical model described by transient diffusion equation considered by Goodman is the "One-phase Stefan problem". It refers to heat conduction involving phase change in medium which is initially ($t = 0$) at its melting temperature, T_{m}, and remains in thermal equilibrium during all the process of phase change [32]. The non-linearity is associated to the equation expressing the jump condition in terms of heat flux at the freezing (melting) front $\delta(t)$. One should note that the latter is a moving boundary. For the classical Stefan problem, the medium holds the half-plane $x > 0$ and the phase change is initiated at the boundary with a sudden temperature decrease (freezing) or increase (melting) and maintained fixed during all the process. The transport properties k and c, thermal conductivity and specific heat, respectively, as well as density ρ are considered constant.

Introducing dimensionless variables, the heat transport equation in the layer $0 < x < \delta(t)$, is given as:

$$\frac{\partial \theta}{\partial \tau} \quad \frac{\partial^2 \theta}{\partial \xi^2}; \quad \tau \quad 0; \quad 0 \quad \xi \quad \Delta(\tau) \tag{1}$$

with initial and boundary conditions

$$\theta(\xi, \tau) \quad 0, \quad \tau \quad 0, \quad 0 \quad \xi \quad \Delta(\tau) \tag{2}$$

$$\theta(\xi, \tau) \quad 1, \quad \tau \quad 0, \quad \xi \quad 0 \tag{3}$$

$$\theta(\xi, \tau) \quad 0, \quad \tau \quad 0, \quad \xi \quad \Delta(\tau) \tag{4}$$

$$\frac{\partial \theta}{\partial \xi} \quad \frac{1}{\mathrm{Ste}} \frac{\mathrm{d}\Delta}{\mathrm{d}\tau}, \quad \tau \quad 0, \quad \xi \quad \Delta(\tau) \tag{5}$$

where $\tau = \alpha t/l_{\mathrm{ref}}^2$, $\xi = x/l_{\mathrm{ref}}$, $\theta = (T - T_\mathrm{m})/(T_0 - T_\mathrm{m})$ and $\Delta = \delta/l_{\mathrm{ref}}$ refer to dimensionless time, space coordinate, temperature, and moving front position, respectively, and $\alpha = k/\rho c$ is the thermal diffusivity .

The Stefan number, Ste, expresses the dimensionless latent heat and defined by the ratio of the heat needed to cool the solid from its melting temperature to T_{ref} ($T_{\mathrm{ref}} = T_0$ in the present section) to the latent heat needed to transform liquid to solid:

$$\mathrm{Ste} \quad \frac{\rho c(T_\mathrm{m} \quad T_0)}{\rho L}$$

The initial condition closing the mathematical model is given by location of the moving front at $t = 0$, that is $\Delta(0) = 0$.

The exact solution of this problem is available in literature [33, 34]. It can be easily derived by using the similarity variable to transform the governing equation from partial differential equation to ordinary differential one. It should be pointed out that our interest is focussed on the moving front position expressed by:

$$\Delta(\tau) \quad 2\lambda\sqrt{\tau} \tag{6}$$

where λ is the freezing constant given by the root of the following transcendental equation:

$$\lambda \mathrm{e}^{\lambda} \mathrm{erf}(\lambda) \quad \frac{\mathrm{Ste}}{\sqrt{\pi}} \tag{7}$$

erf is the error function. The temperature is expressed through: $\theta(\xi, \tau) = 1 - \mathrm{erf}(\xi/2\tau^{1/2})/\mathrm{erf}(\lambda)$.

Goodman HBI solution

Taking into account that the problem is a special case of a non-linear transient diffusion problem, Goodman applied the well known Karman-Pohlhausen method developed in fluid dynamics. This technique, called generally the integral method, was developed, independently, by Th. von Kármán [35] and K. Pohlhausen [36] to study the momentum diffusion in the boundary layer. Analogous to the momentum integral method, the formulation does not require any linearization of the considered equation. The analysis introduces the well known notion of penetration depth, $\delta_\mathrm{p}(t)$ for which the medium is at the equilibrium temperature from $x > \delta_\mathrm{p}(t)$ and there is no heat transferred beyond this point. It is observed that, when no phase change is involved, $\delta_\mathrm{p}(t)$ is set instantaneously to infinity, according to Fourier's law. However, this is not the case in Stefan like-problem since the thermal depth penetration coincides with the phase change front. Then the boundary conditions are well defined as given by eqs. 4 and 5.

Following the procedure outlined in Introduction, the suitable approximating profile for the description of the temperature distribution in the layer is assumed to be quadratic. Enforcing then the boundary conditions (3), and 4), the later takes form:

$$\theta(\xi,\tau) \quad \varsigma\ 1 \quad \frac{\xi}{\Delta} \quad (1 \quad \varsigma)\ 1 \quad \frac{\xi}{\Delta}^{2} \tag{8}$$

where ς is a time-dependent parameter to be determined alike the moving boundary position Δ. Using eq. 8 in the Stefan condition (5), one obtains for the parameter ς:

$$\varsigma \quad \frac{1}{2\mathrm{Ste}}\frac{\mathrm{d}\Delta^{2}}{\mathrm{d}\tau} \tag{9}$$

The technique is based on the approximation of the heat conduction eq. (1) by an overall energy balance in the domain of interest. After space integration, over the layer $0<\xi \quad \Delta(t)$, the heat conduction equation gives:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\int_{\xi\ 0}^{\xi\ \Delta}\theta\,\mathrm{d}\xi \quad \left.\frac{\partial\theta}{\partial\xi}\right|_{\xi\ \Delta} \quad \left.\frac{\partial\theta}{\partial\xi}\right|_{\xi\ 0} \tag{10}$$

The result is referred to as the HBI for the region of interest and expresses the macroscopic heat-balance across this region. It can be seen from the resulting equation that the technique satisfies the heat conduction equation only on the average. Furthermore, derivatives of an arbitrarily profile, as appearing in the right hand side of eq. (10), may induce errors. On the other hand its effectiveness is largely due to the left hand side of the above equation as long as the assumed profile satisfies the boundary conditions.

After substituting the profile (eq. 8) in HBI eq. (10) with takes into account the Stefan condition (eq. 9), the moving boundary location is tracked through the following analytical expression:

$$\Delta(\tau) \quad \sqrt{2[\sqrt{(\mathrm{Ste} \quad 6)^{2} \quad 12\mathrm{Ste}} \quad (\mathrm{Ste} \quad 6)]}\sqrt{\tau} \tag{11}$$

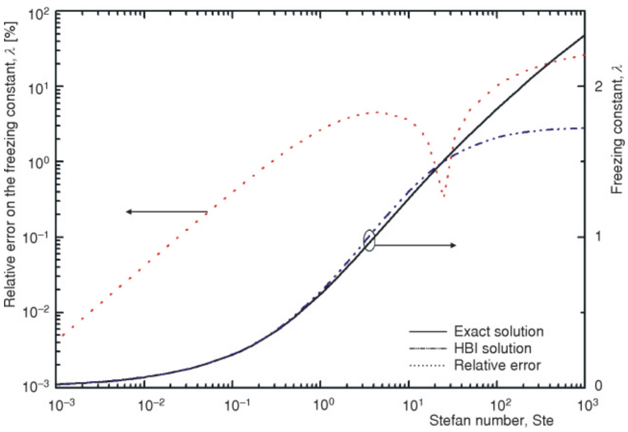

Figure 1. Freezing constant, λ, as given by exact and HBI solutions and its associated relative error $\lambda_{\mathrm{exa}} - \lambda_{\mathrm{app}}\ /\lambda_{\mathrm{exa}}$ [%]

It can be noticed the computational simplicity introduced by the HBI procedure compared to the exact solution which requires evaluations of both the freezing constant through numerical iterative sequences and tabulated functions (the error function). Moreover, as shown in fig. 1, the obtained results from eq. (11) compared to the exact solution are satisfactory. The method predicts the moving front location with an error that does not exceed 2.5% for Stefan number less than unity which covers most usual isothermal phase change material. These features motivated an appreciable number of investigations devoted to refinement procedures.

Further considerations and refinements

The accuracy of the method depends strongly on the substituting profile which, according to Fox [37], must be carefully chosen. The precision does not necessarily increase with the polynomial degree as expected. As a matter of fact, investigations have shown that in some cases lower polynomial degrees gave better results than those at higher order [19, 25]. Then the following important question arises: could it possible to develop a systematic procedure for choosing the appropriate polynomial order of the profile to get the most accurate results?

The above question at first found its answer with Goodman's primary works [19, 22] where the author concluded that there is no *a priori* guarantee that increasing the order of the polynomial will improve the accuracy of the method. Among several studies [24, 25, 37], that confirmed Goodman's conclusion, one can refer to Vujanovic *et al*. [24]. On the other hand, the later authors showed that, for a specific problem, a procedure can be developed to optimize the polynomial degree.

Additional boundary conditions are required to increase the approximations order. Among other investigations devoted to additional mathematical relations the most important are reported hereafter.

Alternative Stefan condition

Goodman [19] developed an alternative Stefan condition similar to the smoothing condition considered in problems without phase change. It is recalled that smoothing condition implies that all successive temperature derivatives vanish at the penetration depth [22]. The alternative Stefan condition is built from the total derivative of temperature on the moving front where the heat conduction equation is assumed to be valid; fact which might not be strictly exact since the interface is isothermal. This procedure leads to:

$$\left.\frac{\partial\theta}{\partial\xi}\right|_{\xi\ \Delta}^{2} \qquad \mathrm{Ste}\left.\frac{\partial^2\theta}{\partial\xi^2}\right|_{\xi\ \Delta} \tag{12}$$

After substituting the temperature profile (eq. 8) in eq. (12), the evolution of the parameter ζ as function of the moving front position Δ is obtained. The HBI eq. (10) is then developed using the original Stefan condition (eq. 5) leading to a new analytical solution (tab. 1). One should observe that this approach provides lower accuracy than the original HBI method (eq. 11) as shown in fig. 2 and confirmed in the Wood's [38] investigation.

The original and alternative forms of the Stefan condition – eqs. (5) and (12) – were both considered by Wood to highlight six alternative pathways in the two-parameter quadratic HBI method implementation. In addition to solutions due to Goodman and discussed above, two new analytical expressions for the freezing constant are obtained (tabs. 1 and 2). The author shows that the solution accuracy is improved through the use of the original Stefan condition as well as in the development of the HBI equation and in the boundary conditions generation. According to the author, the loss of accuracy, with the use of eq. 12, arises from the assumption introduced in its derivation: the application of the heat conduction equation at the isothermal moving front.

Poots considerations

Poots [12] considers the HBI method to treat the freezing front in three configurations: inward solidification of the semi-infinite region, circular cylinder, and sphere. Since the basic quadratic HBI method fails when applied to the circular cylinder and sphere, the author develops two approaches. At first, the author considers the Goodman HBI with a linear one-parameter profile $\theta = 1 - (\xi/\Delta)$ which leads to a simple analytical freezing constant: $\lambda = \ [\mathrm{Ste}/(2 + \mathrm{Ste})]^{1/2}$. The same author proposed then a new approach to make the quadratic HBI method applicable to the above-mentioned configurations. For that purpose, a procedure, due to Tani [39], based on a modification of the Karman-Pohlhausen technique is used. The latter consists on assuming a simple two-parameter quadratic profile satisfying the boundary conditions (3, 4). It is written in a form slightly different from the profile (eq. 8) assumed by Goodman:

$$\theta(\xi,\tau) \quad 1 \quad \frac{\xi}{\Delta} \quad 1 \quad \zeta\frac{\xi}{\Delta} \tag{13}$$

Then a new equation with less physical meaning than HBI equation is developed and used inconjunction with the Goodman HBI (eq. 10). It's derived by multiplying both sides of eq. (1) by($\theta \quad \xi$)dξ and integrating from $\xi = 0$ to $\xi = \Delta$; the alternative Stefan condition (eq. 12) is applied. The substitution of the assumed profile into both equations leads to a pair of first order equations for Δ and ζ. In addition to the initial location of the moving boundary $\Delta(0) = 0$, the required second condition is derived, by Tani procedure, from consideration of the total thermal energy of the layer growth and it is expressed as:

$$\lim_{\Delta \quad 0}(\Delta\zeta) \quad 0 \tag{14}$$

The obtained result is as simple as the one given by one-parameter linear profile (see tab. 1).

Coupled integral equation approach

Mennig and Özisik [29] conducted an investigation to make the method applicable independently of the assumed temperature profile. The authors developed an additional equation to be used in conjunction with the HBI equation. This approach was identified as a coupled integral equation approach and is based on the use of the trapezoidal rule to evaluate integral of both temperature and heat flux by considering their values only at the boundaries of the thermal layer. This additional boundary condition is then derived and used to define the unknown heat flux at fixed boundary as follows:

$$\left.\frac{\partial\theta}{\partial\xi}\right|_{\xi \quad 0} \quad \frac{1}{\mathrm{Ste}}\frac{\mathrm{d}\Delta}{\mathrm{d}\tau} \quad 2\frac{\theta(\Delta,\tau) \quad \theta(0,\tau)}{\Delta} \tag{15}$$

The eq. (15) associated to HBI equation allows to get the freezing constant λ. It should be noted that, even if the authors did not explicitly assume an approximating profile, the approach considers implicitly the temperature distribution to be linear in the left hand side of HBI equation and quadratic in the right hand side since the heat flux is considered varying linearly within the thermal layer.

Double integral approach

In order to bypass the derivative of an arbitrary temperature profile, the heat diffusion equation is integrated twice with respect to space. The result in conjunction with the HBI equation allows removing the heat flux at the fixed boundary. The technique, due to Volkov *et al.* [20], is originally developed to refine Goodman HBI method in the case of transient heat diffusion without phase change. The non-linearity is introduced by the temperature dependency of the material properties. In the Stefan problem, taking into consideration that non-linearity is due to the moving front, this technique [26] leads to:

$$\int_{\xi=0}^{\xi=\Delta}\int_{\xi=0}^{\xi}\frac{\partial\theta}{\partial\tau}\mathrm{d}\xi\,\mathrm{d}\xi \quad 1 \quad \Delta\left.\frac{\partial\theta}{\partial\xi}\right|_{\xi=0} \tag{16}$$

After some algebraic manipulations [39, 40] of eq. (16) in conjunction with HBI eq. (10) and eq. (5), the following is obtained:

$$\frac{\Delta^2}{2\mathrm{Ste}} \int_0^{\Delta}\xi\theta\,\mathrm{d}\xi = \tau \tag{17}$$

Table 1 reports the analytical expressions obtained for the freezing constant when linear or two-parameter quadratic profiles are used. One can note that the result (eq.17) is also derived by Elmas [28], Hamill *et al.* [27], and El-Genk *et al.* [42] using the double integral approach formulated in fixed domain rather than moving one. The domain is fixed trough the use of the Landau space variable $\eta = x/\delta(t)$ leading to:

$$\Delta^2(\tau) \quad \frac{1}{2\mathrm{Ste}} \int_{\eta=0}^{\eta=1}\eta\theta\,\mathrm{d}\eta \quad ^{1} \tau \tag{18}$$

The consideration of the temperature limits $0 \quad \theta \quad 1$ sets $0 \quad \int_0^1\eta\theta\,\mathrm{d}\eta \quad 1/2$ which gives to $\Delta^2(\tau)$ the upper and lower bounds. Two approximations are developed to obtain $\Delta(\tau)$. The first one is based on the arithmetic average of upper and lower limits of $\Delta(\tau)$ [42] while the second is computed from eq. (18) where the average is performed on the denominator [28]. Another approach is developed by El-Genk *et al.* [42] using a four-parameter, including $\Delta(\tau)$, quadratic profile. The result is linearized through the use of the alternative Stefan condition.

Piecewise HBI implementation

Finding a satisfactory approximating profile constitutes a major difficulty of the HBI method. For that purpose Noble [31] suggests that piecewise implementation constitutes a promising way to improve the accuracy. This can be understood since most functions can be piecewise approximated, with an increasing accuracy inversely proportional to the interval size, in addition to the thin layer approximation of the diffusion equation.

In principle the layer is divided into thinner sub-layers in which the profile can be approximated by simple form. The HBI method is then applied using linear [15, 23, 31, 43], quadratic, exponential or Gaussian [23] profiles in each sub-layer. This is expressed for instance in the sub-layer $\xi_j \quad \xi \quad \xi_{j\ 1}$ (j = 0, 1, ..., N – 1) as:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\int_{\xi_j}^{\xi_{j\ 1}}\theta\,\mathrm{d}\xi \quad \theta_{j\ 1}\frac{\mathrm{d}\xi_{j\ 1}}{\mathrm{d}\tau} \quad \theta_j\frac{\mathrm{d}\xi_j}{\mathrm{d}\tau} \quad \left.\frac{\partial\theta}{\partial\xi}\right|_{\xi_{j\ 1}} \quad \left.\frac{\partial\theta}{\partial\xi}\right|_{\xi_{j\ 1}} \tag{19}$$

where

$$\frac{\mathrm{d}\xi_N}{\mathrm{d}\tau} \quad \frac{\mathrm{d}\Delta}{\mathrm{d}\tau} \quad \text{and} \quad \left.\frac{\partial\theta}{\partial\xi}\right|_N \quad \frac{1}{\mathrm{Ste}}\frac{\mathrm{d}\Delta}{\mathrm{d}\tau}$$

Noble [31] considers subdivision of the space independent variable into equal region depth as follows:

$$\xi_j \quad \xi_{j\ 1} \quad \frac{\Delta}{N} \quad \ldots \quad j\frac{\Delta}{N}, \quad j \quad 0,1,\ldots,N \tag{20.a}$$

Bell [30, 43], on the other hand, modifies the method by subdividing the dependent variable, temperature, into N intervals:

$$\theta_j \quad \theta_{j\ 1} \quad \frac{1}{N}(\theta_\mathrm{m} \quad \theta_0) \quad \ldots \quad \theta_0 \quad \frac{j}{N}(\theta_\mathrm{m} \quad \theta_0), \quad j \quad 0, 1,\ldots,N \tag{20.b}$$

As a result of this procedure, a set of non-linear coupled equations is obtained and solved numerically. Note that when high order-approximation is used, the conditions expressing the continuity of heat flux between each sub-layer and its neighbours must be considered together with those defining temperature continuity [23, 34].

The convergence properties of the heat balance integral method, for conduction problem with [44] or without [45] phase-change, are discussed. It is shown that, with a particular mode of subdivision and a piecewise linear profile, the approximate solution converges formally to the exact solution. However, this convergence towards the exact solution requires an increase of sub-regions number increasing then the number of non-linear coupled equations to solve. As result, the HBI method becomes a numerical procedure similar to a finite element approach and losing then its simplicity.

Some analytical expressions for λ

The various approaches developed, using Goodman HBI technique, to track the moving boundary in the case of the one-phase Stefan problem provided several approximate analytical solutions.

Table 1 summarizes some analytical expressions of the freezing constant. On the overall, all expressions provide accurate localisation of the moving boundary for small Stefan numbers or at the beginning of the process (short times). This is due to the fact that the cumu-

Table 1. Analytical freezing constant expressions obtained by analytical methods developed using heat balance integral technique

Freezing constant expression, λ^2	Authors
$\frac{1}{2}\sqrt{(\mathrm{Ste}\quad 6)^2\quad 12\mathrm{Ste}}\quad(\mathrm{Ste}\quad 6)$ $3\frac{1\quad 2\mathrm{Ste}\quad\sqrt{1\quad \mathrm{Ste}}}{5\quad 2\mathrm{Ste}\quad\sqrt{1\quad 2\mathrm{Ste}}}$	Goodman, T. R. [1], [19]
$3\frac{1\quad \mathrm{Ste}\quad\sqrt{1\quad \mathrm{Ste}}}{2\quad \mathrm{Ste}\quad\sqrt{1\quad \mathrm{Ste}}}$	Poots, G. [12]
$6\frac{1\quad \mathrm{Ste}\quad\sqrt{1\quad 2\mathrm{Ste}}}{1\quad 2\mathrm{Ste}\quad\sqrt{1\quad 2\mathrm{Ste}}}$	Wood, A. S. [38]
$\frac{2\mathrm{Ste}}{4\quad \mathrm{Ste}}$	Mennig, J. and Özisik, M. N. [29]
$\frac{\mathrm{Ste}}{2\quad \mathrm{Ste}}$	Hamill, T. D. and Bankoff, S. G. [27], Elmas, M. [28], Poots, G. [12]
$\frac{\mathrm{Ste}(2\quad \mathrm{Ste}\quad 2\sqrt{1\quad \mathrm{Ste}})}{8(1\quad \mathrm{Ste})}$ $\frac{3\mathrm{Ste}}{5\quad \mathrm{Ste}\quad\sqrt{1\quad 2\mathrm{Ste}}}$	El-Genk, M. S. and Cronenberg, A.W. [42]
$\frac{3\mathrm{Ste}}{2(3\quad \mathrm{Ste})}$	Sadoun, N. and Si-Ahmed, E. K. [40]
$\frac{1}{4}\sqrt{(\mathrm{Ste}\quad 6)^2\quad 24\mathrm{Ste}}\quad(\mathrm{Ste}\quad 6)$	Sadoun, N., Si-Ahmed, E. K., and Colinet, P. [41]

lated heat is negligible in both cases. Furthermore for small Ste the stored heat is small compared to latent heat while for short time the crust layer is very thin. In such cases the temperature profile can be well approximated by a simple linear form. Thus, these approaches provide an efficient device to compute starting solution for numerical schemes when required.

The expression of the freezing constant obtained from exponential and Gaussian basic HBI method [23] and refined HBI with simple exponential distribution [46] are summarized in tab. 2. It should be observed that the solution requires solving transcendental equations.

Table 2. Semi-analytical freezing constant expressions obtained by analytical methods developed using heat balance integral technique

Freezing constant expression, λ^2	Authors
$\frac{\text{Ste}}{2}\frac{\zeta e^{\zeta}}{e^{\zeta}\quad 1}$ where ζ is the root of $[(1+\text{Ste})\zeta \quad \text{Ste}]e^{2\zeta} \quad (2\zeta \quad \text{Ste})e^{\zeta} \quad \zeta \quad 0$ $\frac{\text{Ste}}{2}(1 \quad 2\zeta)$ where ζ is the root of $(1+2\zeta)[2(1 \quad \text{Ste})\zeta e^{\zeta} \quad (1 \quad \zeta)\text{Ste}] \quad 2\zeta \quad 0$	Mosally, F., Wood, A. S., and Al-Fhaid, A. [23]
$\frac{\text{Ste}}{2}\zeta$ where $\zeta \quad \{0, 1\}$ is the root of $\text{Ste}\,\zeta^2 \quad (2\text{Ste} \quad 6)\zeta \quad \sqrt{\frac{72(1 \quad \zeta)}{\text{Ste}}} \quad 12 \quad 0$	Wood, A. S. [38]
$\frac{\text{Ste}}{2}\frac{\zeta e^{\zeta}}{e^{\zeta}\quad 1}$ where ζ is the root of $\zeta(e^{\zeta} \quad 1)(\zeta e^{\zeta} \quad e^{\zeta} \quad 1) \quad \text{Ste}[\zeta(\zeta \quad 2)e^{\zeta} \quad 2(e^{\zeta} \quad 1)]e^{\zeta} \quad 0$	Sadoun, N., Si-Ahmed, E. K., and Legrand, J. [46]

Simple exponential or quadratic profile leads to the same order of accuracy in the Goodman's basic HBI. Gaussian profile provides, as expected, more accurate solution. This can be understood since the latter is suggested by the exact solution. As matter of fact, the first term in the power series of the error function erf(η) is described by ηe^{η^2}. However, in most cases the exact solution is unfortunately unknown beforehand.

Some numerical results, obtained using modified Goodman HBI technique, are reported in tab. 3. The first, considers piecewise implementation of the method with linear temperature profile in each cell [23]. Three domains are considered, 10, 20, and 40 sub-regions. As expected, the rate of convergence of the method is numerically established. The second approach is based on the HBI refinement using the double integral technique. Three applications are as well considered linear, quadratic, and exponential profiles to capture the temperature distribution.

The relative error on the freezing constant λ, given in tab. 1, is plotted as function of the Stefan number in fig. 2. Two important remarks are made. Firstly the alternative Stefan condition inhibits obviously the method accuracy. Secondly the double integral technique improves the accuracy while remaining simple and flexible. In addition, the double technique requires only a linear profile to give results as precise as those obtained from the piecewise implementation (tab. 3).

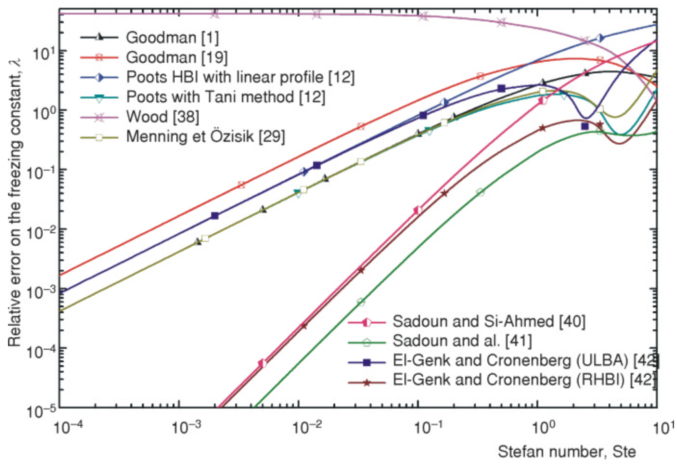

Figure 2. Relative error $\lambda_{exa} - \lambda_{app}$ $/\lambda_{exa}$ [%] on the freezing constant, λ, as given by different approaches constructed using Goodman's HBI technique (color image see on our web site)

Table 3. Values of the freezing constant given by piecewise linear HBI and refined HBI using double integral approach

Ste	Exact solution	Piecewise linear HBI N – number of cells [18]			Refined HBI using double integral approach		
		$N = 10$	$N = 20$	$N = 40$	Linear [40]	Quadratic [41]	Exponential [46]
10^{-3}	0.022357	0.022357	0.022357	0.022357	0.022357	0.022357	0.022357
10^{-2}	0.070593	0.070585	0.070589	0.070591	0.070593	0.070593	0.070593
10^{-1}	0.220016	0.219756	0.219884	0.219950	0.210027	0.219950	0.210028
1	0.620063	0.613967	0.616960	0.618502	0.612372	0.621290	0.621843
10^{2}	1.850946	1.665452	1.756491	1.803263	1.206777	1.641134	2.094461

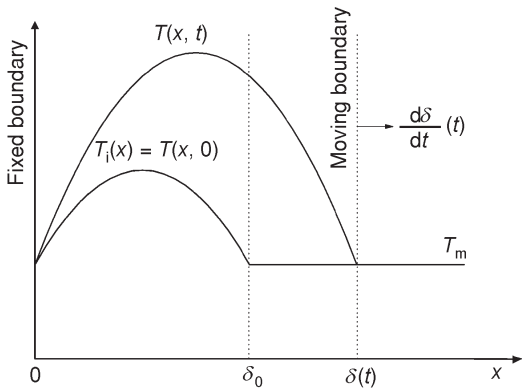

Figure 3. Physical model for one-phase Stefan problem in semi-infinite medium with source/sink in the growth layer (case of melting)

Application to Stefan problem with source term

Consider the cooling of a semi-infinite slab of phase change material that is initially ($t = 0$) at its melting temperature T_m and occupies the region x δ_i ($l_{ref} = \delta_i$ in the present section). The liquid is in contact with a finite slab 0 x δ_i which is solid with initial temperature distribution $T_i(x) = T(x, 0)$. A heat sink is applied to solid region and expressed by the function $S(x, t)$. Initial heat flux at the contact surface causes the liquid to solidify, so that the solid thickness $\delta(t)$ increases as time proceeds (fig. 3).

If constant thermophysical properties are assumed and density change from liquid to solid is ignored, then the heat conduction equation and related initial and boundary conditions are given in the dimensionless form as follows:

$$\frac{\partial\theta}{\partial\tau} \quad \frac{\partial^2 \theta}{\partial\xi^2} \quad \Sigma(\xi,\tau); \quad \tau \quad 0, \quad 0 \quad \xi \quad \Delta(\tau) \tag{21}$$

$$\theta(\xi,\tau) \quad F(\xi), \quad \tau \quad 0, \quad 0 \quad \xi \quad 1 \tag{22}$$

$$\theta(\xi,\tau) = 0, \quad \tau > 0, \quad \xi = 0 \tag{23}$$

$$\theta(\xi,\tau) = 0, \quad \tau > 0, \quad \xi = \Delta(t) \tag{24}$$

$$\frac{\partial\theta}{\partial\xi} \quad \frac{1}{\mathrm{Ste}}\frac{\mathrm{d}\Delta}{\mathrm{d}\tau}, \quad \tau \quad 0, \quad \xi \quad \Delta(\tau) \tag{25}$$

The dimensionless variables $\tau = \alpha t/\delta_i^2$, $\xi = x/\delta_\mathrm{i}$, $\Delta = \delta/\delta_\mathrm{i}$, $\theta = (T - T_\mathrm{m})/(T_\mathrm{ref} - T_\mathrm{m})$, $\Sigma = \delta_i^2 S/k(T_\mathrm{ref} - T_\mathrm{m})$, and $F(\xi) = [T_\mathrm{i}(x) - T_\mathrm{m}]/(T_\mathrm{ref} - T_\mathrm{m})$ refer to dimensionless time, space coordinate, moving front position, temperature, forcing term, and initial temperature distribution, respectively. The initial thickness δ_i is taken as the reference length $l_\mathrm{ref.}$.

The problem is described by a parabolic equation (eq. 21) with non-linear free boundary condition (eq. 25). Closure to the mathematical model is given by the moving boundary initial position $\Delta(\tau) = 1$ for $\tau = 0$.

Approximating profile

Following the HBI implementation procedure outlined above, the first step consists on the choice of the temperature profile in the thermal layer $0 \quad \xi \quad \Delta(\tau)$. Compared to the classical Stefan problem, where the only information available is given by the boundary conditions, the present problem sets the initial temperature distribution $T_\mathrm{i}(x)$ or F in dimensionless form. The profile's shape is assumed to remain the same during the process and given as follows:

$$\theta(\xi,\tau) \quad \zeta(\tau)F \quad \frac{\xi}{\Delta(\tau)} \tag{26}$$

where ζ is a time-dependent function to be determined with the moving front location $\Delta(\tau)$. Its initial value, $\zeta(0) = 1$, is obtained considering the initial temperature distribution (eq. 22). It should be noted that the assumed profile verifies the boundary conditions (23, 24) and its substitution into the Stefan condition (eq. 25) leads to:

$$\frac{\mathrm{d}\Delta^2}{\mathrm{d}\tau} \quad 2\mathrm{Ste}\left.\frac{\partial F}{\partial\eta}\right|_{\eta \quad 1} \zeta \tag{27}$$

where $\eta = \xi/\Delta(\tau)$.

Heat-balance integral equation

The second step establishes the energy balance integral equation by integrating the heat diffusion equation with respect to space and over the layer, $0 \quad \xi \quad \Delta(\tau)$ with the condition that the Stefan equation at the moving boundary is verified, as advised by Wood [38]:

$$\frac{d}{\mathrm{d}\tau}{}_{\xi\ 0}^{\xi\ \Delta}\theta(\xi,\tau)\mathrm{d}\xi \quad \frac{1}{\mathrm{Ste}}\frac{\mathrm{d}\Delta}{\mathrm{d}\tau} \quad \left.\frac{\partial\theta}{\partial\xi}\right|_{\xi\ 0} \quad {}_{\xi\ 0}^{\xi\ \Delta} \quad (\xi,\tau)\mathrm{d}\xi \tag{28}$$

Expression of assumed profile (26) combines with the eq. (28) to give:

$$\frac{\mathrm{d}\zeta}{\mathrm{d}\tau} \quad \frac{\zeta}{\Delta^2} \quad \frac{1}{2}\frac{\mathrm{d}\Delta^2}{\mathrm{d}\tau} \quad \frac{\left.\dfrac{\partial F}{\partial \eta}\right|_{\eta \;\; 1} \quad \left.\dfrac{\partial F}{\partial \eta}\right|_{\eta \;\; 0} \quad \dfrac{\Delta}{\zeta}{}_{\xi \;\; 0}^{\xi \;\; \Delta} \quad (\xi,\tau)\mathrm{d}\xi}{{}_{\eta=0}^{\eta=1} F(\eta)\mathrm{d}\eta} \tag{29}$$

Introducing given expressions of the initial temperature distribution, $F(\eta)$, and forcing term, $\Sigma(\xi,\tau)$ in eqs. (27) and (29) above, a pair of first order ordinary differential equations is obtained. The result is adequate to determine the two unknown functions $\Delta(\tau)$ and $\zeta(\tau)$ subject to the initial conditions:

$$\Delta(\tau = 0) = 1 \tag{30}$$

$$\zeta(\tau = 0) = 1 \tag{31}$$

Test problem

Fasano *et al.* [47] investigated the problem and demonstrated that it has a unique solution provides that the functions $\Sigma(\xi,\tau)$ and $F(\xi)$ are sufficiently regular. In the particular case of Ste = 1 and for the initial temperature distribution and the forcing function defined by the following expressions:

$$F(\xi) \quad \xi(1 \quad \xi) \tag{32}$$

$$\Sigma(\xi,\tau) \quad \xi\mathrm{e}^{\tau} \quad 2 \tag{33}$$

Exact analytical solution is derived from their analysis. It expresses the moving boundary location and the temperature distribution for as follows:

$$\Delta(\tau) = \mathrm{e}^{\tau} \tag{34}$$

$$\theta(\xi,\tau) \quad \xi(\Delta \quad \xi), \quad \text{for } 0 \quad \xi \quad \Delta(\tau) \tag{35}$$

From the initial temperature distribution, one deduces the approximating profile to be considered in this case. Thus:

$$\theta(\xi,\tau) \quad \zeta\frac{\xi}{\Delta} \; 1 \quad \frac{\xi}{\Delta} \tag{36}$$

Setting and making use of eqs. (32) and (33) where $F(\xi)$ and $\Sigma(\xi,\tau)$ are defined, eqs. (28) and (29) lead to the following second order ordinary differential equation:

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}\tau^2} \quad \frac{1}{2\sigma} \quad \frac{\mathrm{d}\sigma}{\mathrm{d}\tau}^{2} \quad \frac{12}{\sigma}\frac{\mathrm{d}\sigma}{\mathrm{d}\tau} \quad 6\mathrm{Ste}(\mathrm{e}^{\tau}\sqrt{\sigma} \quad 4) \tag{37}$$

Time-dependent parameter is deduced from Stefan condition:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\tau} \quad 2\mathrm{Ste}\zeta \tag{38}$$

Combined with initial, conditions (eqs. 30-31) provide starting values $\sigma = 1$ and $\mathrm{d}\sigma/\mathrm{d}\tau =$ $= 1$ at $\tau = 0$, required by integration of differential eq. (37). The temperature distribution is expressed as follows:

$$\theta(\xi,\tau) \quad \frac{1}{2\mathrm{Ste}}\frac{\mathrm{d}\Delta^2}{\mathrm{d}\tau}\frac{\xi}{\Delta} \quad 1 \quad \frac{\xi}{\Delta} \tag{39}$$

Numerical integration can be performed using available softwares like Mathematica or Matlab. For that purpose, eq. (37) was converted to an equivalent coupled system of two first order differential equations. The dimensionless freezing front velocity, ($\mathrm{d}\Delta/\mathrm{d}\tau$), *vs.* the dimensionless time is plotted in fig. 4 for different values of the dimensionless latent heat (Stefan number). The solution given by numerical method based on modified boundary immobilisation technique is plotted for comparison.

It is important to note that for Ste = 1, the HBI method leads to the exact solution. This can be checked by setting $\sigma = \mathrm{e}^{2\tau}$ in eqs. (37) and (39). The exact solution is then obtained.

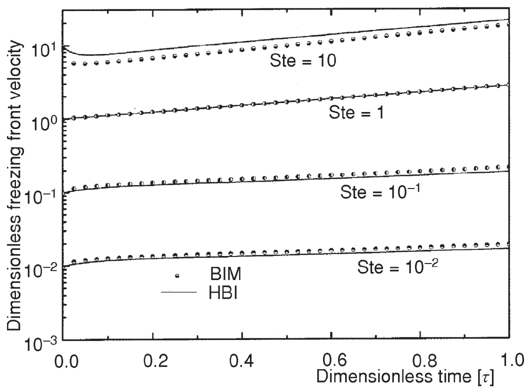

Figure 4. Dimensionless freezing front velocity as given by Goodman heat-balance integral (HBI) and numerical moving immobilisation (BIM) methods

Conclusions

Through the present paper, Goodman HBI method is discussed with illustration by application to a classical one-phase Stefan problem. The main advantage of the HBI method lies in the remarkable association of simplicity, flexibility, and acceptable accuracy. However, the choice of the best trial polynomial for the description of the temperature distribution is recognized to be the important problem of the HBI approach. Investigations conducted to its refinement, lead generally to results with restricted application domains or to solutions losing the simplicity of the method. It should be noted that the double integral method is the approach which, not only decreases the sensitivity to the choice of the trial function but it also keeps the simplicity and flexibility of the HBI method. Application of the method when the form of the temperature distribution is known shows that the technique may lead to the exact solution in some cases.

On the overall, the technique even used in its original form remains a very good tool to generate a satisfactory solution for transient non-linear diffusion problems, with less computing time than a classical numerical approach.

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Nomenclature

c – specific heat, [$\mathrm{Jkg^{-1}K^{-1}}$]
F – dimensionless initial temperature distribution $\{=[T_\mathrm{i}(x) - T_\mathrm{m})/(T_\mathrm{ref} - T_\mathrm{m})]\}$
f – initial temperature distribution
k – thermal conductivity, [$\mathrm{Wm^{-1}K^{-1}}$]
L – specific latent heat of solidification, [$\mathrm{Jkg^{-1}}$]

l – length, [m]
S – forcing term, [Wm^{-1}K^{-1}]
Ste – Stefan number [$= \rho c(T_0 - T_m)/\rho L$], [–]
T – temperature, [K]
t – time, [s]
x – spatial coordinate, [m]

Greek letter

α – thermal diffusivity [$= k/(\rho c)$], [m^{2}s^{-1}]
Δ – dimensionless freezing front position ($= \delta/\delta_i$)
δ – freezing front position, [m]
ζ – dimensionless shape function
η – Landau of variable ($= \xi/\Delta = x/\delta$), [–]
θ – dimensionless temperature [$= (T - T_m)/(T_{ref} - T_m)$]
λ – freezing constant, [–]
ξ – dimensionless spatial coordinate ($= \xi/\delta_i$)
ρ – density, [kgm^{-3}]
Σ – dimensionless forcing term – [$\delta_i^2 S/k(T_{ref} - T_m)$]
σ – intermediate variable ($= \Delta^2$), [–]
τ – dimensionless time ($= \alpha t/\delta_i^2$)

Subscripts

app – approached value
exa – exact value
i – initial value
m – melting or freezing point
ref – reference
0 – relative to $x = 0$

Functions

e – exponential function
erf – error function, $\text{erf}(x) = 2/\pi^{1/2} {}_0^x \text{e}\ {}^{t^2} \text{d}t$

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