OPTIMAL VIBRATION CONTROL OF AN ISOTROPIC BEAM THROUGH BOUNDARY CONDITIONS

by

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An isotropic structure modelled as a Timoshenko beam is considered for the optimal vibration control problem. The beam model to be controlled is described by a distributed parameter system with the selection of Timoshenko's shear correction factor. Control of the vibrations is achieved through a function placed on the boundary conditions. The performance index which seeks to be minimized indicates that the goal is to minimize the magnitude of performance measure without consuming control effort in large quantities. It is shown how to derive the optimal control function using Pontryagin's principle that turns the control problem into solving optimality system of PDE with terminal values. Wellposedness of the optimal solution on the control set is presented and controllability of the problem is analyzed. Numerical simulations are given in terms of computer codes produced in MATLAB© in the forms of graphical and tables in order to show the applicability and effectiveness of the control acting on the boundary conditions.

Key words: *boundary control, isotropic beam, Pontryagin's principle*

Introduction

One of the most widely utilized beam models is the Timoshenko beam model, derived from the effects of shear deformation and rotary inertia [1, 2]. The use of the shear correction factor is one of the leading features of Timoshenko's beam theory. Since Timoshenko's beam theory was introduced in 1921, there have been many studies that define the shear correction factor or try to find its value. Cowper [3] derived shear coefficient value that match the same value as the Timoshenko's value only when the Poisson's ratio is zero. Kaneko reviewed various studies about the calculation of the shear coefficients for the Timoshenko's beams [4]. According to his conclusion, the values included in the study of Timoshenko [2] are closest to experimental results. Hutchinson [5] came to the conclusion that Timoshenko's value is best for long wavelengths, based on the data from his study, where he developed a set of solutions for the free circular cross-section beam. Leissa *et al*. [6] applied a Rayleigh-Ritz solution using Cowper's shear coefficient to a circular cross-section and presented it by comparison with Timoshenko beam theory. Kennedy *et al*. [7] strain moments to solve the beam problem according to the variables of displacement and rotation along the average thickness. Thanks to these strain moments, it is also possible to work with a non-isotropic and non-homogeneous beam KennedyTBT2011.

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Modelling and analyzing of the physical system adequately is important, but controlling the vibrations in the beam by using control actuators is also important [8]. In an optimal control problem, control can be accomplished through a boundary condition on the system or through an internal force [9]. Boundary control is a way to control of a distributed parameter system in which the control action is implemented to the system by means of its boundary conditions. Selection of boundary condition such as Dirichlet, Neumann or Robin leads to several types of boundary controls. Fattorini and Murphy [10] considered non-linear parabolic boundary control systems where control is applied by means of the Dirichlet boundary condition. Fattorini and Murphy [11] also presented Pontryagin's principle for optimal control problems subject to Robin or Neumann boundary conditions. In Nowakowski [12], optimal control of a system governed by a parabolic equation with a control over a boundary determined as Neumann condition is investigated. In the study, sufficient optimality conditions are reached by applying a dual dynamic programming approach. Yildirim *et al*. [13] presented optimal boundary control of the Mindlin-type beam which is modelled according to Einstein's causality principle that the dynamic behavior of the model must be of the same order with respect to both time and spatial parameters. Korpeoglu *et al*. [14] introduced the optimal vibration control of the second strain gradient theory-based beam which captures the size effects of the structures in micro and nanoscale by means of Pontryagin's principle. Yildirim and Kucuk [15] proposed optimal vibration control of a Timoshenko beam using active controlling with piezoelectric patch actuator via Pontryagin's principle.

In this paper, a vibration control strategy which is composed of boundary force for an isotropic beam is designed. Pontryagin's principle is used to get the optimal control solutions. An adjoint variable is defined to reformulate the optimal control problem in terms of Hamiltonian function. The state equations, the adjoint equations, optimality conditions and terminal conditions are expressed in terms of Pontryagin's Hamiltonian. The performance index seeks to minimize the magnitude of the performance measure that is defined as a dynamic responce the beam as well as the control input over the time interval. Numerical results are presented using computer simulation produced in MATLAB© to confirm that the control scheme is effective and applicable.

Since vibrations are undesirable conditions in a structure, control of the vibrations is a significant research area that needs to be studied. There are studies on vibration control and control strategies in the literature, but the feature of the study that recognizes as different from other studies is that vibration control is achieved by means of control action that takes place on the boundary. In addition, the beam is modeled as a PDE that includes fourth order derivatives in terms of both spatial co-ordinates and time that coincides with Einstein's causality. Besides, there is no study on the suppression of the vibrations with an application of boundary conditions for the beam modelled using Timoshenko's shear correction factor via Pontryagin's principle.

Mathematical model

The equations of motion for an isotropic beam model [7] is given, with the notation in tab. 1, as follows:

$$EI\frac{\partial^4\phi}{\partial x^4}+\rho A\frac{\partial^2\phi}{\partial t^2}-\frac{17+10v}{5}\rho I\frac{\partial^4\phi}{\partial t^2\partial x^2}+\frac{12+10v}{5}\left(\frac{\rho I}{A}\right)^2\frac{\partial^4\phi}{\partial\phi t^4}=0 \tag{1}$$

where x and t is the spatial and time variables at:

$$\Phi=\left\{(x,t):x\in(0,\ell),t\in\left(0,t_f\right)\right\}$$

Table 1. Parameters of the beam model (1)

Symbol	Description
E	Young's modulus
A	Cross-sectional area
I	Weighted second moment of area
v	Poisson ratio
ρ	Material density
ℓ	Length of the beam
$\phi(x, t)$	Beam displacement

The initial conditions for $\phi(x)$:

$$\phi(x,0)=\phi_0(x),\quad \phi_t(x,0)=\phi_1(x),\ \phi_{tt}(x,0)=\phi_2(x),\quad \phi_{ttt}(x,0)=\phi_3(x) \tag{2}$$

and the boundary conditions:

$$\phi(0,t)=\phi(\ell,t)=0,\ \phi_{xx}(0,t)=\phi_{xx}(\ell,t)=p(t) \tag{3}$$

where $p(t)$ is the control function be determined.

Wellposedness and controllability

Picard's existence theorem [16] says that there is a solution for the equation system (1)-(3) in the class of analytic functions under the conditions:

$$p(t)\in L^2(0,t_f),\phi_0(x)\in H^1(0,\ell),\phi_i(x)\in L^2(0,\ell),\quad i=1,2,3$$

$$\phi,\frac{\partial^i\phi}{\partial t^i},\frac{\partial^i\phi}{\partial x^i},\frac{\partial^4\phi}{\partial t^2\partial x^2}\in L^2(\mathcal{S}),\quad i=0,1,\dots,4$$

where $L^2(S)$ denote the class of square integrable functions with a usual inner product and norm in the domain S. In addition, eqs. (1)-(3) can be written as ODE form and therefore, eqs. (1)-(3) have a solution under favour of linear Picard-Lindelof existence-uniqueness theorem. Consider the following *Lemma 1* based on energy method for the uniqueness of the solution eqs. (1)-(3).

Lemma 1. The solution the problem (1)-(3) is unique.

Proof 1. Assume that the problem has two solution $\phi_1(x, t) \neq \phi_2(x, t)$ and set the difference function $\varphi(x, t) = \phi_1(x, t) - \phi_2(x, t)$ for the isotropic beam:

$$EI\varphi_{xxxx}+\rho A\varphi_{tt}-\frac{17+10v}{5}\rho I\varphi_{ttxx}+\frac{12+10v}{5}\left(\frac{\rho I}{A}\right)^2\varphi_{tttt}=0,\quad 0\le t\le t_f,\quad 0\le x\le\ell \tag{4}$$

with zero initial conditions

$$\varphi(x,0)=\varphi_t(x,0)=\varphi_{tt}(x,0)=\varphi_{ttt}(x,0)=0 \tag{5}$$

and the following boundary conditions

$$\varphi(0,t)=\varphi(\ell,t)=\varphi_{xx}(0,t)=\varphi_{xx}(\ell,t)=0 \tag{6}$$

when $\varphi(x, t)$ is shown to be equal to zero in S, the uniqueness of the solution is get. Examining the energy integral:

$$E(t)=\frac{1}{2}\int_0^l\left\{\rho A\varphi_{tt}^2+\frac{12+10\nu}{5}\left(\frac{\rho I}{A}\right)^2\varphi_{ttt}^2-\frac{17+10\nu}{5}\rho I\frac{\partial^2}{\partial x^2}\varphi_{tt}^2\right\}\mathrm{d}x \tag{7}$$

and differentiating $E(t)$ with respect to t:

$$\frac{\mathrm{d}E(t)}{\mathrm{d}t}=\int_0^\ell\left\{\rho A\varphi_{tt}\varphi_{ttt}+\frac{12+10\nu}{5}\left(\frac{\rho I}{A}\right)^2\varphi_{ttt}\varphi_{tttt}-\frac{17+10\nu}{5}\rho I\varphi_{ttxx}\varphi_{ttt}\right\}\mathrm{d}x$$
$$=\int_0^\ell\left\{\rho A\varphi_{tt}+\frac{12+10\nu}{5}\left(\frac{\rho I}{A}\right)^2\varphi_{tttt}-\frac{17+10\nu}{5}\rho I\varphi_{xt}\varphi_{ttxx}\right\}\varphi_{ttt}\mathrm{d}x \tag{8}$$

Considering eq. (4) with boundary conditions (6), d$E(t)$/dt is a negative quantity. Therefore, $E(t)$ is decreasing.

Observe that $E(0) = 0$ and $E(t) \geq 0$ to conclude that $E(t)$ is constantly 0. As a result, $\varphi(x, t)$ is identically equal to zero in S, which completes the proof.

Then carrying out the energy method, the problem is well-posed. Based on the unique solution of the beam system, it is determined that the control function is also unique. For this reason, the system being studied has a unique solution and a unique control function, so the system is observable. Briefly, the system defined by eqs. (1)-(3) is controllable according to the Hilbert uniqueness method [17, 18].

Optimal control problem

The objective of the optimal control problem is to determine the optimal control function $p°(t)$ that will cause the process to satisfy the the governing eq. (1) subject to eqs. (2) and (3) and at the same time minimizes the performance index. Performance index is identified by taking the sum of the weighted dynamic response of the beam and the consumption of the control voltage spent during the control period. A control that satisfies the control constraints during the time interval [0, t_f] is called an admissible control and the set of admissible controls by P_{ad} is given:

$$P_{ad}=\left\{p(t)\mid p\in L^2\left(0,t_f\right),\ \left|p(t)\right|\leq a_0<\infty,\ a_0\ \text{ is a constant}\right\} \tag{9}$$

and the performance index over the time interval $0 \leq t \leq t_f$

$$\mathcal{J}[\phi,p]=\int_0^\ell\left[\mu_1\phi^2\left(x,t_f\right)+\mu_2\phi_t^2\left(x,t_f\right)\right]\mathrm{d}x+\int_0^{t_f}\mu_3 p(t)^2\,\mathrm{d}t \tag{10}$$

where μ_1, μ_2, and μ_3 are weight coefficients satisfying $\mu_1 + \mu_2 \neq 0$, $\mu_1, \mu_2 \geq 0$, $\mu_3 > 0$. The performance index is selected as a sum of two integrals. The first integral is the dynamic response of the beam and seeks to minimize the vibrations at the terminal time $t = t_f$. The second integral is the penalty function that minimizes the magnitude of the control over the range $0 \leq t \leq t_f$. The optimal control problem is indicated:

$$\mathcal{J}\left[p^o(t)\right]=\min_{p(t)\in P_{ad}}\mathcal{J}\left[p(t)\right] \tag{11}$$

subject to the system (1)-(3).

Boundary control characterization

In order to illustrate the Pontryagin's approach for determining the optimal control function, let us define the adjoint system with adjoint variable v corresponding to (1)-(3):

$$EIv_{xxxx} + \rho Av_{tt} - \frac{17+10v}{5}\rho Iv_{ttxx} + \frac{12+10v}{5}\left(\frac{\rho I}{A}\right)^2 v_{tttt} = 0 \tag{12}$$

with boundary conditions

$$v(0,t) = v(\ell,t) = 0,\ \ v_{xx}(0,t) = v_{xx}(\ell,t) = 0 \tag{13}$$

and terminal conditions

$$\begin{gathered}\rho Av_t(x,t_f) - \frac{17+10v}{5}\rho Iv_{xxt}(x,t_f) + \frac{12+10v}{5}\left(\frac{\rho I}{A}\right)^2 v_{ttt}(x,t_f) = 2\mu_1\phi(x,t_f)\\ -\rho Av(x,t_f) + \frac{17+10v}{5}\rho Iv_{xx}(x,t_f) + \frac{12+10v}{5}\left(\frac{\rho I}{A}\right)^2 v_{tt}(x,t_f) = 2\mu_2\phi_t(x,t_f)\\ v_t(x,t_f) = 0\\ v(x,t_f) = 0\end{gathered} \tag{14}$$

Optimal control problem is reformulated using Pontryagin's principle, which asserts that a necessary condition for optimal control function that minimizes the Pontryagin's Hamiltonian. Since the Hamiltonian, defined in section Derivation of the Pontryagin's principle, satisfies:

$$\frac{\partial\mathcal{H}}{\partial p}\left[t,\phi^o(t),p^o(t),v^o(t)\right] = 0 \ \text{ and } \ \frac{\partial^2\mathcal{H}}{\partial p^2} > 0$$

the Pontryagin's principle is also sufficient to be an optimal solution. Pontryagin's principle gives a clear statement for the optimal control function by relating the state variable and the optimal control function implicitly. In this context, Pontryagin's principle is applicable to the optimal control problem (11) as described in the next section.

Derivation of the Pontryagin's principle

Theorem 1. If the optimal control function $p^o(t) \in P_{ad}$, which causes the system (1)-(3) minimizes the Hamiltonian so that:

$$\mathcal{H}(t;v,p) = -EIS(t)p(t) + \mu_3 p^2(t) \tag{15}$$

where

$$S(t) = v_x(\ell,t) - v_x(0,t) \tag{16}$$

then

$$\mathcal{J}(p^o) \le \mathcal{J}(p) \tag{17}$$

Proof 2. Let us form an operator:

$$\Gamma(\phi) = EI\phi_{xxxx} + \rho A\phi_{tt} - \frac{17+10v}{5}\rho I\phi_{ttxx} + \frac{12+10v}{5}\left(\frac{\rho I}{A}\right)^2 \phi_{tttt} \tag{18}$$

and differences:

$$\Delta\phi = \phi(x,t) - \phi^o(x,t) \tag{19}$$

$$\Delta p = p(t) - p^o(t) \tag{20}$$

Evaluating the operator and differences gives:

$$\Gamma(\Delta\phi) = 0 \tag{21}$$

with the boundary conditions

$$\Delta\phi(0,t) = \Delta\phi(\ell,t) = 0,\ \Delta\phi_{xx}(0,t) = \Delta\phi_{xx}(\ell,t) = \Delta p(t) \tag{22}$$

and initial conditions:

$$\Delta\phi(x,0) = \Delta\phi_t(x,0) = 0,\ \Delta\phi_{tt}(x,0) = \Delta\phi_{ttt}(x,0) = 0 \tag{23}$$

The following relation yields:

$$\int_0^\ell\int_0^{t_f}\big(\Delta\phi\Gamma(v) - v\Gamma(\Delta\phi)\big)\mathrm{d}t\mathrm{d}x = \int_0^\ell\int_0^{t_f}\left\{\underbrace{\left[EI\left(\Delta\phi v_{xxxx} - v\Delta\phi_{xxxx}\right)\right]}_{\text{I}} + \left[\underbrace{\frac{17+10v}{5}\rho I\left(v\Delta\phi_{ttxx} - \Delta\phi v_{ttxx}\right)}_{\text{II}}\right] + \right.$$

$$\left. + \left[\underbrace{\frac{12+10v}{5}\left(\frac{\rho I}{A}\right)^2\left(\Delta\phi v_{tttt} - v\Delta\phi_{tttt}\right)}_{\text{III}}\right] + \left[\underbrace{\rho A\left(\Delta\phi v_{tt} - v\Delta\phi_{tt}\right)}_{\text{IV}}\right]\right\}\mathrm{d}t\mathrm{d}x = 0 \tag{24}$$

Carrying out integration by parts for I, II, III, IV and using the terminal conditions eq. (14) gives:

$$\text{I} = \int_0^\ell\int_0^{t_f} EI\left(\Delta\phi v_{xxxx} - v\Delta\phi_{xxxx}\right)\mathrm{d}t\mathrm{d}x = EI\int_0^{t_f}\left[\Delta\phi_{xx}(\ell,t)v_x(\ell,t) - \Delta\phi_{xx}(0,t)v_x(0,t)\right]\mathrm{d}t \tag{25}$$

$$\text{II} = \int_0^\ell\int_0^{t_f}\frac{17+10v}{5}\rho I\left(v\Delta\phi_{ttxx} - \Delta\phi v_{ttxx}\right)\mathrm{d}t\mathrm{d}x =$$

$$= \frac{17+10v}{5}\rho I\int_0^\ell\left[v_{xx}(x,t_f)\Delta\phi_t(x,t_f) - v_{xxt}(x,t_f)\Delta\phi(x,t_f)\right]\mathrm{d}x \tag{26}$$

$$\text{III} = \int_0^\ell\int_0^{t_f}\left(\frac{12+10v}{5}\frac{\rho I}{A}\right)^2\left(\Delta\phi v_{tttt} - v\Delta\phi_{tttt}\right)\mathrm{d}t\mathrm{d}x = \frac{12+10v}{5}\left(\frac{\rho I}{A}\right)^2 \cdot$$

$$\cdot\int_0^\ell\left[v_{ttt}(x,t_f)\Delta\phi(x,t_f) - v_{tt}(x,t_f)\Delta\phi_t(x,t_f) + v_t(x,t_f)\Delta\phi_{tt}(x,t_f) - v(x,t_f)\Delta\phi_{ttt}(x,t_f)\right]\mathrm{d}x \tag{27}$$

$$\text{IV} = \int_0^\ell\int_0^{t_f}\rho A\left(\Delta\phi v_{tt} - v\Delta\phi_{tt}\right)\mathrm{d}t\mathrm{d}x = \rho A\int_0^\ell\left\{\Delta\phi(x,t_f)v_t(x,t_f) - \Delta\phi_t(x,t_f)v(x,t_f)\right\}\mathrm{d}x \tag{28}$$

Collecting these four results yields:

$$\int_0^{\ell}\int_0^{t_f}\left[\Delta\phi\Gamma(v)-v\Gamma(\Delta\phi)\right]\mathrm{d}t\mathrm{d}x = EI\int_0^{t_f}\left[v_x(\ell,t)\Delta\phi_{xx}(\ell,t)-v_x(0,t)\Delta\phi_{xx}(0,t)\right]\mathrm{d}t +$$

$$+\int_0^{\ell}\left[\rho A\left\{\Delta\phi(x,t_f)v_t(x,t_f)-\Delta\phi_t(x,t_f)v(x,t_f)\right\}+\frac{17+10v}{5}\rho I\left\{v_{xx}(x,t_f)\Delta\phi_t(x,t_f)-\right.\right.$$

$$\left.-v_{xxt}(x,t_f)\Delta\phi(x,t_f)\right\}+\frac{12+10v}{5}\left(\frac{\rho I}{A}\right)^2\{v_{ttt}(x,t_f)\Delta\phi(x,t_f)-v_{tt}(x,t_f)\Delta\phi_t(x,t_f)+$$

$$\left.\left.+\,v_t(x,t_f)\Delta\phi_{tt}(x,t_f)-v(x,t_f)\Delta\phi_{ttt}(x,t_f)\right\}\right]\mathrm{d}x=0 \tag{29}$$

Bringing into focus on the deviations of performance index:

$$\Delta\mathcal{J}(p)=\mathcal{J}(p)-\mathcal{J}(p^o)$$

$$=\int_0^{\ell}\left\{\mu_1\left[\phi^2(x,t_f)-\phi^{o^2}(x,t_f)\right]+\mu_2\left[\phi_t^2(x,t_f)-\phi_t^{o^2}(x,t_f)\right]\right\}\mathrm{d}x+\mu_3\int_0^{t_f}\left[p^2(t)-p^{o^2}(t)\right]\mathrm{d}t \tag{30}$$

The values of $\phi^2(x, t_f)$ and $\phi_t^2(x, t_f)$ around $\phi^{o^2}(x, t_f)$ and $\phi_t^{o^2}(x, t_f)$ by Taylor Series expansion, respectively:

$$\begin{aligned}\phi^2(x,t_f)-\phi^{o^2}(x,t_f)&=2\phi^o(x,t)\Delta\phi(x,t_f)+r_1\\ \phi_t^2(x,t_f)-\phi_t^{o^2}(x,t_f)&=2\phi_t^o(x,t)\Delta\phi_t(x,t_f)+r_2\end{aligned} \tag{31}$$

in which:

$$r_1=2(\Delta\phi)^2+\ldots>0,\quad r_2=2(\Delta\phi_t)^2+\ldots>0 \tag{32}$$

Substituting eq. (31) into eq. (30):

$$\Delta\mathcal{J}(p)=\int_0^{\ell}\left\{2\mu_1\left[\phi^o(x,t_f)\Delta\phi^o(x,t_f)+r_1\right]+2\mu_2\left[\phi_t^o(x,t_f)\Delta\phi_t^o(x,t_f)+r_2\right]\right\}\mathrm{d}x+$$

$$+\mu_3\int_0^{t_f}\left[p^2(t)-p^{o^2}(t)\right]\mathrm{d}t \tag{33}$$

Applying the fact $2\mu_1 r_1 + 2\mu_2 r_2 \geq 0$ gives:

$$\int_0^{t_f}EI\left\{v_x(0,t)-v_x(\ell,t)\right\}\Delta p(t)\mathrm{d}t+\mu_3\int_0^{t_f}\left\{p^2(t)-p^{o^2}(t)\right\}\mathrm{d}t\geq 0 \tag{34}$$

Then Pontryagin's Hamiltonian is of the form:

$$\mathcal{H}(t;v,p)=-EIS(t)p(t)+\mu_3p^2(t) \tag{35}$$

where $S(t) = v_x(l, t) - v_x(0, t)$.

Thus:

$$\min\mathcal{H}(t;v,p)=\mathcal{H}(t;v^o,p^o) \tag{36}$$

yields

$$\mathcal{J}\left(p^{o}\right) \leq \mathcal{J}\left(p\right)$$

Simulations results and discussions

The efficiency and competence of the boundary control algorithm introduced are simulated through computer codes produced in MATLAB©. Optimal solutions of the isotropic beam for the case with terminal time t_f = 5 seconds and weight coefficient $\mu_3 = 10^{-2}$ are shown in the simulations. The other weight coefficients μ_1 and μ_2 in the first integral of the performance index functional are considered as $\mu_1 = \mu_2 = 1$. The length of the isotropic beam is taken as $\ell = 1$ m. The properties of the isotropic beam are taken as $I = 1 \cdot 10^{-7}$ m^4, $E = 3 \cdot 10^9$ N/m^2, $\nu = 0.3$, $A = 2 \cdot 10^{-2}$ m^2, $\rho = 2 \cdot 10^5$ m^2. The values of the velocity and displacement of the beam are calculated at the exact middle point. The introduced control algorithm is valid even if the coefficients are chosen as desired. The response of the isotropic beam is analyzed subject to the initial conditions:

$$\phi(x,0)=0, \quad \phi_t(x,0)=\sqrt{2}\sin(\pi x), \quad \phi_{tt}(x,0)=\sqrt{2}\cos(\pi x)$$
$$\phi_{ttt}(x,0)=\sqrt{2}\sin(\pi x) \tag{37}$$

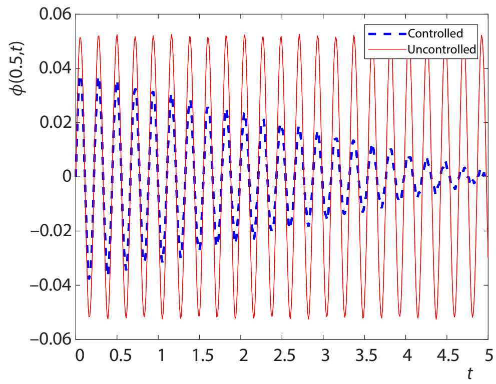

Figure 1. Controlled and uncontrolled displacements

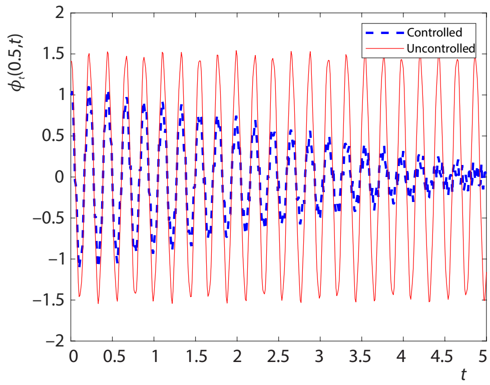

Figure 2. Controlled and uncontrolled velocities

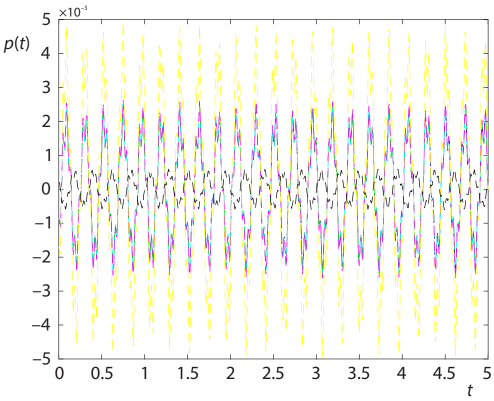

Figure 3. Optimal control solution for the case with $t_f = 5$ and different weight coefficients μ_3

The controlled/uncontrolled displacements at $x = 0.5$ are shown in fig. 1. It is observed that the vibrations are close to zero at the terminal time. The controlled/uncontrolled velocities are also given in fig. 2. Besides, in fig. 3, optimal control function $p(t)$ is presented for different μs. These results, which are also seen with the help of the figures, show how effective the control algorithm is using minimum level of control.

The dynamic responce of the beam which is defined as the performance measure of the beam in case of $\mu_1 = 1$, $\mu_2 = 1$, and $\mu_3 = 0$:

$$\mathcal{J}(\phi) = \int_0^1 \left[\phi^2 \left(x, t_f \right) + \phi_t^2 \left(x, t_f \right) \right] \mathrm{d}x \tag{38}$$

and the penalty function that minimizes the magnitude of the control over $(0, t_f)$:

$$\mathcal{J}(p) = \int_0^{t_f} p(t)^2 \, \mathrm{d}t \tag{39}$$

Choosing the weight coefficients μ_1, μ_2, and μ_3 determines the importance of the performance measure and penalty function. Table 2 shows the values of $\mathcal{J}(\phi)$ and $\mathcal{J}(p)$. for different cases of μ_1, μ_2, and μ_3.

Table 2. The values of $\mathcal{J}(\phi)$ and $\mathcal{J}(p)$ for different values of μ_3

μ_3	$\mathcal{J}(\phi)$	$\mathcal{J}(p)$
10^5	$8.9 \cdot 10^{-4}$	$1.5 \cdot 10^{-20}$
10^3	$3.6 \cdot 10^{-4}$	$1.1 \cdot 10^{-16}$
10^1	$6.7 \cdot 10^{-6}$	$4.2 \cdot 10^{-13}$
10^{-1}	$6.1 \cdot 10^{-10}$	$1.0 \cdot 10^{-12}$
10^{-3}	$6.0 \cdot 10^{-14}$	$5.1 \cdot 10^{-11}$
10^{-5}	$7.9 \cdot 10^{-18}$	$6.3 \cdot 10^{-11}$

As it is examined from the tab. 2 that as the weight factor μ_3 increases, the dynamic response of the beam increases while the corresponding control expenditure decreases. The data in the table presented and figures indicate that vibrations of the beam are damped out with the proposed boundary control algorithm.

Conclusion

In this study, the vibration control problem for the isotropic beam modelled with Timoshenko's shear correction factor is considered. A function $p(t)$ that placed on the boundary conditions is used as an control actuator. Pontryagin's principle is used to get the optimal control solutions. Numerical simulations for the control of the beam are provided in terms of computer codes in MATLAB©. The results presented in figures and table indicates that the boundary control scheme is effective in suppressing vibration of the beam . Also, it is important in terms of being applicable to different types of structures as undesirable vibrations can be suppressed with minimum control cost with the boundary control algorithm introduced in this study.

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