KELVIN-HELMHOLTZ DISCONTINUITY IN TWO SUPERPOSED VISCOUS CONDUCTING FLUIDS IN A HORIZONTAL MAGNETIC FIELD

by

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The Kelvin-Helmholtz discontinuity in two superposed viscous conducting fluids has been investigated in the taking account of effects of surface tension, when the whole system is immersed in a uniform horizontal magnetic field. The streaming motion is assumed to be two-dimensional. The stability analysis has been carried out for two highly viscous fluid of uniform densities. The dispersion relation has been derived and solved numerically. It is found that the effect of viscosity, porosity and surface tension have stabilizing influence on the growth rate of the unstable mode, while streaming velocity has a destabilizing influence on the system.

Key words: *viscosity, porous medium, streaming velocity, magnetic field, surface tension, instability*

Introduction

The problem of the Kelvin-Helmholtz discontinuity between two superposed fluids is of prime importance in various astrophysical, geophysical and laboratory situations. The Kelvin-Helmholtz discontinuity arises when air is blown over mercury or when highly ionized hot plasma is surrounded by slightly cold gas or when a meteor enters the earth's atmosphere. Chandrasekhar [1] has given a detailed account of problems as investigated by different researchers for incompressible fluids. The influence of viscosity on the stability of the plane interface separating two incompressible superposed fluid in uniform horizontal magnetic field, has been studied by Bhatia [2]. He has carried out the stability analysis for two fluids of equal kinematic viscosities and different uniform densities. A good account of hydrodynamic stability problems has also been given by Drazin *et al*. [3] and Joseph [4].

In recent years, the investigation of flow of fluids through porous media have become an important topic due to the recovery of crude-oil from the pores of reservoir rocks. A great number of applications in geophysics may be found in the books by Phillips [5], Ingham *et al*. [6], and Neild *et al*. [7]. When the fluid permeates a porous material, the gross effect is represented by the Darcy's law. The Rayleigh instability of a thermal boundary layer in flow through porous medium has been considered by Wooding [8]. Kumar [9] has studied the stability of two superposed Walters B viscoelastic fluid-particle mixture in porous medium. The stability of two superposed Walters B' viscoelastic fluids in the presence of suspended particles and variable magnetic field in porous medium has been studied by Sharma *et al*. [10]. Khan *et al*. [11] have studied the stability of two superposed viscoelastic fluids in the presence of horizontal magnetic field. More recently, Kumar *et al.* [12] have studied the instability of two rotating viscoelastic superposed fluids with suspended particle in porous medium.

The importance of the Kelvin-Helmholtz problem has been demonstrated recently by Bhatia *et al*. [13] while studying the stability of two superposed viscous fluids. D' Anglo *et al.* [14, 15] have studied the Kelvin-Helmoholtz instability problem in superposed dusty plasma. Benjamin *et al*. [16] have given an excellent reappraisal of the classic Kelvin-Helmholtz problem in hydrodynamics and have given a Hamiltonian formulation of the problem. Allah [17] has investigated the effects of heat and mass transfer on the Kelvin-Helmholtz instability in hydromagnetics.

El-Sayeed [18] recently, investigated the electro-hydrodynamic instability of two superposed viscous streaming fluids through porous medium.

The aim of this paper is to study the Kelvin-Helmholtz discontinuity between two viscous conducting fluids in a uniform horizontal magnetic field through a porous medium taking account of effects of surface tension.

Formulation of the problem and perturbation equations

We consider an incompressible, viscous infinitely conducting fluid having streaming velocity $\mathbf{U} = (U_x, U_y, 0)$.The prevailing magnetic field is also taken to be two-dimensional, uniform, and acting along the direction in which streaming motion takes place *i. e.*, $\mathbf{H} = (H_x, H_y, 0)$.

The relevant linearized perturbation equations are:

$$\frac{\rho}{\varepsilon}\frac{\partial \vec{u}}{\partial t} \quad \frac{\rho}{\varepsilon}(\vec{U} \quad)\vec{u} + \frac{\rho}{\varepsilon}(D\vec{U})w \qquad \delta p \quad \vec{g}\delta\rho \quad (\qquad \vec{h}) \quad \vec{H} \quad D\mu \quad \frac{\partial w}{\partial \vec{x}} \quad \frac{\partial \vec{u}}{\partial z} \quad \frac{1}{\varepsilon}$$

$$\frac{\mu}{\varepsilon} \quad {}^{2}\vec{u} \quad \frac{\mu}{Q}\vec{u} \qquad {}_{s} \qquad T_s \quad \frac{\partial^2}{\partial x^2} \quad \frac{\partial^2}{\partial y^2} \quad \delta z_s \quad \delta(z \quad z_s) \tag{1}$$

$$\varepsilon\frac{\partial}{\partial t}(\delta\rho) \quad (\vec{U} \quad)\delta\rho \quad (\vec{u} \quad)\rho \quad 0 \tag{2}$$

$$\varepsilon\frac{\partial \vec{h}}{\partial t} \quad (\vec{u} \quad)\vec{h} \quad (\vec{H} \quad)\vec{u} \tag{3}$$

$$\begin{aligned} &\vec{h} \quad 0 \\ &\vec{u} \quad 0 \end{aligned} \tag{4}$$

where $\vec{u}(u, v, w)$, $\vec{h}(h_x, h_y, h_z)$, $\delta\rho$, and δp are perturbations in velocity, magnetic field $\mathbf{H}$, density ρ and pressure p, respectively . Here T, μ and $\mathbf{g} = (0, 0, -g)$ are the surface tension, coefficient of viscosity, and acceleration due to gravity, respectively. In above equations $\delta(z - z_s)$ denotes Dirac' function, Q is the permeability of porous medium, and ε is the medium porosity. Analyzing the disturbance into normal modes, we seek solutions whose dependence on x, y, and t is given by:

$$\mathrm{f}(z)\mathrm{e}^{ik_x x \quad ik_y y \quad nt} \tag{5}$$

where f(z) is some function of z, k_x and k_y are the horizontal numbers, $k^2 = (k_x^2 \quad k_y^2)$, and n is the growth rate of harmonic disturbance.

Eliminating some of the variables from the above equations, we obtain an equation in w as:

$$\frac{n}{\varepsilon}[\rho k^2 \quad D(\rho Dw)] \quad \frac{gk^2}{n\,\varepsilon}(D\rho)w \quad \frac{(\vec{k}\vec{H})}{n\,\varepsilon}(D^2 \quad k^2)w \quad \frac{2}{\varepsilon}(D\mu)(D^2 \quad k^2)Dw$$

$$\frac{1}{\varepsilon}(D^2\mu)(D^2 \quad k^2)w \quad \frac{\mu}{\varepsilon}(D^2 \quad k^2)^2 w \quad \frac{1}{Q}[\mu w k^2 \quad D(\mu Dw)]$$

$$\frac{k^4 T}{n\,\varepsilon}\delta(z \quad z_s)w \quad iD \quad \frac{\rho}{\varepsilon} \quad 1 \quad \frac{(\vec{k}\vec{H})^2}{n^{\,2}\rho} \quad D(\vec{k}\vec{U})w \tag{6}$$

where we have written $n' = n + i(\vec{k}\vec{U})$

Superposed fluids

We consider the case when two superposed fluids of uniform densities ρ_1, ρ_2, and viscosities μ_1, μ_2, magnetic fields H_{x1}, H_{y1} and H_{x2}, H_{y2}, and with streaming velocities U_{x1}, U_{y1} and U_{x2}, U_{y2} are separated by a horizontal boundary at the interface $z = 0$. Therefore, in both the regions $z < 0$ and $z > 0$ of constant densities, eq. (6) becomes:

$$(D^2 - k^2)(D^2 - M^2)w = 0 \tag{7}$$

where

$$M^2 \quad k^2 \quad \frac{n}{\nu} \quad 1 \quad \frac{(\vec{k}\vec{H})^2}{n^{\,2}\rho} \quad \frac{\nu\varepsilon}{Qn} \tag{8}$$

and $\nu = \mu/\rho$ is the coefficient of kinematic viscosity.

Since w must be bounded both when $z \quad \infty$ (in upper fluid) and $z \quad -\infty$ (in lower fluid), the solutions of eq. (7) can be written as:

$$w_1 \quad A_1 n_1 e^{kz} \quad B_1 n_1 e^{M_1 z} \, (z \quad 0) \tag{9}$$

$$w_2 \quad A_2 n_2 e^{\;kz} \quad B_2 n_2 e^{M_2 z} \, (z \quad 0) \tag{10}$$

where A_1, A_2, B_1, and B_2 are constants, and M_1, M_2 are positive square roots of eq. (8) for the two regions and:

$$n_1 \quad n \quad i(\vec{k}\vec{U}_1) \qquad n_2 \quad n \quad i(\vec{k}\vec{U}_2) \tag{11}$$

Boundary conditions

The above equations must satisfy certain boundary conditions , requiring that on interface w, Dw and $\mu(D_2 + k_2)w$ must be continuous.

Also by integrating eq. (6) across the interface $z = 0$,we obtain another condition as:

$$\rho_2 \quad \frac{\mu_2}{n_2}(D^2 \quad k^2) \quad \frac{\mu_2\varepsilon}{Qn_2} \quad \frac{(k_x H_{x2} \quad k_y H_{y2})^2}{n_1^{\,2}} \quad Dw_2 \quad {}_{z\;\;0}$$

$$\rho_1 \quad \frac{\mu_1}{n_1}(D^2 \quad k^2) \quad \frac{\mu_1\varepsilon}{n_1} \quad \frac{(k_x H_{x1} \quad k_y H_{y1})^2}{n_1^{\,2}} \quad Dw_1 \quad {}_{z\;\;0}$$

$$gk^2 \quad \frac{\rho_2}{n_2^{\,2}} \quad \frac{\rho_1}{n_1^{\,2}} \quad w_0 \quad 2k^2 \quad \frac{\mu_2}{n_2} \quad \frac{\mu_1}{n_1} \quad (Dw)_0 \quad \frac{k^4 T}{n} \quad \frac{1}{n_2} \quad \frac{1}{n_1} \quad w_0 \quad 0 \tag{12}$$

where w_0 and $(Dw)_0$ are the unique values of w_1, w_2, and Dw_1, Dw_2, at $z = 0$.

Applying the boundary conditions to the solutions 9 and 10, we obtain:

$$A_1 + B_1 = A_2 + B_2 \tag{13}$$

$$kA_1 + M_1B_1 = -kA_2 - M_2B_2 \tag{14}$$

$$\mu_1[2A_1k^2 \quad (M_1^2 \quad k^2)B_1] \quad \mu_2[2A_2k^2 \quad (M_2^2 \quad k^2)B_2] \tag{15}$$

$$k\rho_2 A_2 \quad k\rho_1 A_1 \quad \frac{\mu_2\varepsilon}{Qn_2}(\quad kA_2 \quad M_2B_2) \quad \frac{\mu_1\varepsilon}{Qn_1}(kA_1 \quad M_1B_1)$$

$$\frac{(k_x H_{x2} \quad k_y H_{y2})^2}{n_2} kA_2 \quad \frac{(k_x H_{x1} \quad k_y H_{y1})^2}{n_1{}^2} kA_2$$

$$\frac{gk^2}{2} \quad \frac{\rho_2}{n_2{}^2} \quad \frac{\rho_1}{n_1{}^2} \quad (A_1 \quad B_1 \quad A_2 \quad B_2)$$

$$k^2 \quad \frac{\mu_2}{n_2} \quad \frac{\mu_1}{n_1} \quad (kA_1 \quad M_1B_1 \quad kA_2 \quad M_2B_2) \quad \frac{k^4T}{2n} \quad \frac{1}{n_2} \quad \frac{1}{n_1} \quad (A_1 \quad B_1 \quad A_2 \quad B_2) \tag{16}$$

On eliminating the constants A_1, B_1, A_2, and B_2, and evaluating the determinant of given matrix of the coefficients in eqs. 13 to 16, we obtain characteristic equation:

$$(M_1 \quad k) \; 2k^2(\nu_1\alpha_1 \quad \nu_2\alpha_2) \quad c \quad \frac{M_1}{k} \quad 1 \quad \frac{(kV_{A2})^2}{n_2{}^2} \quad \frac{\nu_2\alpha_2\varepsilon}{Qn_2} \quad \frac{M_2}{k} \quad 1 \quad \alpha_2$$

$$\nu_2\alpha_2(M_2^2 \quad k^2) \; R \quad S \quad 1 \quad \frac{\varepsilon}{Q} \quad \frac{\nu_1\alpha_1}{n_1} \quad \frac{\nu_2\alpha_2}{n_2} \quad \frac{(kV_{A1})}{n_1{}^2} \quad \frac{(kV_{A2})^2}{n_2{}^2}$$

$$2k \quad \nu_1\alpha_1(M_1^2 \quad k^2) \quad c \quad \frac{M}{k} \quad 1 \quad \frac{(kV_{A2})^2}{n_2{}^2} \quad \frac{\nu_2\alpha_2\varepsilon}{Qn_2} \quad \frac{M_2}{k} \quad 1 \quad \alpha_2$$

$$\nu_2\alpha_2(M_2^2 \quad k^2) \; c \quad \frac{M_1}{k} \quad 1 \quad \frac{(kV_{A1})^2}{n_1{}^2} \quad \frac{\nu_1\alpha_1\varepsilon}{Qn_1} \quad \frac{M_1}{k} \quad 1 \quad \alpha_1$$

$$(M_2 \quad k) \; \nu_1\alpha_1(M_1^2 \quad k^2) \; R \quad S \quad 1 \quad \frac{\varepsilon}{Q} \quad \frac{\nu_1\alpha_1}{n_1} \quad \frac{\nu_2\alpha_2}{n_2} \quad \frac{(kV_{A1})^2}{n_1{}^2} \quad \frac{(kV_{A2})^2}{n_2{}^2}$$

$$2k^2(\nu_1\alpha_1 \quad \nu_2\alpha_2) \; c \quad \frac{M_1}{k} \quad 1 \quad \frac{(kV_{A1})^2}{n_1{}^2} \quad \frac{\nu_1\alpha_1\varepsilon}{Qn_1} \quad \frac{M_1}{k} \quad 1 \quad \alpha_1 \qquad 0 \tag{17}$$

where

$$R \quad gk \quad \frac{\alpha_2}{n_2{}^2} \quad \frac{\alpha_1}{\nu_1{}^2}$$

$$c \quad k^2 \quad \frac{\nu_2\alpha_2}{n_2} \quad \frac{\nu_1\alpha_1}{n_1}$$

$$S \quad \frac{k^3 T_1}{n} \quad \frac{1}{n_2} \quad \frac{1}{n_1}, \quad T_1 \quad \frac{T}{\rho_1 \quad \rho_2}$$

$$(kV_{A1})^2 \quad \frac{(k_x H_{x1} \quad k_y H_{y1})^2}{\rho_1 \quad \rho_2}$$

$$(kV_{A2})^2 \quad \frac{(k_x H_{x2} \quad k_y H_{y2})^2}{\rho_1 \quad \rho_2}$$

where V_{A1} and V_{A2} are Alfven velocities in the two fluids. On evaluating the dispersion relation (17), we obtain the characteristic equation which is quite complex, particularly as both M_1 and M_2 involve square roots. Therefore ,we carry out the stability analysis for highly viscous fluids, as in the non-streaming case of superposed fluids earlier by one of the authors (Khan and Bhatia [11]). We can write:

$$M_1 \quad k \; 1 \quad \frac{n_1}{2k^2 \nu_1} \quad \frac{1}{2} \frac{(kV_{A1})^2}{k^2 n_1 \nu_1 \alpha_1} \quad \frac{1}{2} \frac{\varepsilon}{k^2 Q}$$

$$M_2 \quad k \; 1 \quad \frac{n_2}{2k^2 \nu_2} \quad \frac{1}{2} \frac{(kV_{A2})^2}{k^2 n_2 \nu_2 \alpha_2} \quad \frac{1}{2} \frac{\varepsilon}{k^2 Q} \qquad (18)$$

neglecting square and higher order terms in $1/\nu_{1,2}$.

Substituting the values of M_1 and M_2 in eqs. 17 and 18, we obtain the dispersion relation in dimensionless form as:

$$W_9 n^9 + W_8 n^8 + W_7 n^7 + W_6 n^6 + W_5 n^5 + W_4 n^4 + W_3 n^3 + W_2 n^2 + W_1 n + W_0 = 0 \qquad (19)$$

where the coefficients W_9, W_8, W_7, W_6, W_5, W_4, W_3, W_2, W_1, and W_0 are quite complicated. These coefficients are not given here as they are quite lengthy expression involving the wave number k and the parameters characterizing the effects of surface tension, permeability of porous medium, viscosity, and streaming velocity.

Results and discussion

The eq. (19) is quite complex. In order to study the effects of various physical parameters on the growth rate of unstable modes, the numerical solution of this equation has been sought to locate the values of n (positive real part) against wave number k, for several values of the parameters involved. The numberical calculations are presented in figs. 1-6, where we have taken a potentially unstable arrangement by taking $\alpha_1 = 0.20$, $\alpha_2 = 0.80$ for fixed $H_1 = 5.0$, $H_2 = 10.0$, $V_{A1} = 0.2$, $V_{A2} = 0.5$ and $\varepsilon = 0.1$. These calculations are presented in figs. 1-6, where the growth rate (positive real part of n) is plotted against the wave number k for several values of U_1, U_2, T, Q, X_1, and X_2. In figs. 1 and 2 the growth rate is given against the wave number for U_1 and U_2 (streaming velocities) taking fixed values of T (surface tension) = 4.0, Q (permeability) = 1.2, X_1 (viscosity) = 4.0, and X_2 (viscosity) = 8.0. From fig. 1, we find that the growth rate increases on increasing the streaming velocities (U_1 and U_2) for same values of the wave number. We thus find that streaming velocity has destabilizing influence on the Kelvin-Helmholtz instability of the superposed fluids. In fig. 3, we have given the variation of the growth rate against the wave number for the values of surface tension T = 4.0, 8.0, and 12.0. In these curves also the values of the other parameters have been kept fixed. From fig. 3, we find that the surface tension has a sta-

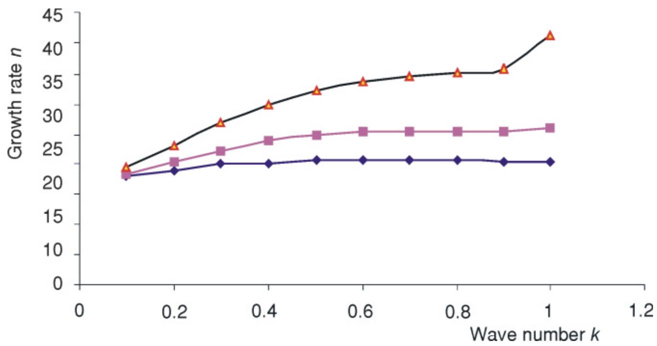

Figure 1. Variation of growth rate *n* against wave number *k* for U_1 = 2.0, 8.0, and 14.0 taking U_2 = 1.0, *T* = 4.0, *Q* = 1.2, X_1 = 4.0, and X_2 = 8.0 (effect of stream velocity U_1)

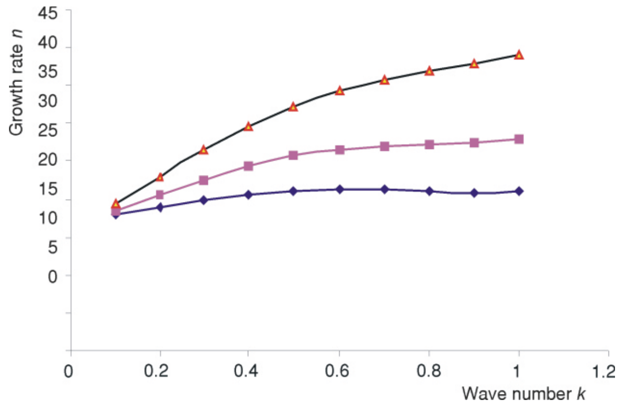

Figure 2. Variation of growth rate *n* against wave number *k* for U_2 = 1.0, 6.0, and 12.0 taking U_1 = 3.0, *T* = 4.0, *Q* = 1.2, X_1 = 4.0, and X_2 = 8.0 (effect of stream velocity U_2)

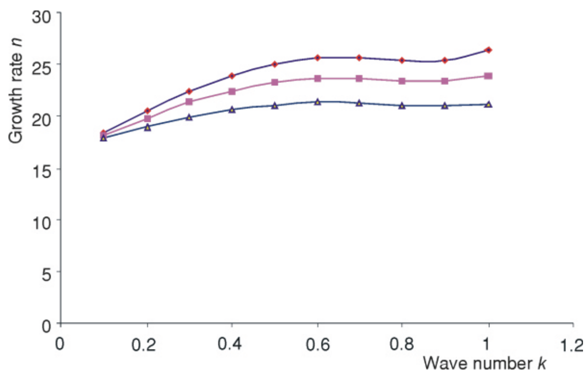

Figure 3. Variation of growth rate *n* against wave number *k* for *T* = 4.0, 8.0, and 12.0 taking U_1 = 3.0, U_2 = 1.0, and X_1 = 4.0, X_2 = 8.0, and *Q* = 1.2 (effect of surface tension *t*)

bilizing influence as the growth rate decreases with increasing surface tension for same values of the wave number. Figure 4 depicts the variation of the growth rate with permeability of porous medium. The curves in the fig. 4 clearly show that the growth rate decreases with increasing value of *Q*, the parameter characterizing effect of the permeability of porous medium. As the growth rate decreases on increasing the values of *Q* for same wave number *k*, the effect of permeability is stabilizing on the growth rate of the unstable configuration. Figures 5 and 6 give the plot of growth rate against the wave number for the values of viscosities X_1 = 4.0, 8.0, 12.0, and X_2 = 5.0, 7.0, 9.0. From figs. 5 and 6, we find that the growth rate decreases on increasing the values of X_1 and X_2 for same wave number *k*. The effect of viscosity is thus stabilizing on the unstable mode of disturbance.

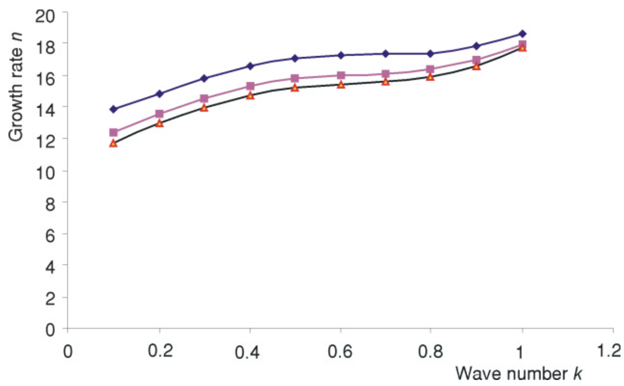

Figure 4. Variation of growth rate *n* against wave number *k* for $Q = 3.0$, 6.0, and 9.0, $U_1 = 3.0$, $U_2 = 1.0$, $T = 4.0$, $X_1 = 4.0$, and $X_2 = 8.0$ (effect of medium permeability *Q*)

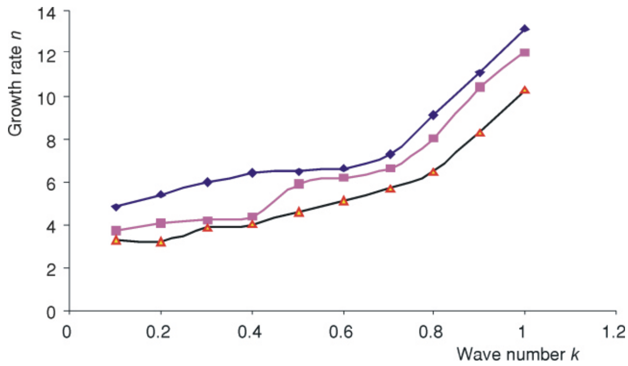

Figure 5. Variation of growth rate *n* against wave number *k* for $X_1 = 4.0$, 8.0, and 12.0, $U_1 = 3.0$, $U_2 = 1.0$, $T = 4.0$, $Q = 1.2$, and $X_2 = 3.0$ (effect of viscosity X_1)

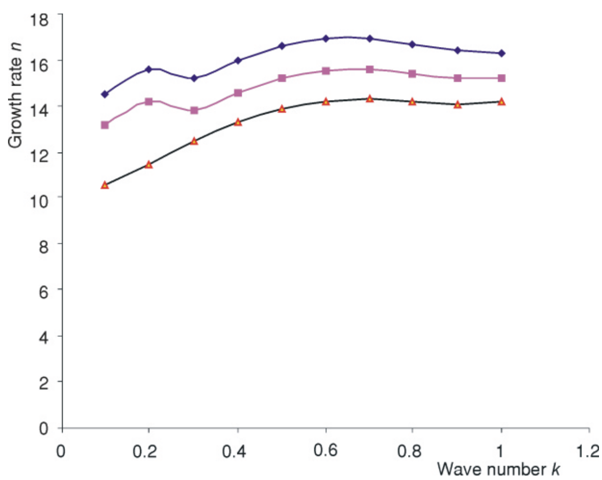

Figure 6. Variation of growth rate *n* against wave number *k* for $X_2 = 5.0$, 7.0, and 9.0, $U_1 = 3.0$, $U_2 = 1.0$, $T = 4.0$, $Q = 1.2$, and $X_1 = 3.0$ (effect of viscosity X_2)

We may, thus, conclude that surface tension, permeability of porous medium and viscosity have stabilizing influence while streaming velocity has destabilizing influence on the system.

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References

[1] Chandrasekhar, S., Hydrodynamic and Hydromagnetic Stability, Dover Publication, New York, USA, 1981

[2] Bhatia, P. K., Rayleigh-Taylor Instability of Two Viscous Superposed Conducting Fluids, *Nuov. Cim. 19B* (1974), 2, pp. 161-168

[3] Drazin, P. G., Reid, W. H., Hydrodynamic Stability, Cambridge University Press, Cambridge, UK, 1981

[4] Joseph, D. D., Stability of Fluid Motion II, Springer-Verlag, New York, USA, 1976

[5] Phillips, O. M., Flow and Reaction in Permeable Rocks, Cambridge University Press, Cambridge, UK, 1991

[6] Ingham, D. B., Pop, I., Transport Phenomena in Porous Medium, Pergamon Press, Oxford, UK, 1998

[7] Nield, D. A., Bejan, A., Convection in Porous Medium, 2nd ed., Springer Verlag, New York, USA, 1999

[8] Wooding, R. A., Rayleigh Instability of a Thermal Boundary Layer in Flow through a Porous Medium, *J. Fluid Mech., 9* (1960), 2, pp. 183-192

[9] Kumar, P., Stability of Two Superposed Viscoelastic (Walters B') Fluid- Particle Mixture in Porous Medium, *Z. Naturforsch., 54a* (1998), 5, pp. 343-347

[10] Sharma, R. C., Kango, S. K., Stability of Two Superposed Walters B' Viscoelastic Fluids in the Presence of Suspended Particles and Variable Magnetic Field in Porous Medium, *Appl. Mech. Engng. 4* (1999), 2, pp. 255-269

[11] Khan, A., Bhatia, P. K., Stability of Two Superposed Viscoelastic Fluid in a Horizontal Magnetic Field, *Indian J. Pure Appl. Math., 32* (2001), 1, pp. 98-108

[12] Kumar, P., Singh, M., Instability of Two Rotating Viscoelastic (Walter B') Superposed Fluid with Suspended Particle in Porous Medium, *Thermal Science, 11* (2007), 1, pp. 93-102

[13] Bhatia, P. K., Sankhla, V. D., Kelvin-Helmholtz Discontinuity in Two Superposed Viscous Conducting Fluids, *Astrophysics and Space Sci., 103* (1984), 1, pp. 33-38

[14] D'Angelo, N., Song, B., Kelvin-Helmholtz Instability in a Plasma with Negative Ions, *IEEE Transectors Plasma Sci., 19* (1991), 1, pp. 42-46.

[15] D'Angelo, N., Song, B., The Kelvin-Helmholtz Instability in Dust Plasmas, *Planet Space Sci., 38* (1990), 12, pp. 1577-1579

[16] Benjamin, T. B., Bridges, T. J., Reappraisal of the Kelvin-Helmholtz Problem, Part I: Hamiltonian Structure, *J. Fluid Mech., 333* (1997), 4, pp. 301-325

[17] Allah, M. H. O., The Effects of Magnetic Field and Mass and Heat Transfer on Kelvin-Helmholtz Stability, *Proc. Nat. Acad. Sci., 68* (1998), II, pp. 163-173

[18] El-Sayed, M. F., Electrohydrodynamic Instability of Two Superposed Viscous Streaming Fluids through Porous Media, *Canad. J. Phys., 75* (1997), 7, pp. 499-508

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