A BRIEF REVIEW OF SEVERAL NUMERICAL METHODS FOR ONE-DIMENSIONAL STEFAN PROBLEMS

by

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This paper describes and compares several effective methods for the numerical solution of one-dimensional Stefan problems. The intention is not to make an exhaustive review and so we restrict our attention to a range of problems and geometries which include melting in the halfplane, outward cylindrical solidification and outward spherical solidification. Effectively, a range of methods is introduced for the solution of Stefan problems, including (1) enthalpy method, (2) boundary immobilization method, (3) perturbation method, (4) nodal integral method, and (5) heat--balance integral method. The above methods are then applied to a selection of test problems. As a result of this comparison some helpful comments can be made and conclusions drawn which may prove valuable in the future use of these methods.

Key words: *moving boundary problem, Stefan problem*

Introduction

Phase change problems, also known as the Stefan problems, occur naturally in many physical processes, such as, freezing and thawing of foods, production of ice, ice formation on pipe surface, solidification of steel, and chemical reaction. Mathematically, melting/solidification problems are special cases of moving boundary problems. Problems in which the solution of a differential equation has to satisfy conditions on the boundary of a prescribed domain are referred to as boundary-value problems. However, in the case of melting/solidification, the boundary of the domain is not known in advance. This means that the solution of such problems requires solving the diffusion or heat conduction equation in an unknown region which has to be determined as part of the solution.

There are very limited analytical solutions to melting/solidification problems and existing closed-form solutions to these significant problems are highly restrictive as to permissible initial and boundary conditions. So numerical solution becomes the main tool in the study of moving-boundary problems. Two conditions are required in order to solve these moving-boundary problems, one to determine the boundary itself and the other to complete the definition of the solution of the differential equation.

This paper involves a brief review of recent key numerical methods for one-dimensional Stefan problems for simple geometries including plane, cylindrical, and spherical. It is important to note that the review is not intended to be exhaustive. Numerical results are obtained from a range of methods researched by the authors, including the enthalpy method, boundary immobilization method (BIM), perturbation method, nodal integral method (NIM), and heat-

-balance integral method (HBIM). By comparing results, and in some cases making comparisons with analytical solutions (where possible), some constructive comments can be made which will provide useful guidelines for the future use of these methods.

Problem formulation

Melting in the half-plane

Consider the melting of certain material initially at its freezing temperature T_f in the half-plane $x > 0$ subject to a time-dependent temperature change at $x = 0$. The dimensionless governing equation for the process is:

$$\frac{\partial T}{\partial t} \quad \frac{\partial^2 T}{\partial x^2}, \quad 0 \quad x \quad s(t), \quad t \quad 0 \tag{1}$$

subject to boundary conditions

$$T(x = 0, t) = \mathrm{f}(t), \quad T\,[x = s(t),\, t] = 0, \tag{2}$$

$$\frac{\mathrm{d}s}{\mathrm{d}t} \quad \alpha \quad \frac{\partial T}{\partial x}_{x \quad s(t)} \tag{3}$$

where T is the temperature, x is the space variable, $s(t)$ is the position of the moving boundary, and $\alpha = c(T_f - T_{ref})/L$ is the Stefan number, where c is the specific heat, L is the latent heat, and T_{ref} is some reference temperature. For example, one can select T_{ref} such that $\mathrm{f}(t = 0) = 1$ or $\max_{0 \quad t \quad t_{final}} \left|\mathrm{f}(t)\right| \quad 1$.

Outward cylindrical solidification

Consider the outward cylindrical solidification of a saturated liquid due to low temperature at the boundary. The problem can be formulated as:

$$\frac{\partial T}{\partial t} \quad \frac{1}{r}\frac{\partial}{\partial r} \quad r\frac{\partial T}{\partial r} \quad , \quad 1 \quad r \quad s(t), \quad t \quad 0 \tag{4}$$

$$T(r = 1,\, t) = \mathrm{f}(t), \qquad T\,[r = s(t),\, t] = 1 \tag{5}$$

$$\frac{\mathrm{d}s}{\mathrm{d}t} \quad \alpha \quad \frac{\partial T}{\partial r}_{r \quad s(t)} \tag{6}$$

Outward spherical solidification

In the case of outward spherical solidification, the corresponding governing equation is:

$$\frac{\partial T}{\partial t} \quad \frac{1}{r}\frac{\partial^2 (rT)}{\partial r^2}, \quad 1 \quad r \quad s(t), \quad t \quad 0 \tag{7}$$

subject to boundary conditions (5) and (6).

Numerical methods

In this section we introduce a range of methods for the solution of Stefan problems. In all methods only the formulation for the plane geometry is described since other applications follow the same idea. Readers may refer to specific papers listed in the references for further details of the methods.

Enthalpy method

The enthalpy formulation is one of the most popular fixed-domain methods for solving the Stefan problem. In the formulation, the enthalpy function is introduced such that the flux condition is automatically satisfied across the phase front, which is realized as a jump discontinuity of the enthalpy. Date [1] has developed an enthalpy method which tracks the phase front easily.

He has applied this method to one and two dimensional problems in plane geometry and has obtained good agreement with existing solutions. More recently, Caldwell *et al*. [2, 3] have also successfully applied the method to cylindrical and spherical geometries. Extension of the method to higher-dimensional problems can be found in Caldwell *et al.* [4].

First, the enthalpy function H is defined by:

$$H = T + \alpha' f_l(T) \tag{8}$$

where $\alpha' = 1/\alpha$ and f_l is the local liquid fraction given by:

$$f_l(T) \quad \begin{matrix} 1 \text{ if } T & 1 \\ 0 \text{ if } T & 0 \end{matrix}$$

Hence H is identical to the temperature except when phase change occurs, in which case H has a jump of α'. Substituting H into the heat equation, we obtain:

$$\frac{\partial H}{\partial t} \quad \frac{\partial^2 H}{\partial x^2} \tag{9}$$

Discretization of eq. (9) will result in a set of non-linear equations. Date [1] introduces a simple method which at the same time provides an effective means of tracking the phase boundary. From eq. (8) we can write $T = H + H'$, where:

$$H \quad \alpha f_l(T) \quad \begin{matrix} \alpha & \text{if } H & \alpha \\ H & \text{if } 0 & H & \alpha \\ 0 & \text{if } H & 0 \end{matrix} \tag{10}$$

Also, we note that $-H'/\alpha'$ is the local liquid fraction while $1 + H'/\alpha'$ is the local solid fraction.

The implicit discretization of eq. (9) is:

$$\frac{H_i^{(k\ 1)} \quad H_i^{(k)}}{\Delta t} \quad \frac{T_{i\ 1}^{(k\ 1)} \quad 2T_i^{(k\ 1)} \quad T_{i\ 1}^{(k\ 1)}}{(\Delta x)^2}, \quad i \quad 1, 2, \ldots, N \quad 1 \tag{11}$$

where x and t represent the space and time steps, respectively. Using the relation $T_i^{(k)} \quad H_i^{(k)} \quad H_i^{(k)}$ with $H_i^{(k)}$ obtained from eq. (10), we have:

$$\gamma H_{i\ 1}^{(k\ 1)} \quad (1 \quad 2\gamma) H_i^{(k\ 1)} \quad \gamma H_{i\ 1}^{(k\ 1)} \quad H_i^{(k)} \quad \gamma [H_{i\ 1}^{(k\ 1)} \quad 2H_i^{(k\ 1)} \quad H_{i\ 1}^{(k\ 1)}] \qquad (12)$$

where $\gamma = \ t/(\ x)^2$. This results in a set of nonlinear equations. To solve this system we employ an iterative scheme, where terms involving H' are set to lag behind terms involving H for one iteration. Using the value of H from the previous time step as the initial guess, the values of H' are calculated from eq. (10). The new value of H is then obtained from eq. (12). This process is continued until the iterations converge. Then we can continue to the next time step. Note that each iteration involves solving a tridiagonal system, and can be done effectively by the Thomas algorithm.

Recalling that H_i/α' is the liquid fraction in the i^{th} control volume, there is a simple way to calculate the position of the phase front. Consider the integral I, which represents the volume of solid in the range $0 \quad x \quad s(t)$:

$$I \quad \int_0^{s(t)} \mathrm{d}x \quad \Delta x \sum_{i\ 1}^{N} \frac{H_{\mathrm{i}}}{\alpha} \quad \frac{\Delta x}{2}$$

The last term is due to the fact that the first cell is always occupied by liquid. Here, the summation can be carried out over all the cells since the cells behind the phase-front give zeros to H_i/α'. Hence we have $s(t) = I$.

Note that in the cases of outward cylindrical and spherical solidification, there are small differences in the formulae for I and $s(t)$. The corresponding equations for outward cylindrical solidification are:

$$I \quad \int_1^{s(t)} r\mathrm{d}r \quad \Delta r \sum_{i\ 1}^{N} r_{\mathrm{i}} \ 1 \quad \frac{H_{\mathrm{i}}}{\alpha} \quad \frac{\Delta r}{2}$$

$$s(t) \quad \sqrt{1 \quad 2I}$$

and the equations for outward sperical solidification are:

$$I \quad \int_1^{s(t)} r^2\mathrm{d}r \quad \Delta r \sum_{i\ 1}^{N} r_{\mathrm{i}}^2 \ 1 \quad \frac{H_{\mathrm{i}}}{\alpha} \quad \frac{\Delta r}{2}$$

$$s(t) \quad \sqrt[3]{1 \quad 3I}$$

Boundary immobilization method (BIM)

With a suitable transformation, it is possible to fix the moving boundary. This method was first applied to a finite difference scheme by Crank [5]. Kutluay *et al*. [6] have also successfully applied the method to various problems.

Under the transformation:

$$x^* \quad \frac{x}{s}, \quad T^*(x^*, t) \quad T(x, t)$$

eqs. (1)-(3) can be transformed to one in the fixed domain $0 \quad r \quad 1$:

$$s^2 \frac{\partial T^*}{\partial t} \quad \frac{\partial^2 T^*}{\partial x^{*2}} \quad x^* s \frac{\mathrm{d}s}{\mathrm{d}t} \frac{\partial T^*}{\partial x^*} \qquad (13)$$

subject to

$$T^*(x^* = 0,\ t) = \mathrm{f}(t),\ T^*(x^* = 1,\ t) = 0, \qquad (14)$$

$$s\frac{\mathrm{d}s}{\mathrm{d}t} \quad \alpha \left.\frac{\partial T^*}{\partial x^*}\right|_{x^* \ 1} \tag{15}$$

A finite difference discretization of eq. (13) implicit in T^* and explicit in s is:

$$a_i^{(k \ 1)} T_{i \ 1}^{(k \ 1)} \quad b_i^{(k \ 1)} T_i^{(k \ 1)} \quad c_i^{(k \ 1)} T_{i \ 1}^{(k \ 1)} \quad (s^{(k)})^2 T_i^{(k)} \tag{16}$$

where

$$\begin{aligned} a_i^{(k \ 1)} \quad & \gamma \ \frac{\Delta x}{2} x_i s^{(k)} \left.\frac{\mathrm{d}s}{\mathrm{d}t}\right|^{(k)} \quad 1 \\ b_i^{(k \ 1)} \quad & (s^{(k)})^2 \quad 2\gamma \\ c_i^{(k \ 1)} \quad & a_i^{(k \ 1)} \quad 2\gamma \\ \left.\frac{\mathrm{d}s}{\mathrm{d}t}\right|^{(k)} \quad & \frac{\alpha}{s^{(k)}} (4T_{N \ 1}^{(k)} \quad T_{N \ 2}^{(k)}) \end{aligned} \tag{17}$$

and $\gamma = \ t/(\ x)^2$. At each time step, the temperature distribution is obtained by solving eq. (16) and the position of the moving boundary is updated via the formula:

$$s^{(k \ 1)} \quad s^{(k)} \quad \left.\frac{\mathrm{d}s}{\mathrm{d}t}\right|^{(k)} \Delta t \tag{18}$$

Note that a starting solution for small time is required by the BIM. For plane geometry one can use the analytic solution for the problem with constant boundary condition as the starting solution [7]. Readers may refer to Caldwell *et al*. [8] for the starting solutions for other geometries.

Perturbation method

The perturbation method only works for small Stefan number. It has been successfully applied to Stefan problems with simple boundary conditions in different geometries see [9-11]. More recently, Caldwell *et al*. [12] successfully applied the method to Stefan problems with time-dependent boundary conditions.

Since $s(t)$ is expected to be a monotonic function of t, we may replace t by s as the second independent variable in the governing equations. By making use of eq. (3), eq. (1) can be written as:

$$\frac{\partial^2 T}{\partial x^2} \quad \alpha \left.\frac{\partial T}{\partial s} \ \frac{\partial T}{\partial x}\right|_{x \ s} \tag{19}$$

On the other hand, the boundary condition at $x = 0$ is written as:

$$T = \mathrm{f}(t) = \mathrm{F}(s) \text{ on } x = 0 \tag{20}$$

We now derive a three term perturbation solution of the form:

$$T(x, s) = T_0(x, s) + \alpha T_1(x, s) + \alpha^2 T_2(x, s) \tag{21}$$

Substituting eq. (21) into eqs. (19) and (20), the governing equations for T_0, T_1, and T_2 are:

$$
\begin{aligned}
&\alpha^0: \quad \frac{\partial^2 T_0}{\partial x^2} \quad 0 \\
&\qquad T_0(x \quad 0,s) \quad \mathrm{F}(s),\ T_0(x \quad s,s) \quad 0 \\
&\alpha: \quad \frac{\partial^2 T_1}{\partial x^2} \quad \frac{\partial T_0}{\partial s} \quad \frac{\partial T_0}{\partial x}_{x\ y} \\
&\qquad T_1(x \quad 0,s) \quad 0,\ T_1(x \quad s,s) \quad 0 \\
&\alpha^2: \quad \frac{\partial^2 T_2}{\partial x^2} \quad \frac{\partial T_0}{\partial s} \quad \frac{\partial T_1}{\partial x}_{x\ s} \quad \frac{\partial T_1}{\partial s} \quad \frac{\partial T_0}{\partial x}_{x\ s} \\
&\qquad T_2(x \quad 0,s) \quad 0,\ T_2(x \quad s,s) \quad 0
\end{aligned}
\tag{22}
$$

The solutions of the above equations are:

$$
\begin{aligned}
T_0(x,s) \quad & \mathrm{F}(s)(1 \quad x^*), \\
T_1(x,s) \quad & \frac{1}{6}\mathrm{F}(s)x^*(x^* \quad 1)[\mathrm{F}(s)(x^* \quad 1) \quad \mathrm{F}(s)s(x^* \quad 2)], \\
T_2(x,s) \quad & \frac{1}{360}\mathrm{F}(s)x^*(x^* \quad 1)[\mathrm{F}(s)(x^* \quad 1)(9x^{*2} \quad 19) \quad 10\mathrm{F}(s)^2 y^2(x^* \quad 4) \\
& 5\mathrm{F}(s)\mathrm{F}(s)s(3x^{*2} \quad 5x^* \quad 17) \quad \mathrm{F}(s)\mathrm{F}(s)s^2(x^* \quad 2)(3x^{*2} \quad 6x^* \quad 4)]
\end{aligned}
\tag{23}
$$

where $x^* = x/s$. Thus, the position of the moving boundary follows the equation:

$$
\begin{aligned}
&\frac{\mathrm{d}s}{\mathrm{d}t} \quad \alpha \frac{\partial T_0}{\partial x} \quad \alpha \frac{\partial T_1}{\partial x} \quad \alpha^2 \frac{\partial T_2}{\partial x}_{x\ s} \\
&\qquad \frac{\alpha}{s}\mathrm{F}(s) \quad \alpha^2 \mathrm{F}(s) \frac{1}{6}\mathrm{F}(s) \quad \frac{1}{3s} \\
&\alpha^3 \mathrm{F}(s) \frac{7}{45s}\mathrm{F}(s)^2 \quad \frac{5}{36}\mathrm{F}(s)^2 \quad \frac{25}{72}\mathrm{F}(s)\mathrm{F}(s) \quad \frac{13}{360}\mathrm{F}(s)\mathrm{F}(s)s
\end{aligned}
\tag{24}
$$

The final step is to substitute back f(t) for F(s). With the relations:

$$
\frac{\mathrm{dF}(s)}{\mathrm{d}s} \quad \frac{\mathrm{df}(t)}{\mathrm{d}t} \quad \frac{\mathrm{d}s}{\mathrm{d}t}^1, \quad \frac{\mathrm{d}^2\mathrm{F}(s)}{\mathrm{d}s^2} \quad \frac{\mathrm{d}^2\mathrm{f}(t)}{\mathrm{d}t} \quad \frac{\mathrm{d}s}{\mathrm{d}t}^2
$$

eq. (24) can be rewritten in the form:

$$
\frac{\mathrm{d}s}{\mathrm{d}t}^3 \quad a(t,s) \frac{\mathrm{d}s}{\mathrm{d}t}^2 \quad b(t,s)\frac{\mathrm{d}s}{\mathrm{d}t} \quad c(t,s) \quad 0
$$

where

$$
\begin{aligned}
a(t,s) \quad & \frac{\alpha \mathrm{f}(t)}{s} \ 1 \quad \frac{\alpha}{3}\mathrm{f}(t) \quad \frac{7\alpha^2}{45}\mathrm{f}(t)^2 \\
b(t,s) \quad & \alpha^2 \mathrm{f}(t)\mathrm{f}(t) \frac{1}{6} \quad \frac{25\alpha}{72}\mathrm{f}(t) \\
c(t,s) \quad & \alpha^3 \mathrm{f}(t)s \frac{5}{36}\mathrm{f}(t)^2 \quad \frac{13}{360}\mathrm{f}(t)\mathrm{f}(t)
\end{aligned}
\tag{25}
$$

By solving eq. (25), the value of ds/dt is obtained and s can be found by numerical integration.

On the other hand, the temperature distribution can be obtained by substituting eq. (16) into eq. (14).

Nodal integral method (NIM)

A semi-analytical nodal method to solve the one-dimensional Stefan problem was developed by Rizwan-Uddin [13]. We give a brief description of the method here. Consider the transformed heat eq. (13). The space-time domain ($0 \quad x^* \quad 1; 0 \quad t \quad t_{\text{final}}$) is first discretized into space-time nodes. Each node is identified by the subscript (*i, k*). The space-averaged, time-dependent temperature and time-averaged, space-dependent temperature for each node are, respectively, defined as:

$$\overline{T}_i^{\,x}(t) \quad \frac{1}{\Delta x}\int_{x_i}^{x_{i\ 1}} T(x^*,t)dx^*, \quad \overline{T}_k^{\,t}(x^*) \quad \frac{1}{\Delta t}\int_{t_{k\ 1}}^{t_k} T^*(x^*,t)\mathrm{d}t$$

First, eq. (15) is integrated over the time step $t_{k-1} \quad t \quad t_k$ to yield:

$$s^2(t) \quad s^2(t_{k\ 1}) \quad 2\alpha\frac{\mathrm{d}\overline{T}_k^{\,t}(x^* \quad 1)}{\mathrm{d}x^*}(t \quad t_{k\ 1}), \quad t_{k\ 1} \quad t \quad t_k$$

Next, for each space-time node, a time-step-averaged, second-order ordinary differential equation (ODE) is obtained for $\overline{T}_k^{\,t}(x^*)$ by operating on eq. (13) with $(1/\Delta t)\int_{t_{k\ 1}}^{t_k}\mathrm{d}t$, and a space-averaged, first-order ODE for $\overline{T}_i^{\,x}(t)$ is obtained by operating on eq. (13) with $(1/\Delta x)\int_{x_i}^{x_{i\ 1}}\mathrm{d}x^*$. After introducing some simplifying assumptions, the second-order ODE in space is solved using the Dirichlet boundary conditions at the left and right edge of the node, leading to a solution of the form:

$$\overline{T}_k^{\,t}(x^*) \quad C_1 \quad C_2x^* \quad g\exp(\ C_3x^*)$$

where C_{m} ($m = 1, 2, 3, \ldots$) are constants. On the other hand, the first-order ODE in time is solved using the initial condition at the beginning of the time step, leading to a solution of the form:

$$\overline{T}_i^{\,x}(t) \quad \overline{T}_i^{\,x}(t_{k\ 1}) \quad C_4\ln[1 \quad C_5(t \quad t_{k\ 1})]$$

A set of coupled algebraic equations is obtained at each time step by requiring continuity of time-step-averaged temperature and its spatial derivative at each node interface. This set of coupled equations is then solved iteratively for time-step-averaged temperature at each location $\overline{T}_k^{\,t}(x^* \quad x_i)$ and space-averaged-temperature at the next time step $\overline{T}_i^{\,k}$ ($t = t_{\text{k}}$).

Further details of the method and its application to two problems with time-dependent boundary conditions are given by Caldwell *et al*. [14]. A comparison of numerical results with those from the enthalpy method is also included.

Heat-balance integral method (HBIM)

The HBIM was first proposed by Goodman [15, 16]. Goodman's idea is to assume a particular temperature profile, and integrate the heat equation over an appropriate interval to obtain a set of heat-balance integral equations. The equations are then solved to obtain the motion of the phase boundary. Later Bell [17] proposed a systematic method to improve the accuracy of HBIM, which we will present below. The main idea is to subdivide the dependent variable *T*, and assume a linear profile within each subdivision. Accuracy can be improved by increasing the number of subdivisions. Refinements of the method have been proposed by several authors, including the exponential HBIM [18] and the refined integral method [19]. More recently, Ren [20] successfully applied the method to the inverse Stefan problem. Convergence analysis of the method can be found in [21].

For simplicity, we only consider melting in the half-plane *i. e*. eqs. (1)-(3), with f(t) = 1. A detailed description of the method for outward cylindrical and spherical solidifications can be found in [22]. First we divide the range [0, 1] into N parts, that is:

$$T_i \quad \frac{i}{N}$$

and denote the corresponding position of the isotherm by Z_i. Assume a linear profile within each subdivision [Z_i, Z_i +1]:

$$T(x) \quad \frac{i \quad 1}{N} \quad \frac{x \quad Z_{i\ 1}}{N(Z_i \quad Z_{i\ 1})}, \quad Z_{i\ 1} \quad x \quad Z_i \tag{26}$$

Integrating eq. (4) over [Z_i + 1, Z_i] gives:

$$\int_{Z_{i\ 1}}^{Z_i} \frac{\partial T}{\partial t} \mathrm{d}x \quad \left.\frac{\partial T}{\partial x}\right|_{Z_i} \quad \left.\frac{\partial T}{\partial x}\right|_{Z_{i\ 1}}$$

Taking the derivative outside the integral sign, we obtain:

$$\frac{d}{\mathrm{d}t} \int_{Z_{i\ 1}}^{Z_i} T \mathrm{d}x \quad Z_i T_i \quad Z_{i\ 1} T_{i\ 1} \quad \left.\frac{\partial T}{\partial x}\right|_{Z_i} \quad \left.\frac{\partial T}{\partial x}\right|_{Z_{i\ 1}}$$

Replacing T by the linear profile and ensuring that the expression representing change in flux is approximated by the discontinuous change in adjacent profile gradients, we obtain a system of ordinary differential equations for the penetration depth Z_i, namely:

$$\begin{gathered} \frac{\mathrm{d}}{\mathrm{d}t} \quad 1 \quad \frac{2N}{\alpha} \; Z_0 \quad Z_1 \quad \frac{2}{Z_0 \quad Z_1} \\ \frac{\mathrm{d}}{\mathrm{d}t}(Z_i \quad Z_{i\ 1}) \quad \frac{2}{Z_i \quad Z_{i\ 1}} \quad \frac{2}{Z_{i\ 1} \quad Z_i}, \quad i \quad 1, \ldots, N \quad 2 \\ \frac{\mathrm{d}}{\mathrm{d}t} Z_{N\ 1} \quad \frac{2}{Z_{N\ 1}} \quad \frac{2}{Z_{N\ 2} \quad Z_{N\ 1}} \end{gathered} \tag{27}$$

It can be seen that the above system is stiff, at least for small t where the distances between adjacent isotherms are small. Hence a stiff ODE solver is required to solve the system. Besides, a starting solution is required by the HBIM. Caldwell *et al*. [23] constructed special starting solutions for outward cylindrical and spherical solidifications. Alternatively, one can also use the starting solutions mentioned in the BIM.

Numerical results and discussions

We present and discuss the numerical results in applying the above methods to different test cases.

Example 1

The first example corresponds to the melting in plane geometry with $\alpha = 0.2$ and f(t) = 1. The analytic solution to the problem is:

$$T(x,t) \quad 1 \quad \frac{\mathrm{erf}\ \dfrac{x}{2\sqrt{t}}}{\mathrm{erf}\,\lambda}, \quad s(t) \quad 2\lambda\sqrt{t} \tag{28}$$

where erf denotes the error function and λ is the solution of the transcendental equation:

$$\sqrt{\pi}\lambda \exp(\lambda^2)\mathrm{erf}(\lambda) \quad \alpha \tag{29}$$

The numerical results for this example are presented in tab. 1. Note that an adaptive ODE solver is used in the perturbation method and HBIM and so the time step is not constant.

Table 1. Melting in plane geometry (α = 0.2, f(t) = 1)

Time	Exact	Enthalpy	BIM	Perturbation	NIM	HBIM
0.200	0.27407	0.27425	0.27456	0.27416	0.27410	0.27367
0.400	0.38759	0.38756	0.38802	0.38772	0.38764	0.38699
0.600	0.47470	0.47487	0.47509	0.47486	0.47476	0.47404
0.800	0.54814	0.54811	0.54850	0.54832	0.54821	0.54747
1.000	0.61284	0.61291	0.61318	0.61304	0.61292	0.61214
1.200	0.67133	0.67135	0.67166	0.67156	0.67142	0.67061
1.400	0.72512	0.72520	0.72543	0.72536	0.72521	0.72437
1.600	0.77519	0.77526	0.77549	0.77545	0.77529	0.77440
1.800	0.82221	0.82225	0.82250	0.82248	0.82232	0.82140
2.000	0.86669	0.86668	0.86697	0.86697	0.86680	0.86585
N		100	100	n. a.	8	32
Δt		0.001	0.001	n. a.	0.01	n. a.

Example 2

The second example corresponds to the melting in plane geometry with $\alpha = 1.0$ and $f(t) = \exp(t) - 1$. The analytic solution to the problem is:

$$T(x, t) = \exp(t - x) - 1, \quad s(t) = t \tag{30}$$

The numerical results for this example are presented in tab. 2.

Table 2. Melting in plane geometry [α = 1.0, f(t) = exp(t) – 1]

Time	Exact	Enthalpy	BIM	NIM
0.100	0.10000	0.10053	0.10002	0.10000
0.200	0.20000	0.20053	0.20003	0.20000
0.300	0.30000	0.30055	0.30005	0.29999
0.400	0.40000	0.40057	0.40006	0.39998
0.500	0.50000	0.50059	0.50008	0.49997
0.600	0.60000	0.60062	0.60009	0.59994
0.700	0.70000	0.70064	0.70010	0.69991
0.800	0.80000	0.80067	0.80011	0.79986
0.900	0.90000	0.90070	0.90013	0.89981
1.000	1.00000	1.00072	1.00014	0.99974
N		100	100	8
Δt		0.001	0.001	0.01

Example 3

This example corresponds to the outward cylindrical solidification with $\alpha = 0.2$ and $f(t) = 0$. There is no known analytical solution to the problem. The numerical results for this example are presented in tab. 3.

Table 3. Outward cylindrical solidification (α = 0.2, f(t) = 0)

Time	Enthalpy	BIM	Perturbation	HBIM
0.200	1.26415	1.26504	1.26421	1.26349
0.400	1.36847	1.36932	1.36869	1.36766
0.600	1.44715	1.44786	1.44735	1.44603
0.800	1.51264	1.51318	1.51274	1.51114
1.000	1.56954	1.57010	1.56971	1.56785
1.200	1.62058	1.62108	1.62073	1.61863
1.400	1.66711	1.66759	1.66727	1.66492
1.600	1.71013	1.71057	1.71027	1.70770
1.800	1.75027	1.75067	1.75040	1.74760
2.000	1.78799	1.78839	1.78814	1.78512
N	100	100	n. a.	32
Δt	0.001	0.001	n. a.	n. a.

Example 4

The last example corresponds to the outward spherical solidification with α = 0.2 and f(t) = 0. There is also no known analytical solution to the problem. The numerical results for this example are presented in tab. 4.

Table 4. Outward spherical solidification (α = 0.2, f(t) = 0)

Time	Enthalpy	BIM	Perturbation	HBIM
0.200	1.25469	1.25564	1.25469	1.25378
0.400	1.35092	1.35176	1.35105	1.34965
0.600	1.42217	1.42284	1.42225	1.42041
0.800	1.48054	1.48115	1.48064	1.47839
1.000	1.53087	1.53142	1.53096	1.52833
1.200	1.57556	1.57603	1.57561	1.57262
1.400	1.61595	1.61640	1.61601	1.61268
1.600	1.65308	1.65345	1.65308	1.64943
1.800	1.68739	1.68780	1.68746	1.68349
2.000	1.71954	1.71991	1.71959	1.71532
N	100	100	n. a.	32
Δt	0.001	0.001	n. a.	n. a.

In the case of plane melting, as reflected in tab. 1 and 2, the methods employed give good results in predicting the position of the moving boundary when compared with the analytic solutions. In the cases of cylindrical and spherical solidification, where the analytic solutions are not available, the methods employed give very similar results, as reflected in tab. 3 and 4. The good agreement achieved gives us confidence in using the methods to solve the Stefan problem numerically for different geometries.

Comments on the methods

Here we give some general comments on the methods, which can serve as a guideline for solving a particular Stefan problem.

The enthalpy method is popular because of its easy formulation. As the governing equation for the enthalpy is very similar to that for temperature, only little extra effort is required

in programming. However, the iterative nature of the solution procedure makes the computational time longer. Besides, normally the enthalpy method produces unphysical oscillating solution near the moving boundary.

The BIM can effectively remove the moving nature of the boundary at the expense of solving a more complicated equation. Besides, a starting solution is required in order that the method can be started. One benefit of the BIM is that the computation time is comparatively short and so it is possible to achieve higher accuracy by refining the mesh size.

The perturbation method can transform the Stefan problem into an ODE for the position of the boundary. However, the formulation of the purturbation method is complicated and cannot be done easily without symbolic mathematics packages. Besides, the perturbation method only works for small Stefan numbers. Since α can be arbitrarily small by selecting T_{ref} close enough to T_{f}, a constraint on f(t) is also required. Experience suggests that requiring

$$\max_{0 \; t \; t_{\text{final}}} |\text{f}(t)| \quad 1$$

is a good constraint. In the case f(t) = 1, it is found that by adding more terms in the perturbation solution the method can work well for Stefan numbers as large as around 0.7.

The NIM can produce better results with comparatively small numbers of intervals. However, as the number of intervals increases the iteration will become more and more difficult to converge. Also the extension of the method to other geometries is difficult.

The HBIM gives good results for problems with constant boundary conditions. However, the extension to time-dependent problems is difficult. The complicated formulation also makes it less attractive. For these reasons the HBIM is normally used for validation purposes.

Nomenclature

c – specifici heat, [kJkg^{-1}K^{-1}]
f_{l} – local liquid fraction, [–]
H – enthalpy function [= $T+\alpha' f_{\text{l}}(T)$], [–]
H' – [= $-\alpha' f_{\text{l}}(T)$], [–]
L – latent heat, [kJkg^{-1}]
N – spatial resolution, [–]
r – radial variable, [–]
s – position of moving boundary, [–]
T – temperature, [–]
T_{f} – freezing temperature, [K]
T_{ref} – reference temperature, [K]
t – temporal variable, [–]
x – spatial variable, [–]
Z – position of isotherm, [–]

Greek letters

α – Stefan number [= $c(T_{\text{f}} - T_{\text{ref}})/L$], [–]
α' – (= $1/\alpha$), [–]
γ – [= $\Delta t/(\Delta x)^2$], [–]

Superscript

* – transformed variable

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