The thermomechanical buckling and postbuckling behavior of layered composite shell type structure are considered with the finite element method (FEM) under the combination of temperature load and applied mechanical loads. To account for through-thickness shear deformation effects, the thermal elastic, Higher-Order Shear deformation Theory (HOST) is used in this study. The refined higher order theories, that takes into account the effect of transverse normal deformation, is used to develop discrete finite element models for the thermal buckling analysis of composite laminates. Attention in this study is focused on analyzing the temperature effects on buckling and postbuckling behavior of thin shell structural components. Special attention in this paper is focused on studying of values of the hole in curved panel on thermal buckling behavior and consequently to expend and upgrade previously conducted investigation. Using FEM, a broader observation of the critical temperature of loss of stability depending on the size of the hole was conducted. The presented numerical results based on HOST can be used as versatile and accurate method for buckling and postbuckling analyzes of thin-walled laminated plates under thermo-mechanical loads.

Key words: Geometric nonlinearity, Buckling, Postbuckling, Thermal loads, FEM, Shells

1. Introduction

Thin-walled layered composite structures are increasingly used in any engineering branch where structural weight is one of the major aspects in the design process. When composite materials are being employed computation methods by using numerical simulation (FEM) are required. These
methods adequately take effects such as material anisotropy, coupling effects and shear deformations into account which are inherent in this class of materials.

The composite laminates due to their high specific strength and stiffness are increasingly used in weight-sensitive applications such as aircraft and space vehicles [1-3]. Most of these vehicles have to operate in hostile thermal environments; as a result, the structural components of these vehicles are subjected to thermal loads [4-6].

Their components are often subjected to combinations of mechanical and thermal loading. In fact, many structures are subjected to high load levels and consequently may result in nonlinear load-deflection relationships due to large deformations of the plate. One of the important problems deserving special attention is the study of their nonlinear response to large deflections and postbuckling.

Many studies according to classical plate theory for the large deflection of multilayered composite plates subjected to mechanical or thermal loading are presented in [7, 8]. Numerous studies involving the application of the shear deformation plate theory to nonlinear bending analysis can be found in [9-11]. In contrast, there have been fewer investigations for the thermal postbuckling of composite laminated plates [12, 13]. The problem of buckling under thermo-mechanical loading has been considered by relatively few investigators [14-16]. The analysis of the buckling and postbuckling behavior of isotropic or composite laminated shells is a topic of considerable technical importance in number of branches of engineering. Such behavior may result from mechanical loading or from thermal loading or from a combination of the two, i.e. from thermo-mechanical loading. A book [17] contains much information on available methods, particularly as related to flat, rectangular plates, and includes details of several hundred pertinent references dating up until the mid-1990s. A large part of this literature, however, is naturally concerned with buckling under mechanical loading but less information exists on the buckling of shells under thermal loading.

In the present paper the particular concern is with the analysis of the buckling and postbuckling behavior under thermo-mechanical loading of isotropic and composite shell type structures. Here the use of FEM in predicting buckling and postbuckling responses of isotropic and composite shells subjected to thermal or mechanical loading or combined thermo-mechanical loading is observed.

It is useful to mention that various geometric and material nonlinearity problems are solved by using FEM based on the First Order Shear deformation Theory (HOST) [18-23] and here the particular adopted approach presents the FEM in the context of HOST.

2. Nonlinear analysis

In the present work, thermal buckling analyses of multilayered composite panel using discrete finite element model is presented. The finite element model is based on the refined higher order theories [24-28] that considers the effect of transverse normal deformation.

The formulation of the presented shell finite element is based on the single-layer 2-D theory because of its ability of an adequate representation of the global behavior (deflections, stresses, buckling loads) of thin composites. The HOST used here, assumes the parabolic distribution of the transverse shear stresses across the laminate thickness. The displacement field for the parabolic transverse shear deformation through the shell thickness is given by:
The relations (1) are obtained assuming that the transverse shear stresses \( \sigma_4 \) and \( \sigma_5 \) are zero on the shell surfaces:

\[
\begin{align*}
\sigma_4(x, y, \pm h/2) &= 0 \\
\sigma_5(x, y, \pm h/2) &= 0
\end{align*}
\]

The parameters \( a \), \( b \) and \( c \) introduced in Eq. (1) can have the values zero and one. By combining their values, the displacement field given by Eq. (1) can very simply describe the third order shear deformation theory, the first order shear deformation theory and the classical Kirchhoff’s plate theory. In that way it is possible to say that Eq. (1) represent the general expressions for the displacements of an arbitrary point of a multi-layered shell for the third order theory. Such a way of presentation of displacements is suitable for subsequent considerations of the formulation of a general shell finite element. This is particularly suitable for computer programme realization, since by combining parameters \( a \), \( b \) and \( c \), it is possible to obtain the desired type of the shell finite element able to describe the thin and thick multi-layered composite shells.

The next governing equation can be used to study the linear/nonlinear static and eigenvalue buckling analysis and can be written as [10, 15, 23]:

\[
\begin{bmatrix} [K] - [K_T] + [K_G] + \frac{1}{2} [N_1(\delta)] + \frac{1}{3} [N_2(\delta)] \end{bmatrix} \{\delta\} = \{F_M\} + \{F_T\}
\]

The governing equation (3) can be used to study the linear/nonlinear static and eigenvalue buckling analysis by neglecting the appropriate terms as [15]:

a) Linear static analysis:

\[
[K]\{\delta\} = \{F_M\} + \{F_T\}
\]

b) Nonlinear static analysis:

\[
\begin{bmatrix} [K] - [K_T] + \frac{1}{2} [N_1(\delta)] + \frac{1}{3} [N_2(\delta)] \end{bmatrix} \{\delta\} = \{F_M\} + \{F_T\}
\]

c) Eigenvalue buckling analysis:

\[
[K]\{\delta\} = \Delta T[K_G^*]
\]

where determining matrix \([ K_G^*] \) means that the linear static analysis of the shell using Eq. (4) have to be carried out. In this analysis the resulting deformation filed is used to determine the initial state of stress resultants. For that purpose Mindlin formulation is used [18, 21]. Mentioned formulation is also used for matrix \([ K_G^*] \) determination. Solution of Eq. (5) can be obtained using Newton-Raphson iteration procedure coupled with displacement control method [18]. To achieve equilibrium
for each load/displacement step Bergan and Clough [19] proposed convergence criteria within the specific tolerance limit of less than 1%.

Following the usual procedure for assembling element stiffness matrices, the equilibrium and stability conditions are expressed as:

\[ [K + \lambda K_c] \delta u = [F] \]  \hspace{1cm} (7)

3. Thermal buckling analysis

Calculating the critical temperature of buckling due to thermal load is a two-stage process. For a specified rise (\(\Delta T\)) in temperature the thermal loads are computed and a linear static analysis is carried out to determine the thermal stress resultants. These stress resultants are then used to compute the geometric stiffness matrix, which is subsequently used in Eq. (7), to determine the least eigenvalue \(\lambda\) and the associated mode shape \(\delta u\). The critical temperature \(T_{CR}\) of the plate is calculated using:

\[ T_{CR} = \lambda \cdot \Delta T \]  \hspace{1cm} (8)

In the present analysis, a 4-node quadrilateral from the ‘serendipity’ family of two-dimensional \(C^0\) continuous isoparametric element with 8 degrees of freedom per node [28] is used. The formulation of a 4-nodes shell finite element that can be good enough also if applied to the thin multilayered plates or shells is by no means an easy matter. The authors’ experience has shown that a good approach to the formulation of a 4-node shell finite element can be based on the application of the Discrete Kirchhoff’s Theory (DKT) for bending behavior. DKT ensures \(C^1\) continuity at discrete points on inter-element boundaries. The improved 4-nodes layered shell element is derived combining HOST and DKT (Fig. 1). More details about that element can be found in [27] and [28].

![Fig. 1. Improved 4-nodes shell finite element](image)

In the \(C^0\) finite element theory the continuum displacement vector within the element is defined by:

\[ a = \sum_{i=1}^{M} N_i(r,s)a_i \]  \hspace{1cm} (9)

In the case of the negligible mid-surface normal stress \(\sigma_z\) the stress-displacement relationships, stress resultants and the constitutive equations associated with HOST are given in [27] and [28].
The total stiffness matrix of the element is obtained by the linear superposition of the following three independent parts:

I. Membrane stiffness matrix \( K_M \)
II. Bending stiffness matrix \( K_B \), and
III. Rotational stiffness matrix \( K_\Theta \)

In order to avoid irregular systems of equations in the case of completely plane systems, a very small rotational stiffness is adjoined to the variable \( \Theta \), defining the rotation about the \( z \)-axis and it causes a larger stiffness of the system. The displacements \((u, v)\) for the membrane element behavior are approximated by 6-term quadratic polynomials as shown in [11] and are defined by the Eq. (10):

\[
\begin{align*}
u &= \sum_{i=1}^{4} N_i(r, s) V_i + (1 - r^2)\beta_1 + (1 - s^2)\beta_2 \\
u &= \sum_{i=1}^{4} N_i(r, s) U_i + (1 - r^2)\alpha_1 + (1 - s^2)\alpha_2
\end{align*}
\]

The displacements \( b = [\alpha_i, \beta_i]^T \) can be taken as some internal displacements having a quadratic effect on actual displacement. The membrane element equilibrium relations are organized in a matrix form:

\[
\begin{bmatrix}
K_{11} & K_{12} \\
K_{21} & K_{22}
\end{bmatrix}
\begin{bmatrix}
d_n \\
b
\end{bmatrix}
= 
\begin{bmatrix}
F_1 \\
F_2
\end{bmatrix}
\]

(11)

By static condensation of the internal variables \( b \), one obtains:

\[
\begin{align*}
K_M d_n &= F_n \\
K_M &= K_{11} - K_{12}K_{22}^{-1}K_{21} \\
F_n &= F_1 - K_{12}K_{22}^{-1}F_2 \\
b &= -K_{22}^{-1}K_{21}d_n + K_{22}^{-1}F_2
\end{align*}
\]

(12)

For the element properties the Gauss Quadratic formulae with 2x2 points are used. By static condensation the internal variables \( \alpha_i, \beta_i \) are eliminated on element level and the total number of membrane degrees of freedom per element is not changed.

4. Numerical examples

In this study three software packages were used. FEM-based commercial software was used for individual validation and comparison of the results, in-house FEM-based software “SAMKE” [29] was used for structural analyses, in which HOST is embedded, and also analytical in-house software was used, primary for the purpose of assessment of the service life of the structures.

This section presents some numerical examples of nonlinear behavior of isotropic and composite structure affected by temperature.

4.1 Thermal buckling and postbuckling behavior of a curved laminated panel with a hole
In this section results of buckling and postbuckling behavior of a curved laminated composite panel with circular hole subjected to thermal loads are shown. The primary goal is to verify the computation analyzes of the loss of stability of the considered panel under the action of thermo-mechanical loads. When considering the stability of a curved composite panel with circular hole the effects of variations in laminate stacking sequence, fiber orientation, number of layers and aspect ratio of the panels are important parameters to their buckling and postbuckling behavior.

Material properties:

<table>
<thead>
<tr>
<th>Material</th>
<th>Type</th>
<th>Properties</th>
</tr>
</thead>
<tbody>
<tr>
<td>Material 1: Isotropic</td>
<td></td>
<td>(E = 200 \text{ [GPa]}) (\nu = 0.33) (\alpha = 9.0 \times 10^{-6} \text{ [1/°C]})</td>
</tr>
<tr>
<td>Material 2: Composite</td>
<td></td>
<td>(E_L = 130.3 \text{ [GPa]}) (E_T = 9.37 \text{ [GPa]}) (G_{LT} = 4.502 \text{ [GPa]}) (\nu_{LT} = 0.33) (\alpha_L = 0.139 \times 10^{-6} \text{ [1/°C]}) (\alpha_T = 9.0 \times 10^{-6} \text{ [1/°C]}) ([\pm 45°/0°/90°]_{2S})</td>
</tr>
</tbody>
</table>

For the purpose of analyzing the effect of a hole size and radius of curvature on panel stability the curved panel subjected to uniform temperature change is considered. For numerical validation geometry and material properties (Material 2) associated with each lamina are given in Fig. 2. Boundary conditions of the model in numerical simulations are: \(x = 0, x = W, y = 0, y = L, v = w = w_y = 0\) (without edge restraints). For this problem, several finite element meshes were tested and practically the same results were obtained but the finest finite element mesh was selected. Established finite element mesh is modeled by 4800 elements, i.e. 4960 nodes and it is presented in Fig. 3.
Additional information is in detail elaborated and depicted in [15], such as elastic-plastic material behavior and stress distribution as a function of linearity and nonlinearity.

The results of buckling and postbuckling behavior of a curved layered composite panel with a circular hole subjected to thermal loads are presented in Figures 4 to 7.

![Fig. 4. The first buckling mode using linear eigenvalue method](image)

These figures suggest that effects of thermal loads on buckling and postbuckling behavior are obvious. The postbuckling temperature increases with increasing the hole size, which could be easily recognised by observing Figures 4 to 7 and also Table 1. Comparison of the results derived from numerical simulations and available results presented in [14] clearly suggests that good agreements between them is achieved. This agreement is depicted in Fig. 5, where thermal buckling and postbuckling responses are presented on diagram on which vertical axis represents temperature and horizontal axis represents relative displacement (w/h). Presented agreements propose that conducted numerical simulations were proven as a credible numeric for thermal buckling and postbuckling prediction and can be use for the purpose of numerical investigation of the geometrical domain between these two observed geometries defined by radius value of the hole of 10 [mm] and 35 [mm]. The main goal of the numerical investigation of the mentioned domain is to study the dependence between radius value of the hole and the critical buckling temperature and to inspect and define nearly the highest critical buckling temperature affected by the maximal radius value after which the loss of stability of composite panel occurs.

![Fig. 5. Thermal buckling and postbuckling responses of a curved panel with a hole](image)
The effect of radius value of the hole ($a$) in a curved layered composite panel, as shown in Fig. 6 (Material 2), on the critical buckling values of temperature ($T_{CR}$) are given in Table 1.

**Table 1. The effect of radius value on critical buckling temperatures**

<table>
<thead>
<tr>
<th>$T_{CR}$ [$^\circ$C]</th>
<th>588</th>
<th>601</th>
<th>619</th>
<th>634</th>
<th>648</th>
<th>660</th>
<th>674</th>
<th>623</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$ [mm]</td>
<td>10</td>
<td>15</td>
<td>20</td>
<td>25</td>
<td>27.5</td>
<td>30</td>
<td>32.5</td>
<td>35</td>
</tr>
</tbody>
</table>

As results suggest, it is obvious (Table 1 and Fig. 6) that radius value of the hole of about 32.5 [mm] presents maximal possible value after which the critical behavior of the composite shell structure occurs. It can be concluded from Fig. 6 that trend between simulated points could be assumed as polynomial. For the simulated domain, using diagram presented in Fig. 6, it is also possible to determine critical buckling temperature for any radius value of the interest which is under the analyzed domain. This means that diagram can be useful as a first iteration in practical thermal buckling analyzes when is necessary to determine critical temperature for exact radius value, but for that purpose, necessary verification procedure has to be conducted for randomly selected hole diameters that differ from those presented in Table 1, which means that percent discrepancies between values calculated using extracted equation, i.e. analytical expression, and simulated (FEM-based) values should be examined.

![Fig. 6. Critical buckling temperature as a function of radius of the hole in a curved panel](image)

Also, along with Table 1, for the purpose of visualization, Fig. 7 is presented to depict the effect of radius value on critical buckling temperatures but only for new six numerical points observed within this study.
Conclusions

Thermal buckling and postbuckling behaviors of curved laminated composite panels with holes have been examined by employing the finite element technique based on the HOST, which is a tool available in in-house FEM-based software “SAMKE”, that allows parabolic description of the transverse shear stresses and therefore the shear correction factors of the usual shear deformation theory are not required. An improved HOST is employed to account for the transverse shear strains by maintaining stress-free top and bottom forces of the panel.

The good agreement between numerical and experimental results was achieved which leads to conclusion that presented finite element results based on HOST can be used as versatile, accurate and trustable numerical method for buckling and postbuckling analyzes of a thin-walled isotropic and composite structural components under thermo-mechanical loads. Within this research, after conducted validation, numerical investigation of the geometrical domain between two radius values was carried out to determine the critical temperature at which the loss of stability of a composite panel occurs, depending on the radius of the hole located in it. Furthermore, within this domain, a verified
analytical expression for the critical buckling temperature, as a function of radius of the hole in a curved composite panel, can be extracted, which presents common but trustable procedure that can be used as a first iteration in practical thermal buckling analyzes when is necessary to determine critical buckling temperature for exact radius value.

Proposed well-validated numerical approach can be also used for practical instability analysis of the structures made of the fiber reinforced laminates. It should be emphasized that presented results were achieved for particular models, but the whole approach is completely universal and can be applied on any isotropic or composite panel. This approach can be widely used in aircraft and spacecraft industry, and also in design process of the structural components that operates in hostile thermal environments in thermal power plants.

**Appendix A. Deformation patterns for curved composite panels with embedded hole subjected to thermal loading**

Figures A1 and A2 depict numerically obtained deformations of a curved composite panels with a hole located in it under the thermal load for radius value of 10 [mm] (Fig. A1) and 35 [mm] (Fig. A2) as values (geometries) used for the purpose of validation.

![Deformation patterns for curved composite panel with hole value of a=10 mm](image1)

\[ T = 600 \, ^\circ\text{C} \]

\[ T = 850 \, ^\circ\text{C} \]

\[ T = 1000 \, ^\circ\text{C} \]

\[ T = 1100 \, ^\circ\text{C} \]

**Fig. A1. Deformation patterns for curved composite panel with hole value of a=10 [mm] [15]**
Appendix B. Buckling and postbuckling of a curved isotropic panel without hole under combined thermal and mechanical loads

This example considers buckling and postbuckling behavior of a curved isotropic panel without hole subjected to combined thermal load and external mechanical load in form of a pressure (Geometry in Fig. 2 for Material 1 and same boundary conditions). Computational results derived from FEM simulations and the effects of combined loads to relative displacement of a curved panel are presented in Fig. B1.

Fig. B1. Effect of thermo-mechanical load to relative displacement (w/h) of a curved panel
Figure B1 shows a nonlinear analysis in the domain of temperature change up to 400 °C \((T = 400 \, ^\circ C)\). The applied thermal loadings have a significant effect on the thermal buckling and postbuckling responses of considered curved panel. In fact, the structure undergoes buckling at lower temperature when the applied thermal field is uniform through the thickness. Thermal stresses developed due to elevated temperature will lead to buckling failure of these slender structural elements.

**Acknowledgment**

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**Nomenclature**

- \(u, v\) - translations of the points in the middle plane \((x, y, z = 0)\) [mm]
- \(w\) - out-of-plane deflection [mm]
- \(\Psi_1, \Psi_2\) - rotations of the normals about the y and x axes [rad]
- \([K]\) - linear stiffness matrix [-]
- \([N_i]\) - nonlinear stiffness matrices [-]
- \([N_2]\) - nonlinear stiffness matrices [-]
- \([K_T]\) - geometric stiffness matrices due to thermal stress [-]
- \([K_G]\) - geometric stiffness matrices due to initial stress [-]
- \([F_M]\) - mechanical load vector [-]
- \([F_T]\) - thermal load vector [-]
- \(\delta\) - vector of degrees of freedom associated to the displacement field in a finite element discretization [-]
- \([K_G^*]\) - geometric stiffness matrix due initial state of stress [-]
- \(\Delta T\) - temperature rise [°C]
- \(\alpha_L\) - coefficient of thermal expansion in longitudinal direction [1/°C]
- \(a\) - nodal displacement vector of the plate [mm]
- \(\gamma\) - least eigenvalue [-]
- \(\delta u\) - associated mode shape [-]
- \(N_i(r, s)\) - interpolation function associated with the node \(i\) and expressed through the normalized coordinates \((r, s)\) [-]
- \(M\) - is the number of nodes in the element [-]
- \(L\) - length of panel [cm]

\(\{F\}\) - global load vectors [N]
\(\lambda\) - least eigenvalue [-]
\(\delta u\) - associated mode shape [-]

\(\{K\}\) - linear stiffness matrix [-]
\(\{N_i\}\) - nonlinear stiffness matrices [-]
\(\{d_n\}\) - nodal variables [-]
\(\{b\}\) - non-conforming modes [-]

\(\{F_M\}\) - corresponding equivalent load components [N]
\(\{F_T\}\) - thermal load vector [-]
\(\{E_T\}\) - modulus of elasticity in transversal direction [GPa]
\(\{E\}\) - modulus of elasticity [GPa]

\(\{K\}\) - linear stiffness matrix [-]
\(\{N_i\}\) - nonlinear stiffness matrices [-]
\(\{d_n\}\) - nodal variables [-]
\(\{b\}\) - non-conforming modes [-]

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\(\{K\}\) - linear stiffness matrix [-]
\(\{N_i\}\) - nonlinear stiffness matrices [-]
\(\{d_n\}\) - nodal variables [-]
\(\{b\}\) - non-conforming modes [-]
\( a_i \) - generalized displacement vector in the mid-surface [nm]
\( W \) - width of panel [cm]
\( \nu \) - Poisson's coefficient [-]
\( \nu_{LT} \) - Poisson's coefficient of orthotropic plate [-]
\( \alpha \) - coefficient of thermal expansion [1/°C]
\( \frac{w}{h} \) - relative displacement
\( R \) - radius of hole [cm]
\( T_{CR} \) - critical temperature [°C]

References


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