STABILITY OF TWO SUPERPOSED VISCOUS-VISCOELASTIC FLUIDS

by

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The Rayleigh-Taylor instability of a Newtonian viscous fluid overlying a Rivlin-Ericksen viscoelastic fluid is considered. Upon application of normal mode technique, the dispersion relation is obtained. As in both Newtonian viscous-viscous fluids the system is stable in the potentially stable case and unstable in the potentially unstable case, this holds for the present problem also. The behaviour of growth rates with respect to kinematic viscosity and kinematic viscoelasticity parameters are examined numerically and it is found that both kinematic viscosity and kinematic viscoelasticity have stabilizing effect.

Key words: *Rayleigh-Taylor instability, normal mode technique, linearized theory, Rivlin-Ericksen viscoelastic fluid*

Introduction

The instability of the plane interface separating two Newtonian fluids when one is accelerated towards the other or when one is superposed over the other has been studied by several authors, and Chandrasekhar [1] has given a detailed account of these investigations. The influence of viscosity on the stability of the plane interface separating two incompressible superposed fluids of uniform densities, when the whole system is immersed in a uniform horizontal magnetic field, is studied by Bhatia [2]. He has carried out the stability analysis for two fluids of equal kinematic viscosities and different uniform densities. A good account of hydrodynamic stability problems has also been given by Drazin and Reid [3], and Joseph [4].

The fluids have been considered to be Newtonian in all the above studies. Sharma and Sharma [5] have studied the stability of the plane interface separating two viscoelastic (Oldroydian) superposed fluids of uniform densities. In another study, Sharma [6] has studied the instability of the plane interface between two Oldroydian viscoelastic superposed conducting fluids in the presence of a uniform magnetic field. Fredricksen [7] has given a good review of non-Newtonian fluids. Molten plastics, petroleum oil additives and whipped cream are examples of incompressible viscoelastic fluids. There are many non-Newtonian fluids that cannot be characterized by Oldroyd's [8] constitutive relations. The Rivlin-Ericksen elastico-viscous fluid is one such fluid.

Srivastava and Singh [9] have studied the unsteady flow of a dusty elastico-viscous Rivlin-Ericksen fluid through channels of different cross-sections in the presence of a time dependent pressure gradient. In another study Garg *et al.* [10] have studied the rectilinear oscillations of a sphere along its diameter in a conducting dusty Rivlin-Ericksen fluid in the presence of a uniform magnetic field. Thermal instability in Rivlin-Ericksen elastico-viscous fluid in presence of magnetic field and rotation, separately, has been investigated by Sharma and Kumar [11, 12]. In another study, Sharma and Kumar [13] have studied the hydromagnetic stability of two Rivlin-Ericksen elastico-viscous superposed conducting fluids and the analysis has been carried out, for two highly viscous fluids of equal kinematic viscosities and equal kinematic viscoelasticities.

It is this class of elastico-viscous fluids we are interested in, particularly to study the stability of the plane interface between viscous and viscoelastic (Rivlin-Ericksen) fluids.

The stability of the plane interface between viscous (Newtonian) and viscoelastic (Rivlin-Ericksen) fluids may find applications in geophysics, chemical technology, and bio-mechanics and is therefore, studied in the present paper.

Formulation of the problem and perturbation equations

Consider a static state in which an incompressible Rivlin-Ericksen viscoelastic fluid is arranged in horizontal strata and the pressure p and the density ρ are functions of the vertical co-ordinate z only. The character of the equilibrium of this initial static state is determined, as usual, by supposing that the system is slightly disturbed and then by following its further evolution.

Let $\vec{v}(u, v, w)$, $\delta\rho$, and δp denote the perturbation in fluid velocity (0, 0, 0), density ρ and pressure p, respectively. Then the linearized perturbation equations relevant to the problem are:

$$\rho\frac{\partial\vec{v}}{\partial t} \quad \delta p \quad \vec{g}\delta\rho \quad \rho \quad \nu \quad \nu\frac{\partial}{\partial t} \quad {}^{2}\vec{v} \quad \frac{\mathrm{d}\mu}{\mathrm{d}z} \quad \frac{\partial}{\partial t}\frac{\mathrm{d}\mu}{\mathrm{d}z} \quad \frac{\partial w}{\partial \bar{x}} \quad \frac{\partial\vec{v}}{\partial z} \tag{1}$$

$$\vec{v} \quad 0 \tag{2}$$

$$\frac{\partial}{\partial t}\delta\rho \quad wD\rho \tag{3}$$

where $\nu(\mu/\rho)$ and $\nu (\mu /\rho)$ denote the kinematic viscosity and the kinematic viscoelasticity of the fluid, $\vec{g}(0,0, g)$ is the acceleration due to gravity, $\bar{x} \quad (x, y, z)$ and $D \quad \mathrm{d}/\mathrm{d}z$. Equation (3) ensures that the density of every particle remains unchanged as we follow it with its motion.

Analyzing the disturbances into normal modes, we seek solutions whose dependence on x, y, and t is given by:

$$\exp(ik_x x + ik_y y + nt) \tag{4}$$

where k_x, and k_y are horizontal wave numbers, $k^2 = k_x^2 + k_y^2$, and n is a complex constant.

For perturbations of the form (4), eqs. (1) – (3) give:

$$\rho n u = -ik_x \delta p + \rho(\nu + \nu' n)(D^2 - k^2)u + (ik_x w + Du)(D\mu + nD\mu') \tag{5}$$

$$\rho n v = -ik_y \delta p + \rho(\nu + \nu' n)(D^2 - k^2)v + (ik_y w + Dv)(D\mu + nD\mu') \tag{6}$$

$$\rho n w = -D\delta p - g\delta\rho + \rho(\nu + \nu' n)(D^2 - k^2)w + 2Dw(D\mu + nD\mu') \tag{7}$$

$$ik_x u + ik_y v + Dw = 0 \tag{8}$$

$$n\delta\rho = -wD\rho \tag{9}$$

Eliminating δp between eqs. (5)-(7) and using eqs. (8) and (9), we obtain:

$$\begin{aligned} & n[D(\rho Dw) - k^2\rho w] - \{D[\rho(\nu + \nu' n)(D^2 - k^2)Dw] \\ & - k^2\rho(\nu + \nu' n)(D^2 - k^2)w\} = -\frac{gk^2}{n}(D\rho)w \\ & + \{D[(D\mu + nD\mu')(D^2 - k^2)w] - 2k^2(D\mu + nD\mu')(Dw)\} = 0 \end{aligned} \tag{10}$$

Two uniform viscous and viscoelastic (Rivlin-Ericksen) fluids separated by a horizontal boundary

Consider the case of two uniform fluids of densities, viscosities; ρ_2, μ_2 (upper Newtonian fluid) and ρ_1, μ_1 (lower Rivlin-Ericksen viscoelastic fluid) separated by a horizontal boundary at $z = 0$. Then, in each region of constant ρ and constant μ, μ', eq. (10) becomes:

$$(D^2 - k^2)(D^2 - q^2)w = 0 \tag{11}$$

where $q^2 = k^2 + n/(\nu + \nu' n)$.

Since w must vanish both when $z \to +\infty$ (for the upper fluid) and $z \to -\infty$ (for the lower fluid), the general solution of eq. (11) can be written as:

$$w_1 = A_1 e^{kz} + A_2 e^{q_1 z} \quad (z < 0) \tag{12}$$

$$w_2 \quad A_3 \mathrm{e}^{\;kz} \quad A_4 \mathrm{e}^{\;q_2 z} \tag{13}$$

where A_1, A_2, A_3, and A_4 are constants of integration,

$$q_1 \quad \sqrt{k^2 \quad \frac{n}{\nu_1 \quad \nu_1 n}} \quad \text{and} \quad q_2 \quad \sqrt{k^2 \quad \frac{n}{\nu_2}} \tag{14}$$

In writing the solutions (12) and (13), it is assumed that q_1 and q_2 are so defined that their real parts are positive.

Boundary conditions

The solutions (12) and (13) must satisfy certain boundary conditions. Clearly, all three components of velocity and tangential viscous stresses must be continuous. The continuity of Dw follows from eq. (8) and the continuity of u and v. Since:

$$\tau_{xz} \quad 2\mu \quad 2\mu' \frac{\partial}{\partial t} \quad e_{xz} \quad (\mu \quad \mu' n)(Du \quad ik_x w)$$

and

$$\tau_{yz} \quad 2\mu \quad 2\mu' \frac{\partial}{\partial t} \quad e_{yz} \quad (\mu \quad \mu' n)(Dv \quad ik_y w)$$

are continuous,

$$ik_x \tau_{xz} \quad ik_y \tau_{yz} \quad (\mu \quad \mu' n)(D^2 \quad k^2)w$$

is continuous across an interface between the two fluids. Hence, the boundary conditions to be satisfied at the interface $z = 0$ are that:

$$w \tag{15}$$

$$Dw \tag{16}$$

and

$$(\mu \quad \mu' n)(D^2 \quad k^2)w \tag{17}$$

must be continuous. Integrating eq. (10) across the interface $z = 0$, we obtain another condition:

$$[\rho_2 Dw_2 \quad \rho_1 Dw_1]_{z \; 0} \quad \frac{1}{n}\mu_2 (D^2 \quad k^2) Dw_2 \quad \frac{1}{n}(\mu_1 \quad \mu_1' n)(D^2 \quad k^2) Dw_1 \Big|_{z \; 0}$$

$$\frac{gk^2}{n^2}[\rho_2 \quad \rho_1] w_0 \quad \frac{2k^2}{n}[\mu_2 \quad \mu_1 \quad n\mu_1'](Dw)_0 \tag{18}$$

where w_0, $(Dw)_0$ are the common values of w_1, w_2 and Dw_1, Dw_2 respectively, at $z = 0$.

Dispersion relation and discussion

Applying the boundary conditions (15)-(18) to the solutions (12) and (13), and eliminating the constants A_1, A_2, A_3, and A_4 from resulting equations, we obtain:

$$\det(a_{ij}) = 0 \tag{19}$$

where $i, j = 1, 2, 3, 4$ and

$$\begin{gathered}
a_{11} \quad a_{12} \quad 1,\ a_{13} \quad a_{14} \quad 1,\ a_{21} \quad a_{23} \quad k,\ a_{22} \quad q_1,\ a_{24} \quad q_2,\ a_{31} \quad 2k^2(\mu_1 \quad \mu_1 n) \\
a_{32} \quad (\mu_1 \quad \mu_1 n)(q_1^2 \quad k^2),\ a_{33} \quad 2k^2\mu_2,\ a_{34} \quad \mu_2(q_2^2 \quad k^2) \\
a_{41} \quad \alpha_1 \quad \frac{R}{2} \quad \frac{k^2}{n}(\nu_2\alpha_2 \quad \nu_1\alpha_1 \quad n\nu_1\alpha_1),\ a_{42} \quad \frac{R}{2} \quad \frac{k}{n}(\nu_2\alpha_2 \quad \nu_1\alpha_1 \quad n\nu_1\alpha_1)q_1, \\
a_{43} \quad \alpha_2 \quad \frac{R}{2} \quad \frac{k^2}{n}(\nu_2\alpha_2 \quad \nu_1\alpha_1 \quad n\nu_1\alpha_1),\ a_{44} \quad \frac{R}{2} \quad \frac{k}{n}(\nu_2\alpha_2 \quad \nu_1\alpha_1 \quad n\nu_1\alpha_1)q_2, \\
\alpha_{1,2} \quad \frac{\rho_{1,2}}{\rho_1 \quad \rho_2} \text{ and } R \quad \frac{gk}{n^2}(\alpha_2 \quad \alpha_1)
\end{gathered} \tag{20}$$

Eq. (19) yields the following characteristic equation:

$$\begin{gathered}
(q_1 \quad k) \quad 2k^2(\nu_2\alpha_2 \quad \nu_1\alpha_1 \quad n\nu_1\alpha_1) \quad \frac{k}{n}(\nu_2\alpha_2 \quad \nu_1\alpha_1 \quad n\nu_1\alpha_1)(q_2 \quad k) \quad \alpha_2 \\
(R \quad 1)(\nu_2\alpha_2)(q_2^2 \quad k^2) \quad 2k \quad (\nu_1\alpha_1 \quad n\nu_1\alpha_1)(q_1^2 \quad k) \\
\frac{k}{n}(\nu_2\alpha_2 \quad \nu_1\alpha_1 \quad n\nu_1\alpha_1)(q_2 \quad k) \quad \alpha_2 \\
(\nu_2\alpha_2)(q_2^2 \quad k^2) \frac{k}{n}(\nu_2\alpha_2 \quad \nu_1\alpha_1 \quad n\nu_1\alpha_1)(q_1 \quad k) \quad \alpha_1 \\
(q_2 \quad k) \quad (\nu_1\alpha_1 \quad n\nu_1\alpha_1)(q_1^2 \quad k^2) \quad (R \quad 1) \\
2k^2(\nu_2\alpha_2 \quad \nu_1\alpha_1 \quad n\nu_1\alpha_1) \quad \frac{k}{n}(\nu_2\alpha_2 \quad \nu_1\alpha_1 \quad n\nu_1\alpha_1)(q_1 \quad k) \quad \alpha_1 \qquad 0
\end{gathered} \tag{21}$$

The dispersion relation (21) is quite complicated, as the values of q_1 and q_2 involve square roots. We, therefore, make the assumption that the fluids are of high viscosity and high viscoelasticity. Under this assumption, we have:

$$q \quad k\sqrt{1 \quad \frac{n}{k^2(\nu \quad \nu n)}} \quad k\;1 \quad \frac{1}{2}\frac{n}{k^2(\nu \quad \nu n)} \quad k \quad \frac{n}{2k(\nu \quad \nu n)} \tag{22}$$

so that

$$q_1 \quad k \quad \frac{n}{2k(\nu_1 \quad \nu n)} \quad \text{and} \quad q_2 \quad k \quad \frac{n}{2k\nu_2} \tag{23}$$

Substituting the values of $q_1 - k$ and $q_2 - k$ from eqs. (22) and (23) in eq. (21), we obtain the dispersion relation:

$$\begin{aligned} &\alpha_1\nu_1[2k^2\alpha_1\nu_1 \quad 1]n^3 \quad [(\alpha_1\nu_1 \quad \alpha_2\nu_2)(1 \quad 4k^2\alpha_1\nu_1)]n^2 \\ &[4k^2\nu_1\nu_2\alpha_1\alpha_2 \quad 2k^2(\alpha_1^2\nu_1^2 \quad \alpha_2^2\nu_2^2) \\ &\alpha_1\nu_1 gk(\alpha_2 \quad \alpha_1)]n \quad gk(\alpha_2 \quad \alpha_1)(\alpha_1\nu_1 \quad \alpha_2\nu_2) \quad 0 \end{aligned} \tag{24}$$

Assuming $\nu_1 = \nu_2 = \nu$ ([1], p. 443), as this simplifying assumption does not obscure any of the essential features of the problem, the dispersion relation (24) becomes:

$$\begin{aligned} &\alpha_1\nu_1[2k^2\alpha_1\nu_1 \quad 1]n^3 \quad [\nu(1 \quad 4k^2\alpha_1\nu_1)]n^2 \\ &[2k^2\nu^2 \quad \alpha_1\nu_1 gk(\alpha_2 \quad \alpha_1)]n \quad gk\nu(\alpha_2 \quad \alpha_1) \quad 0 \end{aligned} \tag{25}$$

Stable case

For the *potentially stable arrangement* ($\alpha_2 < \alpha_1$), all the coefficients of eq. (25) are positive. So, all the roots of eq. (25) are either real and negative or there are complex roots (which occur in pairs) with negative real parts and the rest negative real roots. The system is, therefore, stable in each case. Hence the potentially stable arrangement remains stable for the stability of two superposed viscous-viscoelastic (Rivlin-Ericksen) fluids. This result is also true when the fluids are viscous [1].

Unstable case

For the *potentially unstable arrangement* ($\alpha_2 > \alpha_1$), the constant term in eq. (25) is negative. The eq. (25), therefore, allows at least one change of sign and so has at least one positive real root. The occurrence of positive root implies that system is unstable for disturbances of all wave numbers. The system is, therefore, unstable for potentially unstable case.

We now examine the behaviour of growth rates with respect to kinematic viscosity and kinematic viscoelasticity numerically. We have plotted the growth rate *n* (positive real value) versus the wave number *k* for several values of the kinematic viscosity and the kinematic viscoelasticity in fig. 1 and fig. 2, respectively.

In fig. 1, the growth rate *n* is plotted against wave number *k*, for fixed value of ν_1 2, α_1= 0.38, α_2 = 0.62 and for ν = 1, 2, 5. The growth rate decreases with increase in kinematic viscosity showing its stabilizing effect on the system. In fig. 2, growth rate *n* is

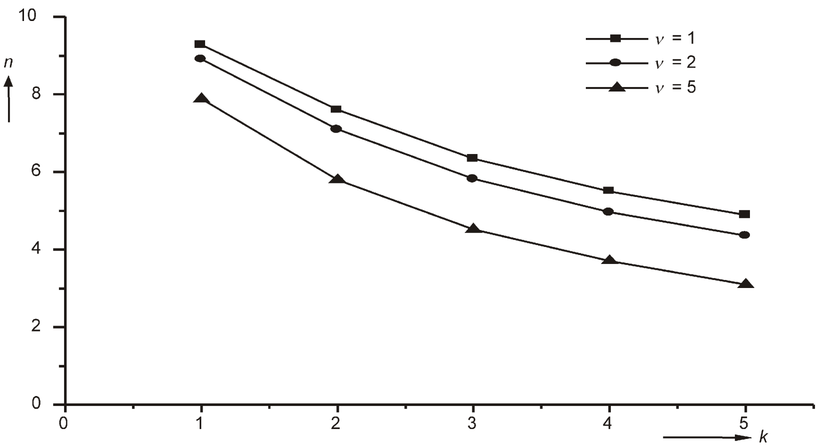

Figure 1. The variation of the growth rate *n* (positive real value) with the wave number *k* for the kinematic viscosities ν = 1, 2, 5 when α_1 = 0.38, α_2 = 0.62, and ν_1 = 2

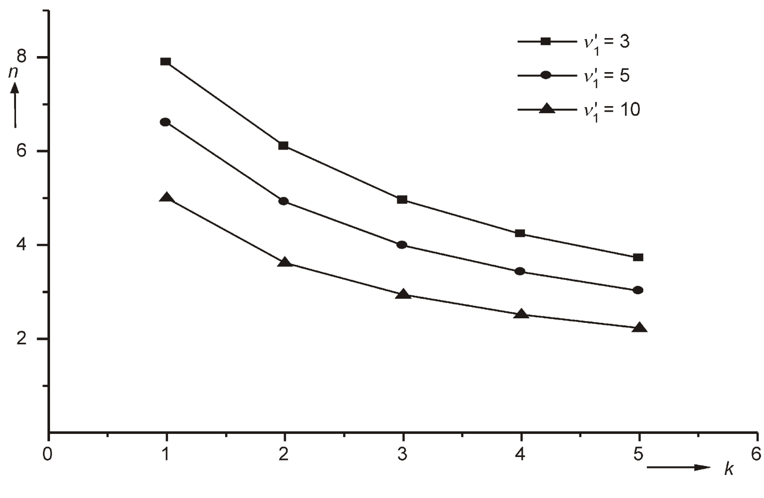

Figure 2. The variation of the growth rate *n* (positive real value) with the wave number *k* for the kinematic viscoelasticities ν_1= 3, 5, 10 when α_1 = 0.38, α_2 = 0.62, and ν = 2

plotted against wave number for fixed $\nu = 2, \alpha_1 = 0.38, \alpha_2 = 0.62$ and for ν_1 3, 5, 10. It is seen that for the same wave number k, the growth rate n decreases as the kinematic viscoeasticity ν_1 increases, showing the stabilizing character of kinematic viscoelasticity.

Conclusions

A detailed account of stability of superposed Newtonian fluids, under varying assumptions of hydrodynamics, was given by Chandrasekhar [1]. With the growing importance of non-Newtonian fluids in chemical engineering, modern technology and industry, the investigations on such fluids are desirable. The Rivlin-Ericksen fluid is one such important non-Newtonian (viscoelastic) fluid. The Rayleigh-Taylor instability of a Newtonian viscous fluid overlying Rivlin-Ericksen viscoelastic fluid is considered in the present paper. Following the linearized perturbation theory and normal mode analysis, the dispersion relation is obtained. The stability analysis has been carried out, for mathematical simplicity, for two highly viscous and viscoelastic fluids of equal kinematic viscosities. As in both Newtonian viscous-viscous fluids, the system is found to be stable for stable configuration and unstable for unstable configuration for the present problem also. The dispersion relation is also solved numerically and it is found that both kinematic viscosity and kinematic viscoelasticity have stabilizing effect on the growth rate.

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Nomenclature

g – acceleration due to gravity, [m/s^2]
$\vec{g}$ – gravity field, [m/s^2]
k – wave-number, [1/m]
k_x, k_y – horizontal wave-numbers, [1/m]
n – growth rate, [1/s]
p – fluid pressure, [Pa]
t – time, [s]
$\vec{v}$ – fluid velocity, [m/s]

Greek letters

μ – dynamic viscosity [kg/ms]
ρ – density, [kg/m^3]
ν – kinematic viscosity, [m^2/s]
ν' – kinematic viscoelasticity, [m^2/s]

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