EXACT SOLUTIONS OF FRACTIONAL NON-LINEAR EQUATIONS BY GENERALIZED BELL POLYNOMIALS AND BILINEAR METHOD

by

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For numerous fluids between elastic and viscous materials, the fractional derivative models have an advantage over the integer order models. On the basis of conformable fractional derivative and the respective useful properties, the bilinear form of time fractional Burgers equation and Boussinesq-Burgers equations are obtained using the generalized Bell polynomials and bilinear method. The kink soliton solution, anti-kink soliton solution, and the single-soliton solution for different fractional order are derived, respectively. The time fractional order system possesses property of time memory. Higher oscillation frequency appears as the time fractional order increasing. The fractional derivative increases the possibility of improving the control performance in complex systems with fluids between different elastic and viscous materials.

Key words*: fractional derivative, Bell polynomials, Burgers equation, classical Boussinesq-Burgers equations, Bilinear form*

Introduction

Fractional differential equations have been used to represent many natural processes in physics, engineering material, finance and other mathematically oriented sciences [1, 2]. An effective method for such equations is imperative. Unlike from the ODE, finding exact analytical solutions of fractional differential equations is still difficult owning to the difference between the fractional calculus and the familiar ordinary calculus.

Since L'Hospital proposed the problem of what $\mathrm{d}^n f/\mathrm{d}x^n$ means if $n = 1/2$, various of fractional derivatives have been proposed, such as Riemann-Liouville and Caputo derivative. Whereas, the kernels of Riemann-Liouville and Caputo derivative are singular which cause some effect when modelling the practical problems [3]. A new well-behaved fractional derivative called *the conformable fractional derivative* depending on the basic limit definition of the derivative possesses the basic properties as the usual derivative, especially the product rule and chain rule. Exact solution from the reduced equation can be further derived by different methods. Chung [4] studied the fractional Newtonian mechanics with the conformable fractional derivative. Eslami *et al.* [5] constructed the exact solutions to the space-time non-linear conformable fractional Bogoyavlenskii equations by using the first integral method. Akbulut and Kaplan [6] obtained the analytical solutions of (2+1)-D conformable time-fractional Zoomeron equation and third order modified KdV equation by auxiliary equation method. Kaya *et al.* [7]

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modeled population density of a bacteria species in a microcosm by conformable fractional order differential equations with piecewise constant arguments.

As non-linear PDE simulating shock wave propagation and reflection, Burgers equation [8] is applied widely in traffic flow, shock wave, turbulence problem and continuous stochastic process. The classical Boussinesq-Burgers equations [9, 10] describe the propagation of shallow water waves. For numerous fluids between elastic or viscous materials, the fractional models have an advantage over the integer order models. The studies of the exact solutions to fractional dynamical equations are meaningful.

In this paper, based on the new definition of fractional derivative and utilizing the link between the Bell polynomials and bilinear operators, we get the bilinear form of the time fractional Burgers equation and classical Boussinesq-Burgers equations.

Preliminaries

Definition 1. [7] Let f: $[0, \infty) \to \mathbb{R}$. Then the *conformable fractional derivative* of f of order α is defined:

$$T_t^\alpha f(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}$$

for all $t > 0$, $\alpha \in (0, 1)$. If f is α-differential in $(0, a)$, $a > 0$, and

$$\lim_{t \to 0^+} T_t^\alpha f(t) \text{ existis, } T_t^\alpha f(0) = \lim_{t \to 0^+} T_t^\alpha f(t)$$

Definition 2. [11, 12] The Hirota bilinear operator D_x is defined:

$$D_t^m D_x^n FG = (\partial_t - \partial_{t'})^m (\partial_x - \partial_{x'})^n F(t,x) G(t',x')\Big|_{t'=t, x'=x}, \; m, n \in \mathbb{N} \tag{1}$$

e. g.

$$D_x FG = F_x G - FG_x, \; D_x^2 FG = F_{xx} G - 2F_x G_x + FG_{xx}, \; D_x^3 FG = F_{xxx} G - 3F_{xx} G_x + 3F_x G_{xx} - FG_{xxx}$$

Using the property of conformable fractional derivative and *Definition 2*, we define operator D_t^α:

$$D_t^\alpha FG = T_t^\alpha FG - FT_t^\alpha G, \quad \alpha \in (0,1] \tag{2}$$

Generalized Binary-Bell-polynomial form for dynamical equations

Generalized Binary-Bell-polynomial form

Consider a C^∞ function $p(x)$ and $q(t)$. The variable $p_{nx} = \partial_x^n p(x)$, $n = 1, 2,...$ with Bell's exponential polynomials $Y_{nx}(p) \equiv \mathrm{e}^{-p} \partial_x^n \mathrm{e}^p$. The variable $q_{\alpha t} = T_t^\alpha q(t)$ with respective fractional exponential polynomials:

$$Y_{\alpha t}(q) \equiv \mathrm{e}^{-q} T_t^\alpha \mathrm{e}^q$$

So we have:

$$Y_{\alpha t}(q) = T_t^\alpha q, \; Y_x(p) = p_x, \; Y_{2x}(p) = p_{2x} + p_x^2, \; Y_{3x}(p) = p_{3x} + 3p_x p_{2x} + p_x^3, ...$$

This generalized Bell polynomials can be extended to more dimensions by introducing more independent variables [13]. Such as the 2-D extension:

$$Y_{mt,nx}(q) \equiv Y_{m,n}(q_{r,s}) = \mathrm{e}^{-q} \partial_t^m \partial_x^n \mathrm{e}^q, \; q_{r,s} = \partial_t^r \partial_x^s q(t,x) \tag{3}$$

e. g.

$$Y_{\alpha t,x}(q) = T_t^\alpha q_x + T_t^\alpha q \cdot q_x, \; Y_{t,x}(q) = q_{t,x} + q_t q_x, \; Y_{t,2x}(q) = q_{t,2x} + q_t q_{2x} + 2q_x q_{t,x} + q_t q_x^2, ...$$

Lemma 1. [13] Setting $F = e^f$ and $G = e^g$, then the link between Hirota operators and Bell polynomial:

$$(FG)^{-1} D_t^m D_x^n F \cdot G = \mathcal{Y}_{mt,nx}\left[\ln\left(\frac{F}{G}\right), \ln(FG)\right]$$

where

$$\mathcal{Y}_{mt,nx}(u', v') \equiv Y_{mt,nx}(h)\Big|_{h_{r,s}=\begin{cases} u'_{r,s} & \text{if } r+s \text{ is odd} \\ v'_{r,s} & \text{if } r+s \text{ is even}\end{cases}}$$

Proof. From (3), we have:

$$Y_{mt,nx}(f+g) = \sum_{p=0}^{m}\sum_{q=0}^{n}\binom{m}{p}\binom{n}{q} Y_{(m-p)t,(n-q)x}(f) Y_{pt,qx}(g)$$

and

$$Y_{m,n}[(-1)^{r+s} q_{r,s}] = (-1)^{m+n} Y_{m,n}(q_{r,s})$$

Using the Hirota bilinear operator (1):

$$(FG)^{-1} D_t^m D_x^n F \cdot G = e^{-f} e^{-g} (\partial_t - \partial_{t'})^m (\partial_x - \partial_{x'})^n e^{f(t,x)} e^{g(t',x')}\Big|_{t'=t, x'=x} =$$

$$= \sum_{p=0}^{m}\sum_{q=0}^{n} (-1)^{p+q} \binom{m}{p}\binom{n}{q} Y_{(m-p)t,(n-q)x}(f) Y_{pt,qx}(g) =$$

$$= Y_{m,n}(h_{r,s})\Big|_{h_{r,s}=f_{r,s}+(-1)^{r+s} g_{r,s} \equiv \mathcal{Y}_{mt,nx}(f-g, f+g) = \mathcal{Y}_{mt,nx}[\ln\left(\frac{F}{G}\right), \ln(FG)]}$$

Lemma 2. Setting $F = e^f$ and $G = e^g$, the link between generalized Hirota operators and Bell polynomial with the conformable fractional derivative:

$$(FG)^{-1} D_t^\alpha D_x FG = \mathcal{Y}_{\alpha t,x}\left[\ln\left(\frac{F}{G}\right), \ln(FG)\right]$$

where

$$\mathcal{Y}_{\alpha t,x}(u', v') \equiv Y_{\alpha t,x}(h)\Big|_{h_{r,s}=\begin{cases} u'_{r,s} & \text{if } r+s \text{ is } \alpha \text{ or } 1 \\ v'_{r,s} & \text{if } r+s \text{ is } \alpha+1\end{cases}}$$

Specially:

$$(FG)^{-1} D_t^\alpha F \cdot G = \mathcal{Y}_{\alpha t} \ln\left(\frac{F}{G}\right), \quad \mathcal{Y}_{\alpha t}(u') \equiv \mathcal{Y}_{\alpha t,0}(u', v')$$

Proof.

$$(FG)^{-1} D_t^\alpha D_x F \cdot G = e^{-f} e^{-g} (T_t^\alpha - T_{t'}^\alpha)(\partial_x - \partial_{x'}) e^{f(t,x)} e^{g(t',x')}\Big|_{t'=t, x'=x} =$$

$$= \sum_{q=0}^{1} (-1)^q \binom{1}{q} Y_{\alpha t,(1-q)x}(f) Y_{0,qx}(g) + \sum_{q=0}^{1} (-1)^q \binom{1}{q} Y_{0,(1-q)x}(f) Y_{\alpha t,qx}(g) =$$

$$= \mathcal{Y}_{\alpha t,x}(f-g, f+g) = \mathcal{Y}_{\alpha t,x}\left[\ln\left(\frac{F}{G}\right), \ln(FG)\right]$$

Specially:

$$(FG)^{-1} D_t^{\alpha} F \cdot G = \mathrm{e}^{-f} \mathrm{e}^{-g} (T_t^{\alpha} - T_{t'}^{\alpha}) \mathrm{e}^{f(t,x)} \mathrm{e}^{g(t',x')} \Big|_{t'=t, x'=x} =$$
$$= \mathcal{Y}_{\alpha t,0}(f-g, f+g) = \mathcal{Y}_{\alpha t}(f-g)$$

Generalized Binary-Bell-polynomial form for Burgers equation

For (1+1)-D time fractional Burgers equation with constant coefficient [14, 15]:

$$T_t^{\alpha} v - 2Avv_x - Bv_{xx} = 0 \tag{4}$$

We introduce the dimensionless field p by setting:

$$v = cp_x \tag{5}$$

with $c = B/A$. The equations for p can be derived from eq. (4):

$$\mathcal{Y}_{\alpha t}(p) - B\mathcal{Y}_{2x}(p) = 0 \tag{6}$$

Substituting $p = \ln(f/g)$ into expressions eq. (6), the bilinear form of eq. (4) can be obtained:

$$\left(D_t^{\alpha} - BD_x^2\right) f \cdot g = 0 \tag{7}$$

where D_x is the Hirota bilinear operator defined by (1), D_t^{α} is the defined by (2), f and g are functions of x and t which can be expanded as the power series of a small parameter ε:

$$\begin{aligned} f &= 1 + \varepsilon f_1(t,x) + \varepsilon^2 f_2(t,x) + \varepsilon^3 f_3(t,x) + \cdots \\ g &= 1 + \varepsilon g_1(t,x) + \varepsilon^2 g_2(t,x) + \varepsilon^3 g_3(t,x) + \cdots \end{aligned} \tag{8}$$

Substituting eq. (8) into eq. (7) and collecting the coefficients of the same power of ε:

$$\begin{aligned} \varepsilon &: (D_t^{\alpha} - BD_x^2)(f_1 \cdot 1 + 1 \cdot g_1) = 0 \\ \varepsilon^2 &: (D_t^{\alpha} - BD_x^2)(f_2 \cdot 1 + f_1 \cdot g_1 + 1 \cdot g_2) = 0 \\ \varepsilon^3 &: (D_t^{\alpha} - BD_x^2)(f_3 \cdot 1 + f_1 \cdot g_2 + f_2 \cdot g_1 + 1 \cdot g_3) = 0 \end{aligned} \tag{9}$$

For eq. (8), choosing $f_1(t, x) = a\exp(\xi_1)/\varepsilon$, $f_i(t, x) = 0$ for $i = 2, 3, 4,...$ and $g_i(t, x) = 0$ for $i = 1, 2, 3,...$:

$$f = 1 + a\mathrm{e}^{\xi_1}, \quad g = 1 \quad \text{with} \quad \xi_1 = k_1 x + k_2 t^{\alpha} \tag{10}$$

where a, k_1, and k_2 are arbitrary constants. Substituting eq. (10) into eq. (9) and using eq. (5):

$$v = cp_x = \frac{B}{A} p_x = \frac{B}{A} \frac{ak_1 \exp(k_1 x + k_2 t^{\alpha})}{1 + a\exp(k_1 x + k_2 t^{\alpha})} \tag{11}$$

where $k_2 = (B/\alpha)k_1^2$. In order to have an intuitive understanding of the solution to the fractional Burgers equation eq. (4), setting $a = k_1 = 1$ in eq. (11), we give the corresponding figs. 1-3 for v in eq. (11) where $A = B = 1$, and figs. 4-6 for v in eq. (11) where $A = -1$, $B = 1$.

Remark 1. From the soliton solution (11) to the fractional Burgers equation eq. (4) and respective figs. 4-6, we can find that:

- *As* $AB > 0$, v is a kink soliton solution; *As* $AB < 0$, v is an anti-kink soliton solution.
- The time fractional order system possesses property of time memory. As the increase of the time fractional order, the system becomes increasingly sensitive to time changes.

– The fractional derivative improves the control performance in complex systems for fluids between different elastic and viscous materials.

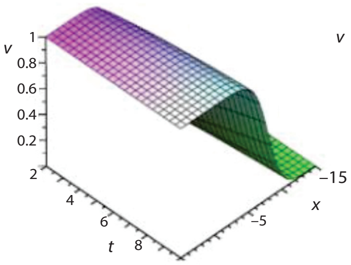

Figure 1. The α = 0.2

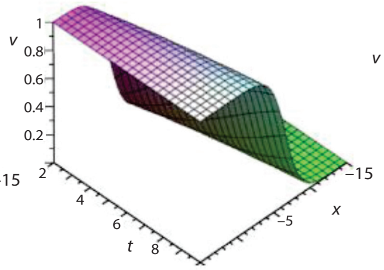

Figure 2. The α = 0.6

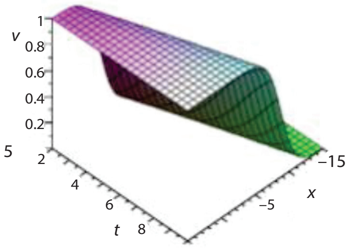

Figure 3. The α = 0.9

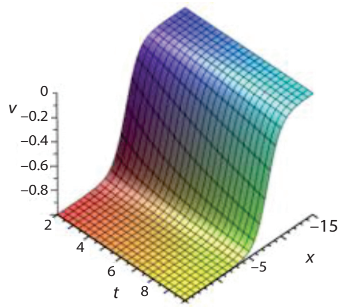

Figure 4. The α = 0.2

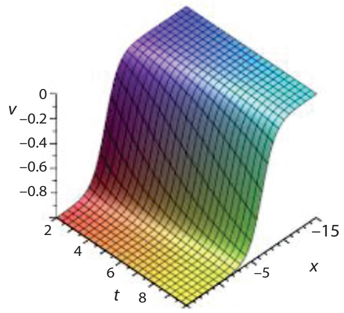

Figure 5. The α = 0.6

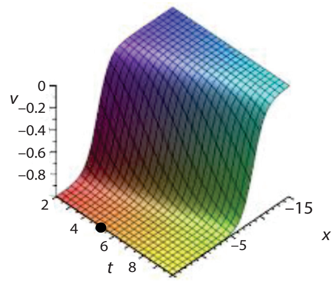

Figure 6. The α = 0.9

Generalized Binary-Bell-polynomial form for fractional classical Boussinesq-Burgers equations

Consider the classical Boussinesq-Burgers equations in fractional form:

$$\begin{aligned} &T_t^{\alpha}u-\frac{1}{2}(\beta-1)u_{xx}-2uu_x-\frac{1}{2}v_x=0 \\ &T_t^{\alpha}v-\beta\left(1-\frac{\beta}{2}\right)u_{xxx}-\frac{1}{2}(1-\beta)v_{xx}-2(uv)_x=0 \end{aligned} \tag{12}$$

Introduce p and q by setting:

$$u=\frac{1}{2}p_x, \quad v=\frac{1-\beta}{2}p_{xx}+\frac{1}{2}q_{xx} \tag{13}$$

The equations for p and q can be derived from eq. (12):

$$\mathcal{Y}_{\alpha t}(p,q)-\frac{1}{2}\mathcal{Y}_{2x}(p,q)=0, \quad 2\mathcal{Y}_{\alpha t,x}(p,q)-\mathcal{Y}_{3x}(p,q)=0 \tag{14}$$

Substituting:

$$p=\ln\left(\frac{f}{g}\right), q=\ln(fg) \tag{15}$$

into expressions eq. (14), where f and g are functions of x and t, the bilinear form of eq. (12) can be obtained:

$$\left(D_t^{\alpha}-\frac{1}{2}D_x^2\right)f\cdot g=0, \quad \left(2D_t^{\alpha}D_x-D_x^3\right)f\cdot g=0 \tag{16}$$

where D_x is the Hirota bilinear operator defined by (1), D_t^α is defined by (2), f and g can be expanded as the power series of a small parameter ε:

$$\begin{aligned} f &= \varepsilon f_1(t,x) + \varepsilon^2 f_2(t,x) + \varepsilon^3 f_3(t,x) + \cdots \\ g &= 1 + \varepsilon g_1(t,x) + \varepsilon^2 g_2(t,x) + \varepsilon^3 g_3(t,x) + \cdots \end{aligned} \tag{17}$$

Substituting eq. (17) into eq. (16) and collecting coefficients of the same power of ε:

$$\begin{aligned} &\varepsilon : (D_t^\alpha - \frac{1}{2} D_x^2)(f_1 \cdot 1) = 0 \\ &\varepsilon : (2D_t^\alpha D_x - D_x^3)(f_1 \cdot 1) = 0 \\ &\varepsilon^2 : (D_t^\alpha - \frac{1}{2} D_x^2)(f_2 \cdot 1 + f_1 \cdot g_1) = 0 \\ &\varepsilon^2 : (2D_t^\alpha D_x - D_x^3)(f_2 \cdot 1 + f_1 \cdot g_1) = 0 \\ &\varepsilon^3 : (D_t^\alpha - \frac{1}{2} D_x^2)(f_3 \cdot 1 + f_1 \cdot g_2 + f_2 \cdot g_1) = 0 \\ &\varepsilon^3 : (2D_t^\alpha D_x - D_x^3)(f_3 \cdot 1 + f_1 \cdot g_2 + f_2 \cdot g_1) = 0 \end{aligned} \tag{18}$$

For eq. (17), choosing $f_1(t,x) = p_1\exp(\xi_2)/\varepsilon$, $g_1(t,x) = q_1\exp(\xi_2)/\varepsilon$, $f_i(t,x) = g_i(t,x) = 0$ for $i = 2, 3, 4,...$, we obtain:

$$f = p_1 \mathrm{e}^{\xi_2}, \quad g = 1 + q_1 \mathrm{e}^{\xi_2} \tag{19}$$

with $\xi_2 = k_1x + k_2t^\alpha$. Substituting eq. (19) into eq. (18) and using eqs. (13) and (15), we obtain the soliton solution:

$$\begin{aligned} u &= \frac{1}{2} \frac{k_1}{1 + q_1 \exp(k_1 x + k_2 t^\alpha)} \\ v &= \frac{\beta k_1^2 q_1}{2} \frac{\exp(k_1 x + k_2 t^\alpha)}{\left[1 + q_1 \exp(k_1 x + k_2 t^\alpha)\right]^2} \end{aligned} \tag{20}$$

where $k_2 = k_1^2/(2\alpha)$.

In order to have an intuitive understanding of the solution to the fractional classical Boussinesq-Burgers equations eq. (12), setting $p_1 = 1$, $q_1 = 0.1$, $k_1 = 1$ in (20), we give the corresponding figs. 7-9 for u in eq. (20), figs. 10-12 for overview of u in eq. (20), figs. 13-15 for v in eq. (20) where $\beta = 4$ and figs. 16-18 for v in eq. (20) where $\beta = -4$.

Remark 2. The anti-kink soliton and single-soliton solution eq. (20) to the fractional classical Boussinesq-Burgers equations eq. (12) are obtained. From the exact solutions and respective figs. 7-18, we can find that:

– As $\beta > 0$, v is a bright soliton solution. As $\beta < 0$, v is a dark soliton solution.
– The time fractional order system possesses property of time memory. The times of oscillation become more within the same time interval, which means higher oscillation frequency with the time fractional order increasing. The fractional derivative increases the possibility of improving the control performance in complex systems by changing the fractional order with fluids between different elastic and viscous materials.

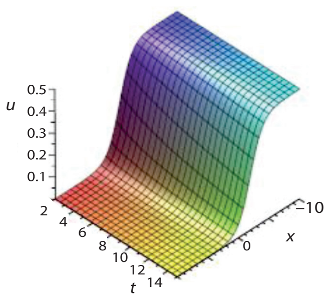

Figure 7. The $\alpha = 0.2$

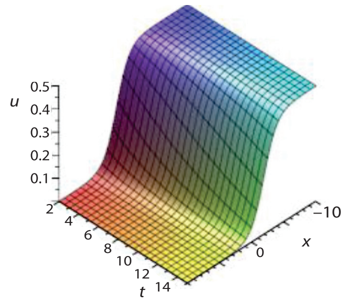

Figure 8. The $\alpha = 0.6$

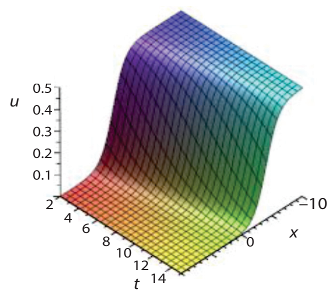

Figure 9. The $\alpha = 0.9$

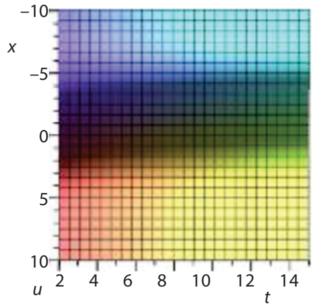

Figure 10. The $\alpha = 0.2$

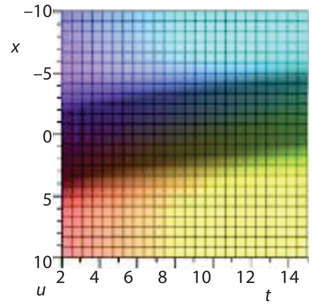

Figure 11. The $\alpha = 0.6$

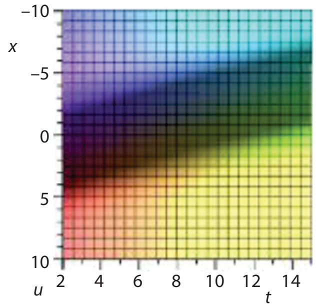

Figure 12. The $\alpha = 0.9$

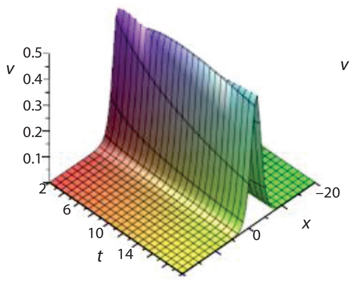

Figure 13. The $\alpha = 0.2$

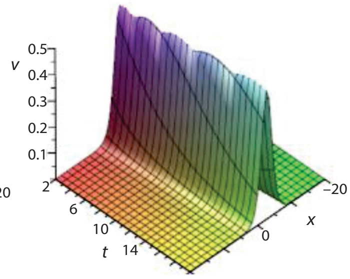

Figure 14. The $\alpha = 0.6$

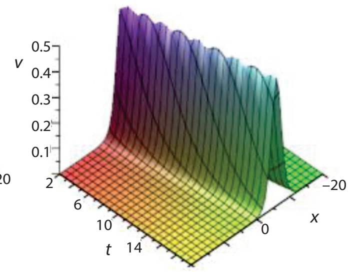

Figure 15. The $\alpha = 0.9$

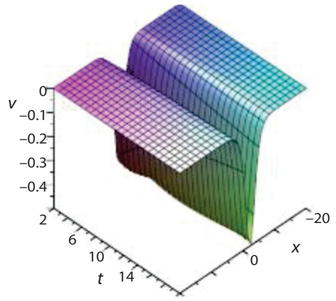

Figure 16. The $\alpha = 0.2$

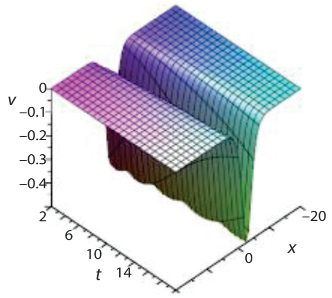

Figure 17. The $\alpha = 0.6$

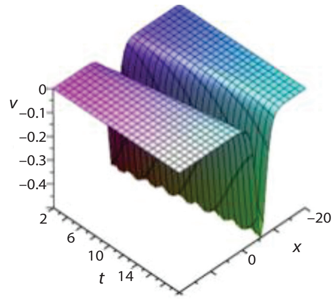

Figure 18. The $\alpha = 0.9$

Conclusion

Based on the conformable fractional derivative, we obtain the bilinear form of the time fractional Burgers and classical Boussinesq-Burgers equations. The kink soliton solution, anti-kink soliton solution and the single-soliton solution are obtained, respectively. The performance of the proposed method in this paper is illustrated through analytical solution and computer simulations. The fractional derivative systems could be used in modelling or fitting fluids between elastic and viscous materials or the properties of various stability augmentation systems such as the dampers. The applications of the method to other fractional derivative dynamic equations which possess practical meaning are worthy of further study.

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