Effects of thermal radiation and heat transfer over an unsteady stretching surface embedded in a porous medium in the presence of heat source or sink

by

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The effects of thermal radiation and heat transfer over an unsteady stretching surface embedded in a porous medium in the presence of heat source or sink are studied. The governing time dependent boundary layer equations are transformed to ordinary differential equations containing radiation parameter, permeability parameter, heat source or sink parameter, Prandtl number, and unsteadiness parameter. These equations are solved numerically by applying Nachtsheim-Swinger shooting iteration technique together with Rung-Kutta fourth order integration scheme. The velocity profiles, temperature profiles, the skin friction coefficient, and the rate of heat transfer are computed and discussed in details for various values of the different parameters. Comparison of the obtained numerical results is made with previously published results.

Key words: Boundary Layer, Stretching Surface, Porous Medium, Thermal Radiation, Heat Source or Sink

Introduction

Boundary layer flow on a continuous moving surface has many practical applications in industrial manufacturing processes. Examples for such applications are: Aerodynamic extrusion of plastic sheets, cooling of infinite metallic plate in a cooling bath, the boundary layer along a liquid film in condensation processes, and a polymer sheet or filament extruded continuously from a dye. Applications also include paper production and glass blowing. It is very important to control the drag and the heat flux for better product quality. The continuous moving surface heat transfer problem has many practical applications in industrial manufacturing processes. Since the pioneering work of Sakiadis [1, 2], various aspects of the problem have been investigated by many authors. Most studies have been concerned with constant surface velocity and temperature (see Tsou et al [3]), but for many practical applications the surface undergoes stretching and cooling or heating that cause surface velocity and temperature variations.

Crane [4], Velggaar [5], and Gupta [6] have analyzed the stretching problem with a constant surface temperature, while Soundalgekar and Ramana [7] have investigated the constant surface velocity case with a power-law temperature variation. Grubka and Bobba [8] have analyzed the stretching problem for a surface moving with a linear velocity and with a variable surface temperature. Ali [9] has reported flow and heat characteristics on a stretched surface subject to power-law velocity and temperature distributions. The flow field of a stretching surface with power-law velocity variations was discussed by Banks [10]. Ali [11] and Elbashbeshy [12] extended Banks's work for a porous stretched surface for
different values of the injection. Elbashbeshy [13] have analyzed the stretching problem which was discussed by Elbashbeshy [12] to include a uniform porous medium. The unsteady heat transfer problems over a stretching surface, which is stretched with a velocity that depends on time are considered by Andersson et al. [14], a new similarity solution for the temperature field is devised, which transforms the time dependent thermal energy equation to an ordinary differential equation. Elbashbeshy and Bazid [15] studied the heat transfer over an unsteady stretching surface. Recently, Ishak et al [16] have studied the heat transfer over an unsteady stretching vertical surface. Ishak et al [17] have also investigated the unsteady laminar boundary layer over a continuously stretching permeable surface. While Ali and Mehmood [18] have presented a study of homotopy analysis of unsteady boundary layer flow adjacent to permeable stretching surface in a porous medium.

On the other hand, the effect of thermal radiation on boundary layer flow and heat transfer problems can be quite significant at high operating temperature. In view of this Elbashbeshy and Demain [19] and Hossain et al [20, 21] have studied the thermal radiation of a gray fluid which is emitting and absorbing radiation in non-scattering medium. Later, Chien-Hsin Chen [22] analyzed mixed convection of a power-law fluid past a stretching surface in the presence of thermal radiation and magnetic field. Battaler [23] has recently studied the effect of thermal radiation on the laminar boundary layer about a flat-plate. Fang and Zhang [24] have recently considered the thermal boundary layer over a shrinking sheet. They have obtained an analytic solution for the boundary layer energy equation for two cases including a prescribed power-law wall temperature case and a prescribed power-law wall heat flux case. The unsteady viscous flow over a continuously shrinking surface with mass transfer has been studied by Fang et al [25]. Ali and Magyari [26] have studied the problem of unsteady fluid and heat flow induced by a submerged stretching surface while its steady motion is slowed down gradually. Sharma and Singh [27] have studied the effects of Ohmic heating and viscous dissipation on steady MHD flow near stagnation point on an isothermal stretching sheet.

The present work is to study heat transfer over an unsteady stretching surface embedded in a porous medium in the presence of thermal radiation and heat source or sink. It may be remarked that the present analysis is an extension of and a complement to the earlier papers [15] and [17].

Formulation of the Problem

Consider an unsteady, two dimensional flow on a continuous stretching surface embedded in a porous medium, with surface temperature \( T_w \), and velocity

\[
U_w = \frac{b x}{(1 - \gamma t)} \quad \text{(see Andersson et al [14]).}
\]

The \( x \)-axis is taken along the continuous surface in the direction of the motion with the slot as the origin, and the \( y \)-axis is perpendicular to it. The conservation equations of the laminar boundary layer are

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (1)
\]
\[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} - \frac{\nu}{K} u, \]  
\[ \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{\kappa}{\rho c_p} \frac{\partial^2 T}{\partial y^2} - \frac{1}{\rho c_p} \frac{\partial q_r}{\partial y} + \frac{Q}{\rho c_p} (T - T_\infty), \]

with the associated boundary conditions:

\[ \begin{align*}
  y = 0 & : \quad u = U_\infty (x,t), \quad v = 0, \quad T = T_w (x,t), \\
  y \to \infty & : \quad u = 0, \quad T = T_\infty.
\end{align*} \]

Where \( u \) and \( v \) are the velocity components in the \( x \) and \( y \) directions respectively, \( t \) is time, \( \nu \) is the kinematic viscosity, \( K \) is the permeability, \( T \) is temperature inside the boundary layer, \( \kappa \) is the thermal conductivity, \( \rho \) is the fluid density, \( c_p \) is the specific heat at constant pressure, \( Q \) is the heat source when \( Q > 0 \) or heat sink when \( Q < 0 \), \( T_w \) is the surface temperature, and \( T_\infty \) is the free stream temperature.

It is assumed that the viscous dissipation is neglected; the physical properties of the fluid are constants. Using the Rosseland approximation for radiation \[22\], radiative heat flux is simplified as

\[ q_r = -\frac{4 \sigma}{3 \alpha^*} \frac{\partial T^4}{\partial y}, \]

where \( \sigma \) and \( \alpha^* \) are the Stefan-Boltzman constant and the mean absorption coefficient respectively. We assume that the temperature differences within the flow are such that the term \( T^4 \) may be expressed as a linear function of temperature. Hence, expanding \( T^4 \) in a Taylor series about \( T_\infty \) and neglecting higher-order terms, we get

\[ T^4 \approx 4T_\infty^3 T - 3T_\infty^4. \]

Using equations (5) and (6), the energy equation (3) becomes

\[ \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{\kappa}{\rho c_p} \frac{\partial^2 T}{\partial y^2} + 16 \sigma T_\infty^3 \frac{\partial T}{\partial y} + \frac{Q}{\rho c_p} (T - T_\infty). \]

The equation of continuity is satisfied if we choose a stream function \( \psi(x,y) \) such that \( u = \frac{\partial \psi}{\partial y} \) and \( v = -\frac{\partial \psi}{\partial x} \). The mathematical analysis of the problem is simplified by introducing the following dimensionless coordinates:

\[ \eta = \frac{b}{\sqrt{v(1 - \gamma t)}} y, \]

\[ \psi(x,y) = \frac{\sqrt{v b}}{1 - \gamma t} x f(\eta). \]

\[ \theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty}, \quad T_w - T_\infty = \frac{b}{2v x^2} (1 - \gamma t)^{3/2}. \]

Substituting equations (8)-(10) into equations (2) and (3), we obtain

\[ f''' + f'' f' - f' - A(f' + \frac{1}{2} \eta f'') - \lambda f' = 0. \]
\[
\left( 1+ \frac{4}{3R} \right) \theta'' + \text{Pr} \left[ f \theta' + 2f' \theta - \frac{A}{2} (3 \theta + \eta \theta') + \delta \theta \right] = 0. \quad (12)
\]

With the boundary conditions:
\[
\eta = 0: \quad f = 0, \quad f' = 1, \quad \theta = 1,
\]
\[
\eta \to \infty: \quad f' = 0, \quad \theta = 0, \quad (13)
\]

Where the prime denotes differentiation with respect to \( \eta \). \( A = \frac{\gamma}{b} \) is the parameter that measures the unsteadiness, \( \lambda = \frac{\nu^2}{K} \) Re is the permeability parameter, \( \text{Re}_x = \frac{U_e x}{\nu} \) is the local Reynolds number, \( R = \frac{\kappa \alpha \text{Re}_x}{4sT} \) is the thermal radiation parameter, \( \text{Pr} = \frac{\mu c_p}{\kappa} \) is the Prandtl number, \( \delta = \frac{Q \kappa \text{Re}_x}{\mu c_p \text{Re}_x^2} \) is the dimensionless heat source or sink, \( \mu \) is the dynamic viscosity, and \( \text{Re}_k = \frac{U_e \sqrt{\kappa}}{\nu} \).

**Numerical Solutions and Discussions**

Equations (11) and (12) with the boundary conditions (13) are solved using Runge-Kutta fourth order technique along with shooting technique. We first convert the two equations (11) and (12) into the following simultaneous linear equations of first order

\[
w_1' = w_2 \quad (14)
\]
\[
w_2' = w_3 \quad (15)
\]
\[
w_3' = w_2^2 + A(w_2 + 0.5 \eta w_3) + \lambda w_2 - w_1 w_3 \quad (16)
\]
\[
w_4' = w_5 \quad (17)
\]
\[
w_5' = \frac{3R \text{Pr}}{4+3R} (0.5A(3w_4 + \eta w_5) - w_1 w_5 - 2w_2 w_4 - \delta w_4) \quad (18)
\]

Where \( w_1 = f, \quad w_2 = f', \quad w_3 = f'', \quad w_4 = \theta, \quad w_5 = \theta' \)

Then the shooting technique is applied to transform the problem into initial value one. Where the initial conditions are

\[
w(0) = 0, \quad w_1(0) = 1, \quad w_2(0) = 1, \quad w_3(0) = m, \quad w_4(0) = n \quad (19)
\]

Hence \( m \) and \( n \) priori unknown and to be determined as a part of the numerical solution. Once the problem is reduced to initial value problem, then it is solved using Runge-Kutta fourth order technique. The computations have been carried out for various values of thermal radiation \( R \), heat source or sink parameter \( \delta \), permeability parameter \( \lambda \), Prandtl number \( \text{Pr} \), and unsteadiness parameter \( A \).
The accuracy of the numerical method is checked by performing various comparisons at different conditions with previously published works. The parameters of physical interest for the present problem are the local skin friction coefficient $C_f$ and the local Nusselt number $Nu_x$, which are defined as

$$C_f = \mu \frac{\partial u}{\partial y}_{y=0},$$

$$Nu_x = \frac{-x \frac{\partial T}{\partial y}_{y=0}}{T_w - T_\infty},$$
or

$$C_f \sqrt{Re_x} = f''(0),$$

$$\frac{Nu_x}{\sqrt{Re_x}} = -\theta'(0).$$

To validate the numerical method used in this study, the steady-state flow case $A = 0, \lambda = \frac{1}{\delta} = \frac{1}{R} = 0$, and $Pr = 1$ was considered and the results for the heat transfer rate at the surface $\theta'(0)$ are compared with those reported in references [8], [9], [15], and [17].

The quantitative comparison is shown in Table 1 and found to be in a very good agreement.

**Table 1. Comparison of $\theta'(0)$ for $A = \lambda = \delta = \frac{1}{R} = 0$ and $Pr = 1$.**

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0054</td>
<td>0.9999</td>
<td>1.0000</td>
</tr>
</tbody>
</table>

**Table 2. The values of $-f''(0)$ and $Nu_x/\sqrt{Re_x}$ for various values of $\lambda$ with $Pr = 10$, $A = 0.8$, $\delta = -0.5$, $R = 0.3$.**

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>0.1</th>
<th>0.3</th>
<th>0.5</th>
<th>0.7</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-f''(0)$</td>
<td>1.30035</td>
<td>1.37550</td>
<td>1.44668</td>
<td>1.51445</td>
<td>1.61071</td>
</tr>
<tr>
<td>$Nu_x/\sqrt{Re_x}$</td>
<td>0.98601</td>
<td>1.00056</td>
<td>1.01364</td>
<td>1.02552</td>
<td>1.04152</td>
</tr>
</tbody>
</table>

**Table 3. The values of $-f''(0)$ and $Nu_x/\sqrt{Re_x}$ for various values of $A$ with $Pr = 10$, $\lambda = 0.1$, $\delta = -0.5$, $R = 0.3$.**

<table>
<thead>
<tr>
<th>$A$</th>
<th>0.4</th>
<th>0.6</th>
<th>0.8</th>
<th>1</th>
<th>1.2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-f''(0)$</td>
<td>1.17853</td>
<td>1.24054</td>
<td>1.30035</td>
<td>1.35799</td>
<td>1.41357</td>
</tr>
<tr>
<td>$Nu_x/\sqrt{Re_x}$</td>
<td>0.53765</td>
<td>0.77808</td>
<td>0.98601</td>
<td>1.16997</td>
<td>1.33570</td>
</tr>
</tbody>
</table>

**Table 4. The values of $-f''(0)$ and $Nu_x/\sqrt{Re_x}$ for various values of $R$ with $Pr = 10$, $\lambda = 0.1$, $\delta = -0.2$, $A = 0.8$.**

<table>
<thead>
<tr>
<th>$R$</th>
<th>0.1</th>
<th>0.3</th>
<th>0.5</th>
<th>0.7</th>
<th>1</th>
</tr>
</thead>
</table>
Table 5. The values of $-f''(0)$ and $\frac{Nu_s}{\sqrt{Re_s}}$ for various values of $\delta$ with $Pr = 10$, $\lambda = 0.1$, $R = 0.3$, $A = 0.8$.

<table>
<thead>
<tr>
<th>$\delta$</th>
<th>$-f''(0)$</th>
<th>$Nu_s/\sqrt{Re_s}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.5</td>
<td>1.30035</td>
<td>0.89601</td>
</tr>
<tr>
<td>-0.2</td>
<td>1.30035</td>
<td>0.73898</td>
</tr>
<tr>
<td>0</td>
<td>1.30035</td>
<td>0.54473</td>
</tr>
<tr>
<td>0.2</td>
<td>1.30035</td>
<td>0.31343</td>
</tr>
<tr>
<td>0.4</td>
<td>1.30035</td>
<td>0.02044</td>
</tr>
</tbody>
</table>

Table 6. The values of $-f''(0)$ and $\frac{Nu_s}{\sqrt{Re_s}}$ for various values of $Pr$ with $\delta = -0.2$, $\lambda = 0.1$, $R = 0.7$, $A = 0.4$.

<table>
<thead>
<tr>
<th>$Pr$</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-f''(0)$</td>
<td>1.17847</td>
<td>1.17847</td>
<td>1.17847</td>
<td>1.17847</td>
<td>1.17847</td>
</tr>
<tr>
<td>$Nu_s/\sqrt{Re_s}$</td>
<td>0.23865</td>
<td>0.20787</td>
<td>0.15527</td>
<td>0.09498</td>
<td>0.03148</td>
</tr>
</tbody>
</table>

From Table 2, we note that, the effect of permeability parameter, $\lambda$, is to decrease the skin frction due to internal heat absorption parameter $\delta < 0$. It is interesting to note that the rate of heat transfer decreases with increasing the permeability parameter.

Also, we note from Table 3 that the surface gradient $f''(0)$ decreases with the increase of the unsteadiness parameter $A$, that that rate of heat transfer decreases with unsteadiness parameter.

From tables 4, 5, and 6, we note that the effect of the radiation parameter $R$, is to decrease the rate of heat transfer while the rate of heat transfer increases with the heat source parameter and decreases with heat sink parameter. Further it is noted that the rate of heat transfer at the surface decreases with Prandtl number. It is also evident that the temperature gradient $\theta'(0)$ is negative for all parameters values considered in this study which means that there is a heat flow from the wall.

Figures 2 and 3 present the velocity profiles, for various values of $A$ and $\lambda$ respectively, while the other parameters are kept constant. From these figures, we note that the velocity decreases with increasing the value of the unsteadiness parameter $A$ and the permeability parameter $\lambda$.

Figures 4 - 7 present the temperature profiles for various values of $A, \delta, R$, and $Pr$ respectively while the other parameters are kept constant. From these figures, we note that the temperature decreases with the increase of the value of the unsteadiness parameter $A$, the radiation parameter $R$, and the Prandtl number $Pr$, while the temperature increases with increasing the value of the heat source or sink parameter $\delta$.

Conclusions

Numerical solutions have been obtained for the effects of the thermal radiation and heat transfer over an unsteady stretching surface embedded in a
porous medium in the presence of heat source or sink. An appropriate similarity transformed was used to transform the system of time-dependent partial differential equations to a set of ordinary differential equations. These equations are solved numerically by applying the Nachtsheim-Swinger shooting technique together with Runge-Kutta fourth order integration scheme. Numerical computations show that the present values of the rate of heat transfer are in a close agreement with those obtained by previous investigation in the absence of porous medium, thermal radiation, heat source or sink, and unsteadiness parameter. The following results are obtained:

1. The velocity decreases with an increase in the value of unsteadiness parameter and permeability parameter.
2. The temperature decreases with an increase in the value of unsteadiness parameter, radiation parameter, and Prandtl number while it increases with an increase in the value of heat source or sink parameter.
3. The surface gradient $f''(0)$ is negative and decreases with increasing unsteadiness parameter and permeability parameter.
4. The rate of heat transfer $\theta'(0)$ is negative decreases with increasing the unsteadiness parameter, permeability parameter, and radiation parameter and increases with increasing heat source or sink parameter.

**Nomenclature**

- $A$ - unsteadiness parameter $\left(=\frac{\gamma}{b}\right)$, [-]
- $b$ - positive constant, $[s^{-1}]$
- $C_f$ - local skin-friction coefficient $\left(=2\tau_w/\rho U^2_\infty\right)$
- $c_p$ - specific heat due to constant pressure, $[Jkg^{-1}K^{-1}]$
- $f$ - dimensionless stream function, [-]
- $K$ - Permeability, $[m^2]$
- $Nu_x$ - Nusselt number, [-]
- $Pr$ - Prandtl number $\left(=\frac{\mu c_p}{k}\right)$, [-]
- $Q$ - heat source or sink
- $q_r$ - radiation heat flux $\left(=\frac{-4\sigma T^4}{3\alpha}\right)$, $[kgm^{-2}]$
- $R$ - thermal radiation parameter $\left(=\frac{k \alpha}{4\sigma T^3_\infty}\right)$, [-]
- $Re_x$ - local Reynolds number $\left(=\frac{U_w x}{\nu}\right)$, [-]
- $T$ - temperature of the fluid, $[K]$
- $t$ - time, $[s]$
- $T_w$ - surface temperature, $[K]$
\( T_w \) - free stream temperature, \([K]\)
\( U_w \) - surface velocity, \([ms^{-1}]\)
\( u \) - fluid velocity in \( x \) direction, \([ms^{-1}]\)
\( v \) - fluid velocity in \( y \) direction, \([ms^{-1}]\)
\( x, y \) - Cartesian coordinates along the surface and normal to it, respectively, \([m]\)

\textit{Greek letters}

\( \alpha^* \) - mean absorption coefficient, \([m^{-1}]\)
\( \gamma \) - stretching rate, \([s^{-1}]\)
\( \delta \) - dimensionless heat source or sink \( \left( \frac{Qv^2 \text{Re}_s}{\mu c_p \text{U}_m^2} \right) \), [-]
\( \eta \) - similarity variable, [-]
\( \theta \) - similarity temperature function, [-]
\( \kappa \) - thermal conductivity, \([kg.m.s^{-3}.K^{-1}]\)
\( \lambda \) - permeability parameter \( \left( \frac{v^2}{K} \right) \), \([m^2.s^{-2}]\)
\( \mu \) - dynamic viscosity of the fluid, \([Nsm^{-2}]\)
\( \nu \) - kinematic viscosity \( \left( \frac{\mu}{\rho} \right) \), \([m^2.s^{-1}]\)
\( \rho \) - density of fluid, \([kgm^{-3}]\)
\( \sigma \) - Stefan-Boltzman constant, \([kgm^{-2}.K^{-4}]\)
\( \tau_w \) - skin friction, \([Nm^{-2}]\)
\( \psi \) - stream function, \([m^2.s^{-1}]\)

\textit{Superscript}

- differentiation with respect to \( \eta \)

\textit{Subscripts}

\( w \) - surface conditions
\( \infty \) - conditions far away from the surface

\textbf{Figure captions}

Figure 1: physical model and coordinate system.
Figure 2: Velocity profiles \( f'(\eta) \) for various values of \( A \) with \( \text{Pr} = 10, \text{R} = 0.3, \lambda = 0.1, \text{and} \ \delta = -0.5 \)
Figure 3: Velocity profiles \( f'(\eta) \) for various values of \( \lambda \) with \( \text{Pr} = 10, \text{R} = 0.3, \text{A} = 0.8, \text{and} \ \delta = -0.5 \)
Figure 4: Temperature profiles \( \theta(\eta) \) for various values of \( A \) with \( \text{Pr} = 10, \text{R} = 0.3, \lambda = 0.1, \text{and} \ \delta = -0.5 \)
Figure 5: Temperature profiles \( \theta(\eta) \) for various values of \( \delta \) with \( \text{Pr} = 10, \text{R} = 0.3, \lambda = 0.1, \text{and} \ A = 0.8 \)
Figure 6: Temperature profiles $\theta(\eta)$ for various values of $R$ with $Pr = 10$, $\delta = -0.2$, $\lambda = 0.1$, and $A = 0.4$

Figure 7: Temperature profiles $\theta(\eta)$ for various values of $Pr$ with $R = 0.7$, $\lambda = 0.1$, $\delta = -0.2$ and $A = 0.4$

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