STUDYING HEAT CONDUCTION IN A SPHERE CONSIDERING HYBRID FRACTIONAL DERIVATIVE OPERATOR

by

Abass H. ABDEL KADER[a], Mohamed. S. ABDEL LATIF[a*], and Dumitru BALEANU[b,c]

[a] Mathematics and Engineering Physics Department, Faculty of Engineering, Mansoura University, Mansoura, Egypt
[b] Department of Mathematics, Faculty of Arts and Sciences, Cankaya University, Ankara, Turkey
[c] Institute of Space Sciences, Magurele-Bucharest, Romania

Original scientific paper
https://doi.org/10.2298/TSCI200524332K

In this paper, the fractional heat equation in a sphere with hybrid fractional derivative operator is investigated. The heat conduction is considered in the case of central symmetry with heat absorption. The closed form solution in the form of three parameter Mittag-Leffler function is obtained for two Dirichlet boundary value problems. The joint finite sine Fourier-Laplace transform is used for solving these two problems. The dynamics of the heat transfer in the sphere is illustrated through some numerical examples and figures.

Key words: *heat conduction with absorption, hybrid fractional derivative operator, three parameter Mittag-Leffler function, finite fourier transform, laplace transform*

Introduction

Recently, fractional calculus is used to study many real world problems formulated in the form of fractional PDE [1-6]. Many definitions for the fractional derivative are proposed in the literature [7-12]. Some examples of these definitions are Caputo, Riemann-Liouville, He's fractional derivative, generalized fractional derivatives, and Reisz definitions [1-20].

Fractal calculus is very useful in modeling phenomena in hierarchical or porous media and it can reveal hidden structures that continuum mechanics would never be able to find [16]. In order to deal with problems in porous media, He [15, 20] developed a new generalized fractional derivative which is given by:

$$ {}_0D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}x} \int_0^t [f_0(\tau) - f(t)](\tau - t)^{-\alpha} \mathrm{d}\tau, \quad 0 < \alpha < 1 $$

where f_0 is the solution of the continuous problem with the same conditions of the fractal problem.

Very recently, the hybrid fractional derivative is proposed in [21]. The definition of this new derivative is given by:

---

* Corresponding author, e-mail: m_gazia@mans.edu.eg

$$ {}_{0}^{\mathrm{CPC}}D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t [k_1(\alpha) f(\tau) + k_0(\alpha) f'(\tau)](t-\tau)^{-\alpha} \,\mathrm{d}\tau $$

which is a linear combination of Caputo derivative and Riemann-Liouville integral. This new definition is widely used in modeling many phenomena in science and engineering, see for example [22-24].

In this paper, we consider the following time-fractional heat conduction equation with heat absorption term in spherical coordinates in the case of central symmetry [6]:

$$ {}_{0}^{\mathrm{CPC}}D_t^{\alpha} T(r,t) = a\left(T_{rr} + \frac{2}{r}T_r\right) - bT \,, \quad 0 \le r < R \tag{1} $$

with the following two cases of Dirichlet conditions:

$$ T(r,0) = 0, \quad T(R,t) = p_0 \delta(t) \tag{2a} $$

$$ T(r,0) = 0, \quad T(R,t) = T_o t^p \tag{2b} $$

where p_0, T_0 are arbitrary constants and $\delta(t)$ is the Dirac delta function.

In the next section, we use the joint finite sin Fourier-Laplace transform to solve eq. (1) with conditions (2a) and (2b).

Exact solution of eq. (1) with condition (2a)

Applying finite sin Fourier transform, see eq. (A1) in the *Appendix*, to eq. (1), we obtain:

$$ \mathcal{F}\left[{}_{0}^{\mathrm{CPC}}D_t^{\alpha} T(r,t) \right] = \mathcal{F}\left[a\left(T_{rr} + \frac{2}{r}T_r\right) - bT \right] \tag{3} $$

Using eq. (A3) in *Appendix*, we obtain:

$$ {}_{0}^{\mathrm{CPC}}D_t^{\alpha} T(\xi_k,t) = a[-\xi_k^2 T(\xi_k,t) + (-1)^{k+1} R T(R,t)] - bT(\xi_k,t) \tag{4} $$

Using the condition $T(R,t) = p_0\delta(t)$, we obtain:

$$ {}_{0}^{\mathrm{CPC}}D_t^{\alpha} T(\xi_k,t) = a[-\xi_k^2 T(\xi_k,t) + (-1)^{k+1} R\, p_0 \delta(t)] - bT(\xi_k,t) \tag{5} $$

Or equivalently:

$$ {}_{0}^{\mathrm{CPC}}D_t^{\alpha} T(\xi_k,t) = a(-1)^{k+1} R\, p_0 \delta(t) - (a\xi_k^2 + b) T(\xi_k,t) \tag{6} $$

Applying Laplace transform to eq. (6), we obtain:

$$ \mathcal{L}[{}_{0}^{\mathrm{CPC}}D_t^{\alpha} T(\xi_k,t)] = \mathcal{L}[a(-1)^{k+1} R\, p_0 \delta(t) - (a\xi_k^2 + b) T(\xi_k,t)] \tag{7} $$

Using eq. (A5) in the *Appendix*, eq. (7) becomes:

$$ \left[\frac{k_1(\alpha)}{s} + k_0(\alpha)\right] s^{\alpha} T(\xi_k,s) - k_0(\alpha) s^{\alpha-1} T(\xi_k,0) = a(-1)^{k+1} R\, p_0 - (a\xi_k^2 + b) T(\xi_k,s) \tag{8} $$

Using the condition $T(r, 0) = 0$ we obtain:

$$\left[\frac{k_1(\alpha)}{s} + k_0(\alpha)\right] s^\alpha T(\xi_k, s) = a(-1)^{k+1} R\, p_0 - (a\xi_k^2 + b) T(\xi_k, s) \tag{9}$$

Solving eq. (9) with respect to $T(\xi_k, s)$, we obtain:

$$T(\xi_k, s) = \frac{a(-1)^{k+1} R\, p_0}{k_0(\alpha) s^\alpha + k_1(\alpha) s^{\alpha-1} + a\xi_k^2 + b} \tag{10}$$

Equation (10) can be rewritten in the form:

$$T(\xi_k, s) = \frac{a(-1)^{k+1} R\, p_0}{a\xi_k^2 + b} \frac{1}{\dfrac{k_0(\alpha)}{a\xi_k^2 + b} s^\alpha + \dfrac{k_1(\alpha)}{a\xi_k^2 + b} s^{\alpha-1} + 1} \tag{11}$$

Taking the inverse Laplace transform of both sides of eq. (11), we obtain:

$$\mathcal{L}^{-1}[T(\xi_k, s)] = T(\xi_k, t) = \frac{a(-1)^{k+1} R\, p_0}{a\xi_k^2 + b} \mathcal{L}^{-1}\left[\frac{1}{\dfrac{k_0(\alpha)}{a\xi_k^2 + b} s^\alpha + \dfrac{k_1(\alpha)}{a\xi_k^2 + b} s^{\alpha-1} + 1}\right]$$

Using eq. (A7) in *Appendix*, we obtain:

$$T(\xi_k, t) = \frac{a(-1)^{k+1} R\, p_0}{a\xi_k^2 + b} \sum_{n=0}^{\infty} \frac{(b + a\xi_k^2)\left[-\dfrac{k_1(\alpha)}{k_0(\alpha)}\right]^n}{k_0(\alpha)} t^{-1+n+\alpha} E_{\alpha,n+\alpha}^{1+n}\left[-\frac{b + a\xi_k^2}{k_0(\alpha)} t^\alpha\right] =$$

$$= \frac{a(-1)^{k+1} R\, p_0}{k_0(\alpha)} \sum_{n=0}^{\infty} \left[-\frac{k_1(\alpha)}{k_0(\alpha)}\right]^n t^{-1+n+\alpha} E_{\alpha,n+\alpha}^{1+n}\left[-\frac{b + a\xi_k^2}{k_0(\alpha)} t^\alpha\right] \tag{12}$$

Applying the inverse finite sin Fourier, see eq. (A4) in the *Appendix*, to (12), we obtain:

$$\mathcal{F}^{-1}[T(\xi_k, t)] = \mathcal{F}^{-1}\left\{\frac{a(-1)^{k+1} R\, p_0}{k_0(\alpha)} \sum_{n=0}^{\infty} \left[-\frac{k_1(\alpha)}{k_0(\alpha)}\right]^n t^{-1+n+\alpha} E_{\alpha,n+\alpha}^{1+n}\left[\frac{b + a\xi_k^2}{k_0(\alpha)} t^\alpha\right]\right\}$$

$$T(r, t) = \frac{2a\, p_0}{k_0(\alpha)} \sum_{k=1}^{\infty} \sum_{n=0}^{\infty} \xi_k (-1)^{k+1} \left[-\frac{k_1(\alpha)}{k_0(\alpha)}\right]^n t^{-1+n+\alpha} E_{\alpha,n+\alpha}^{1+n}\left[\frac{b + a\xi_k^2}{k_0(\alpha)} t^\alpha\right] \frac{\sin(\xi_k r)}{r} \tag{13}$$

Figure 1 shows the distribution of the temperature through the sphere at different values of the order of the fractional derivative, α, and the time, t, when the boundary condition is taken in the form of Dirac delta function. Figure 2 shows the effect of the parameter, k_0, on the distribution of the temperature through the sphere at different values of the order of the fractional derivative, α, when the boundary condition is taken in the form of Dirac delta function. From fig. 2, we can realize that the temperature increases with increasing α. The temperature profile changes with changing the parameter k_0. At large radius the temperature profile increases with increasing k_0.

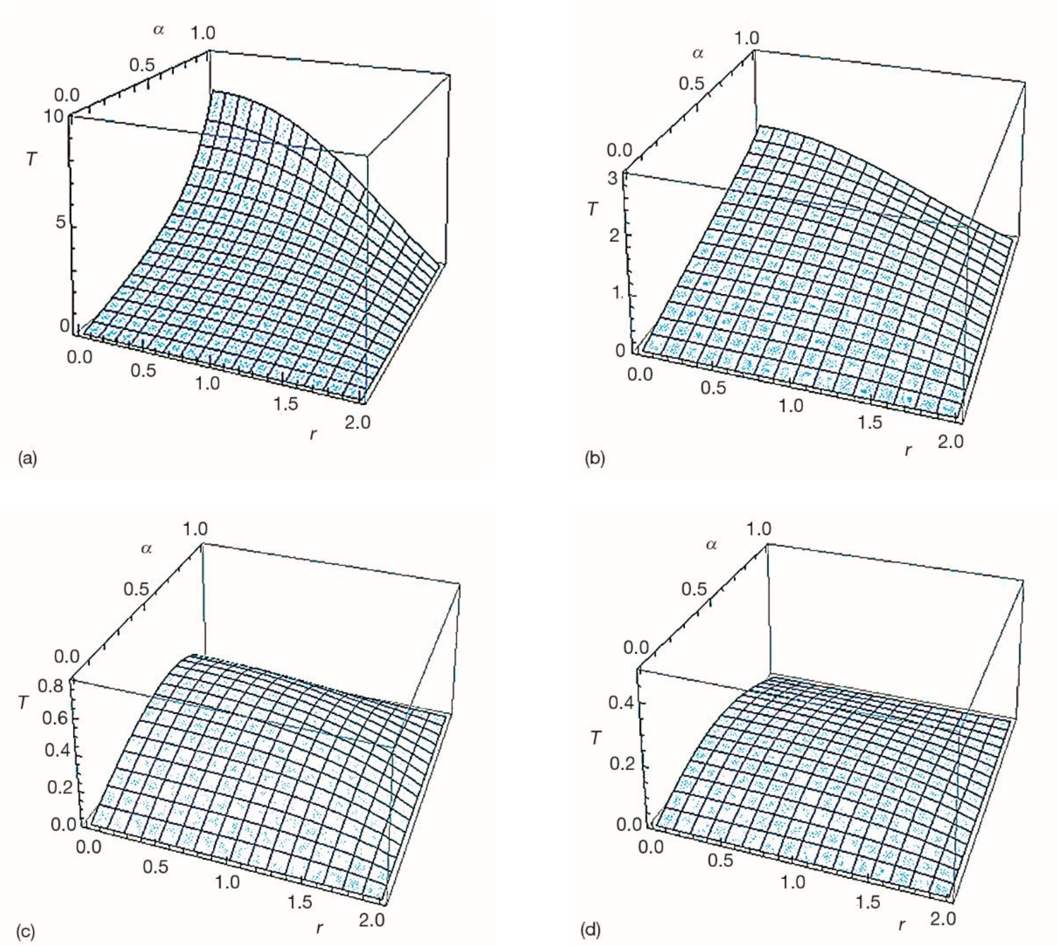

Figure 1. Plot of the solution (13) when $a = b = k_0(\alpha) = 1$, $p_0 = 10$, $R = 2$, $k_1(\alpha) = 0$ for different values of t; (a) $t = 0.5$, (b) $t = 1$, (c) $t = 2$, and (d) $t = 4$

Exact solution of eq. (1) with condition (2b)

Applying the finite sin Fourier transform to eq. (1) and using the condition $T(R,t) = T_0 t^p$, we obtain:

$$ {}^{\text{CPC}}_{\ \ 0}D_t^{\alpha} T(\xi_k, t) = a(-1)^{k+1} R T_0\, t^p - (a\xi_k^2 + b) T(\xi_k, t) \tag{14} $$

Applying the Laplace transform to eq. (14), we obtain:

$$ \mathcal{L}[{}^{\text{CPC}}_{\ \ 0}D_t^{\alpha} T(\xi_k, t)] = \mathcal{L}[a(-1)^{k+1} R T_0\, t^p - (a\xi_k^2 + b) T(\xi_k, t)] \tag{15} $$

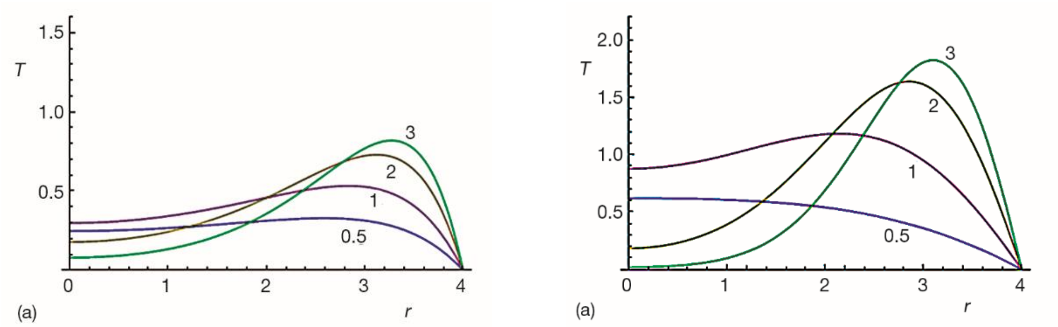

Figure 2. Plot of the solution (13) when *a* = *b* = *t* = 1, p_0 = 10, *R* = 4, $k_1(\alpha)$ = 0 for different values of k_0 and *α*; (a) *α* = 0.5 and (b) *α* = 0.9

Using eq. (A5) in the *Appendix*, we get:

$$\left[\frac{k_1(\alpha)}{s}+k_0(\alpha)\right]s^{\alpha}T(\xi_k,s)-k_0(\alpha)s^{\alpha-1}T(\xi_k,0)=$$

$$a(-1)^{k+1}RT_0\frac{\Gamma[p+1]}{s^{p+1}}-(a\xi_k^2+b)T(\xi_k,s) \tag{16}$$

Using the condition $T(r,0)=0,$ eq. (16) becomes:

$$\left[\frac{k_1(\alpha)}{s}+k_0(\alpha)\right]s^{\alpha}T(\xi_k,s)=a(-1)^{k+1}RT_0\frac{\Gamma[p+1]}{s^{p+1}}-(a\xi_k^2+b)T(\xi_k,s) \tag{17}$$

Solving eq. (17) with respect to $T(\xi_k,s),$ we obtain:

$$T(\xi_k,s)=\frac{a(-1)^{k+1}RT_0\dfrac{\Gamma[p+1]}{s^{p+1}}}{k_0(\alpha)s^{\alpha}+k_1(\alpha)s^{\alpha-1}+a\xi_k^2+b} \tag{18}$$

Equation (18) can be rewritten in the form:

$$T(\xi_k,s)=\frac{a(-1)^{k+1}RT_0\Gamma[p+1]}{a\xi_k^2+b}\frac{s^{-p-1}}{\dfrac{k_0(\alpha)}{a\xi_k^2+b}s^{\alpha}+\dfrac{k_1(\alpha)}{a\xi_k^2+b}s^{\alpha-1}+1} \tag{19}$$

Taking the inverse Laplace transform of both sides of eq. (19), we obtain:

$$\mathcal{L}^{-1}[T(\xi_k,s)]=T(\xi_k,t)=\frac{a(-1)^{k+1}RT_0\Gamma[p+1]}{a\xi_k^2+b}\mathcal{L}^{-1}\left[\frac{s^{-p-1}}{\dfrac{k_0(\alpha)}{a\xi_k^2+b}s^{\alpha}+\dfrac{k_1(\alpha)}{a\xi_k^2+b}s^{\alpha-1}+1}\right] \tag{20}$$

Using eq. (A7) in the *Appendix*, we get:

$$T(\xi_k,t)=\frac{a(-1)^{k+1}RT_0\Gamma[p+1]}{a\xi_k^2+b}\sum_{n=0}^{\infty}\frac{b+a\xi_k^2}{k_0(\alpha)}\left[-\frac{k_1(\alpha)}{k_0(\alpha)}\right]^n t^{n+p+\alpha}E_{\alpha,1+n+p+\alpha}^{n+1}\left[-\frac{b+a\xi_k^2}{k_0(\alpha)}t^\alpha\right] \quad (21)$$

Using the inverse finite sin Fourier, eq. (21) becomes:

$$\mathcal{F}^{-1}[T(\xi_k,t)]=\mathcal{F}^{-1}\left\{\frac{a(-1)^{k+1}RT_0\Gamma[p+1]}{a\xi_k^2+b}\sum_{n=0}^{\infty}\frac{b+a\xi_k^2}{k_0(\alpha)}\left[-\frac{k_1(\alpha)}{k_0(\alpha)}\right]^n t^{n+p+\alpha}E_{\alpha,1+n+p+\alpha}^{n+1}\left[-\frac{b+a\xi_k^2}{k_0(\alpha)}t^\alpha\right]\right\}$$

$$T(r,t)=\frac{2aT_0\Gamma[p+1]}{k_0(\alpha)}\sum_{k=1}^{\infty}\sum_{n=0}^{\infty}\xi_k(-1)^{k+1}\left[-\frac{k_1(\alpha)}{k_0(\alpha)}\right]^n t^{n+p+\alpha}E_{\alpha,1+n+p+\alpha}^{n+1}\left[-\frac{b+a\xi_k^2}{k_0(\alpha)}t^\alpha\right]\frac{\sin(\xi_k r)}{r} \quad (22)$$

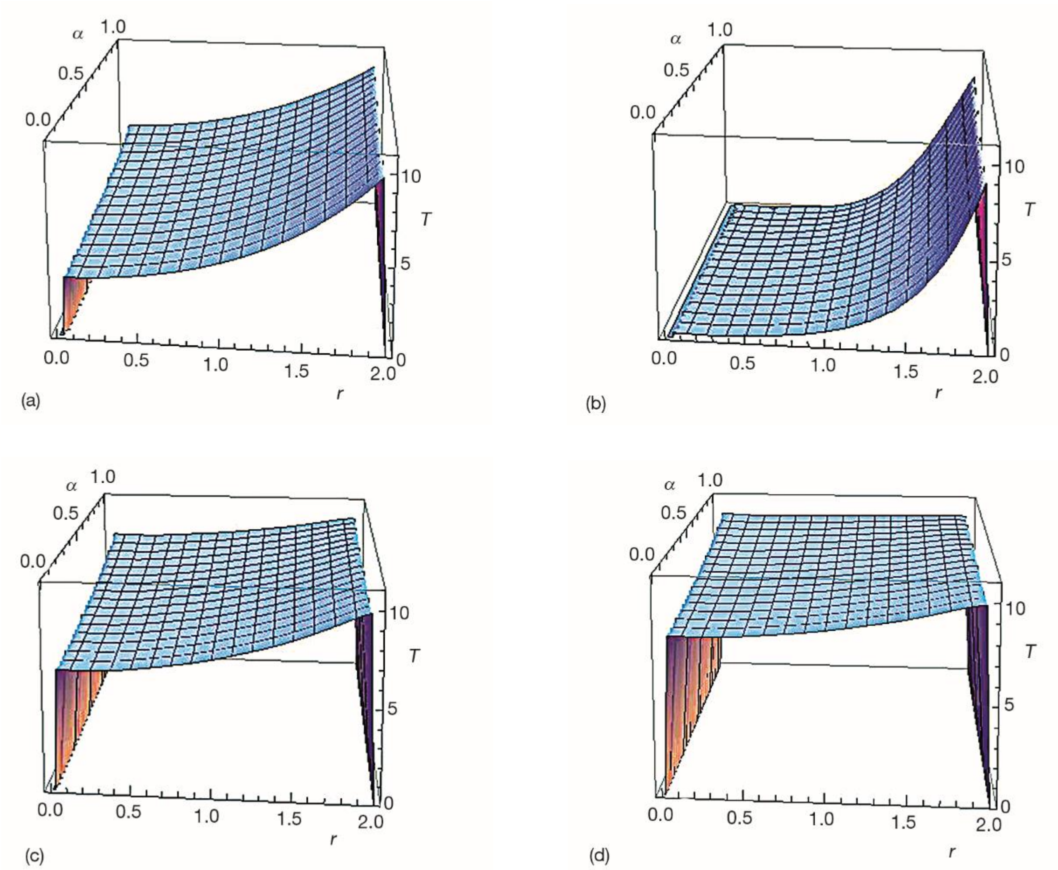

Figure 3. Plot of the solution (22) when $k_0(\alpha) = t = 1$, $T_0 = 10$, $p = 0$, $R = 2$, $p = k_1(\alpha) = 0$ for different values of a and b; (a) $a = b = 1$, (b) $a = b = 0.1$, (c) $a = 2$, $b = 0.5$, and (d) $a = 3$, $b = 0.1$

Figure 3 shows the distribution of the temperature through the sphere at different values of the order of the fractional derivative, α, and the parameters a and b in the case of constant boundary condition. Figure 4 shows the effect of the parameter, k_0, on the distribution of the temperature through the sphere at different values of the order of the fractional derivative α. Figure 4 shows that the temperature T increases when the parameter k_0 decreases. Also figs. 3 and 4 show that the temperature through the sphere T increases with increasing the radius r.

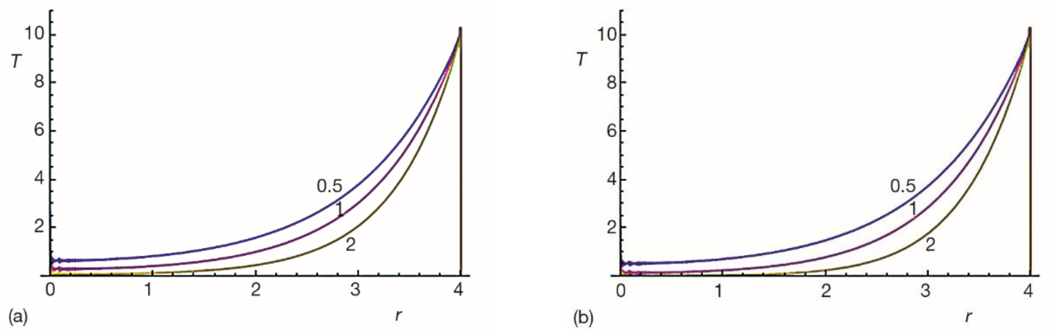

Figure 4. Plot of the solution (22) when $a = b = t = p = 1$, $T_0 = 10$, $R = 4$, $k_1(\alpha) = 0$ for different values of k_0 and α; (a) $\alpha = 0.5$ and (b) $\alpha = 0.9$

Conclusion

We have successfully used joint finite sine Fourier-Laplace transform to get the closed form solution of the fractional heat conduction problem in a sphere with heat absorption and central symmetry. The new hybrid fractional derivative operator is used to investigate the heat distribution inside the sphere. Two Dirichlet boundary value problems are investigated. The obtained solutions are in the form of three parameter Mittag-Leffler function. The results obtained in [6] can be considered as a special solutions of our results. Particularly, when we put $k_0(\alpha) = 1$ and $k_1(\alpha) = 0$ in eqs. (13) and (22) we retrieve the results obtained in [6].

References

[1] Podlubny, I., *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications*, Elsevier, Amsterdam, The Netherland, 1998
[2] Baleanu, D., *et al*., *Fractional Dynamics and Control*, Springer Science & Business Media, New York, USA, 2011
[3] Baleanu, D., *et al*., *Fractional Calculus: Models and Numerical Methods,* Vol. 3, World Scientific, 2012
[4] Elsaid, A., *et al*., Similarity Solutions of Fractional Order Heat Equations with Variable Coefficients, *Miskolc Mathematical Notes*, *17* (2016), 1, pp. 245-254
[5] Elsaid, A., *et al*., Similarity Solutions for Multiterm Time-Fractional Diffusion Equation, *Advances in Mathematical Physics*, *2016* (2016), ID548126
[6] Povstenko, Y., Klekot, J., Fractional Heat Conduction with Heat Absorption in a Sphere Under Dirichlet Boundary Condition, *Computational and Applied Mathematics*, *37* (2018), 4, pp. 4475-4483
[7] Elsaid, A., *et al*., Similarity Solutions for Solving Riesz Fractional Partial Differential Equations, *Progr. Fract. Differ. Appl.*, *2* (2016), 4, pp. 293-298
[8] Khan, M. F., Some New Hypergeometric Transformations Via Fractional Calculus Technique, *Appl. Math. Inf. Sci.*, *14* (2020), 2, pp. 177-190

[9] Cernea, A., Continuous Family of Solutions for Fractional Integro-Differential Inclusions of Caputo-katugampola Type, *Progr. Fract. Differ. Appl., 5* (2020), 1, pp. 37-42
[10] Mouzakis, D. E., Lazopoulos, A. K., Fractional Modelling and the LEIBNIZ (L-fractional) Derivative as Viscoelastic Respondents in Polymer Biomaterials, *Progr. Fract. Differ. Appl., 5* (2020), 1, pp. 43-48
[11] Srivastava, H. M., Saad, K. M., New Approximate Solution of the Time-Fractional Nagumo Equation Involving Fractional Integrals without Singular Kernel, *Appl. Math. Inf. Sci., 14* (2020), 1, pp. 1-8
[12] Chatzarakis, G., *et al*., Oscillatory Properties of a Certain Class of Mixed Fractional Differential Equations, *Appl. Math. Inf. Sci., 14* (2020), 1, pp. 123-131
[13] Abdel Kader, A. H., *et al.,* Some Exact Solution of a Variable Coefficients Fractional Biological Population Model, *Math. Meth. Appl. Sci., 44* (2021), 16, pp. 4701-4714
[14] Abdel Latif, M. S., *et al*., The Invariant Subspace Method for Solving Nonlinear Fractional Partial Differential Equations with Generalized Fractional Derivatives, *Adv. Difference Equ., 2020* (2020), 119, pp. 267-282
[15] He, J. H., A Tutorial Review on Fractal Spacetime and Fractional Calculus, *Internat. J. Theoret. Phys., 53* (2014), 11, pp. 3698-3718
[16] He, J. H., Fractal Calculus and Its Geometrical Explanation, *Results Phys., 10* (2018), Sept., pp. 272-276
[17] He, J. H., Ain Q. T., New Promises and Future Challenges of Fractal Calculus: From Two-Scale Thermodynamics to Fractal Variational Principle, *Thermal Science, 24* (2020), 2A, pp. 659-681
[18] Wang, K. L., *et al.,* Physical Insight of Local Fractional Calculus and Its Application to Fractional Kdv-Burgers-Kuramoto Equation, *Fractals, 27* (2019), 7, 1950122
[19] He, J. H., El-Dib, Y. O., Periodic Property of the Time-Fractional Kundu-Mukherjee-Naskar equation, *Results Phys., 19* (2020), Dec., 103345
[20] Wang, K. L., Liu, S. Y., He's Fractional Derivative and Its Application for Fractional Fornberg-Whitham Equation, *Thermal Science, 21* (2017), 5, pp. 2049-2055
[21] Baleanu, D., *et al*., On a Fractional Operator Combining Proportional and Classical Differintegrals. *Mathematics, 8* (2020), 3, 360
[22] Akgül, E. K., *et al.,* Laplace Transform Method for Economic Models with Constant Proportional Caputo Derivative, *Fractal Fract., 4* (2020), 3, 30
[23] Asjad, M. I., *et al.,* Analysis of MHD Viscous Fluid Flow Through Porous Medium with Novel Power Law Fractional Differential Operator, *Phys. Scr., 95* (2020), 11, 115209
[24] Chu, Y. M., *et al*., Influence of Hybrid Nanofluids and Heat Generation on Coupled Heat and Mass Transfer Flow of a Viscous Fluid with Novel Fractional Derivative, *J. Therm. Anal. Calorim., 144* (2021), 6, pp. 2057-2077
[25] Povstenko, Y., *Linear Fractional Diffusion-Wave Equation for Scientists and Engineers*, Switzerland: Springer International Publishing, New York, USA, 2015
[26] Sandev, T. , Tomovski, Ž., Fractional Equations and Models: Theory and Applications. Vol. 61, Springer Nature, New York, USA, 2019
[27] Tomovski, Z., *et al*., Fractional and Operational Calculus with Generalized Fractional Derivative Operators and Mittag-Leffler Type Functions, *Integral Transforms and Special Functions, 21* (2010), 11, pp. 797-814

Appendix

Definition 1. [22] The finite sin-Fourier transform of the function *f*(*r*) is defined:

$$\mathcal{F}[f(r)]=\int_0^R r\,f(r)\frac{\sin(\xi_k r)}{\xi_k}\mathrm{d}r,\quad \xi_k=\frac{k\pi}{R} \tag{A1}$$

Theorem 1. The finite sin-Fourier transform of the hybrid fractional derivative operator ${}^{\mathrm{CPC}}_{\ 0}D_t^{\alpha}T(r,t)$ is defined:

$$\mathcal{F}[{}^{\mathrm{CPC}}_{\ 0}D_t^{\alpha}T(r,t)]={}^{\mathrm{CPC}}_{\ 0}D_t^{\alpha}T(\xi_k,t) \tag{A2}$$

Proof

$$\mathcal{F}[{}^{\mathrm{CPC}}_{0}D_t^{\alpha}T(r,t)] = \int_0^R r\, {}^{\mathrm{CPC}}_{0}D_t^{\alpha}T(r,t)\frac{\sin(\xi_k r)}{\xi_k}\mathrm{d}r =$$

$$= \int_0^R r\frac{1}{\Gamma(1-\alpha)}\int_0^t\left[k_1(\alpha)T(r,\tau)+k_0(\alpha)\frac{\partial T(r,\tau)}{\partial\tau}\right](t-\tau)^{-\alpha}\mathrm{d}\tau\frac{\sin(\xi_k r)}{\xi_k}\mathrm{d}r =$$

$$= \frac{1}{\Gamma(1-\alpha)}\int_0^R\int_0^t r\left[k_1(\alpha)T(r,\tau)+k_0(\alpha)\frac{\partial T(r,\tau)}{\partial\tau}\right](t-\tau)^{-\alpha}\frac{\sin(\xi_k r)}{\xi_k}\mathrm{d}\tau\,\mathrm{d}r =$$

$$= \frac{1}{\Gamma(1-\alpha)}\int_0^t\left\{\int_0^R r\left[k_1(\alpha)T(r,\tau)+k_0(\alpha)\frac{\partial T(r,\tau)}{\partial\tau}\right]\frac{\sin(\xi_k r)}{\xi_k}\mathrm{d}r\right\}(t-\tau)^{-\alpha}\mathrm{d}\tau =$$

$$= \frac{1}{\Gamma(1-\alpha)}\int_0^t\left[k_1(\alpha)T(\xi_k,\tau)+k_0(\alpha)\frac{\partial T(\xi_k,\tau)}{\partial\tau}\right](t-\tau)^{-\alpha}\mathrm{d}\tau = {}^{\mathrm{CPC}}_{0}D_t^{\alpha}T(\xi_k,t)$$

Theorem 2. [25]

$$\mathcal{F}\left(T_{rr}+\frac{2}{r}T_r\right) = -\xi_k^2 T(\xi,t)+(-1)^{k+1}RT(R,t) \tag{A3}$$

Definition 2. [25] The Finite sin-Fourier transform of the function *f*(*r*) is defined:

$$\mathcal{F}^{-1}[f(\xi_k)] = f(r) = \frac{2}{R}\sum_{k=1}^{\infty}\xi_k f(\xi_k)\frac{\sin(\xi_k r)}{r} \tag{A4}$$

Theorem 3. [21] The Laplace transform of the hybrid fractional derivative operator is given by:

$$\mathcal{L}[{}^{\mathrm{CPC}}_{0}D_t^{\alpha}f(t)] = \left[\frac{k_1(\alpha)}{s}+k_0(\alpha)\right]s^{\alpha}F(s)-k_0(\alpha)s^{\alpha-1}f(0) \tag{A5}$$

Lemma 1. [26] The Inverse Laplace of $\dfrac{s^{\alpha_1\gamma-\beta}}{(s^{\alpha_1}+\lambda)^{\gamma}}$ is defined:

$$\mathcal{L}^{-1}\left[\frac{s^{\alpha_1\gamma-\beta}}{(s^{\alpha_1}+\lambda)^{\gamma}}\right] = t^{\beta-1}E_{\alpha_1,\beta}^{\gamma}(-\lambda t^{\alpha_1}) \tag{A6}$$

Lemma 2. [27] The Inverse Laplace of $\dfrac{s^{\alpha_3}}{1+As^{\alpha_1}+Bs^{\alpha_2}}$ is given by:

$$\mathcal{L}^{-1}\left(\frac{s^{\alpha_3}}{1+As^{\alpha_1}+Bs^{\alpha_2}}\right) = \sum_{k=0}^{\infty}\frac{(-B)^k}{A^{k+1}}t^{(1+k)\alpha_1-k\alpha_2-\alpha_3-1}E_{\alpha_1,(1+k)\alpha_1-k\alpha_2-\alpha_3}^{k+1}\left(-\frac{1}{A}t^{\alpha 1}\right) \tag{A7}$$

Paper submitted: May 24, 2020
Paper revised: April 4, 2021
Paper accepted: April 30, 2021