# A MODIFIED EXP-FUNCTION METHOD FOR FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS

by

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*This paper proposes a novel exponential rational function method, a modification of the well-known exp-function method, to find exact solutions of the time fractional Cahn-Allen equation and the time fractional Phi-4 equation. The solution procedure is reduced to solve a system of algebraic equations, which is then solved by Wu's method. The results show that the present method is effective, and can be applied to other fractional differential equations.*

Key words: *Wu's method, exp-function method, exact solution, solitary solution*

## Introduction

In the real world, we need fractional partial differential equations (FPDE) to analyze and model a large amount of problems [1-4]. The fractional calculus is used in many fields like optics, control theory, fluid mechanics, biology, material science and others. In the past decades, there has been significant progress in the development of various methods for finding exact solutions of FPDE, such as the exp-function method [5-8], the fractional residual method [9], the homotopy perturbation method [10-12], the variational iteration method [13-16], and other methods [17, 18].

In this paper, exact solutions of the time fractional Cahn-Alle n equation [19] and the time fractional Phi-4 equation [19] are considered. The solution procedure of the novel exponential rational function method [20] is reduced to solve a large system of algebraic equations, which is solved by Wu's method [21].

## Novel exponential rational function method

In this section, we outline the main steps of the novel exponential rational function method [20], which is a modification of the exp-function method [5-8].

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*Step 1.* Consider the FPDE in the following form:

$$Q(u,u_x,u_{xx},\cdots,D_t^{\alpha}u,D_t^{2\alpha}u,\cdots)=0,\quad 0<\alpha\le 1 \tag{1}$$

The fractional complex transform [22-27] is used:

$$u(x,t)=U(\xi),\qquad \xi=c\left(x-\frac{vt^{\alpha}}{\Gamma(1+\alpha)}\right) \tag{2}$$

Equation (1) is then changed into a non-linear ODE:

$$R(U,cU',c^2U'',\cdots,-cvU',c^2v^2U'',\cdots)=0 \tag{3}$$

*Step 2.* Similar to the exp-function method [5-8], we suppose that the solution of eq. (3) can be expressed:

$$u(\xi)=\frac{c_0+c_1\varphi(\xi)+\cdots+c_N\varphi^N(\xi)}{d_0+d_1\varphi(\xi)+\cdots+d_N\varphi^N(\xi)},\ \varphi(\xi)=\frac{1}{1+da^{\xi}} \tag{4}$$

where $c_i$ $(0\le i\le N)$ and $d_1(0\le i\le N)$ are unknown to be evaluated.

*Step 3.* The value of $N$ in eq. (3) can be determined by the homogeneous balance principle.

*Step 4.* Substituting eq. (4) into eq. (3) and equating all the coefficients of same power of $a^{\xi}$ to zero, we obtained a system of algebraic equations, the obtaining system can be solved to find the value of $c$, $v$, $c_i(0\le i\le N)$ and $d_i(0\le i\le N)$.

## Exact solutions of the time fractional Cahn-Allen equation

In this section, we consider the time fractional Cahn-Allen equation [19]:

$$D_t^{\alpha}u-u_{xx}+u^3-u=0,\qquad 0<\alpha\le 1 \tag{5}$$

with the aid of transformation (2), eq. (5) can be converted to an ODE:

$$-cvU'-c^2U''-U+U^3=0 \tag{6}$$

Considering the homogeneous balance between the highest order derivative and non-linear term in eq. (6), we have $N-1$, then eq. (6) has the following solution:

$$u(\xi)=\frac{c_0+c_1\varphi(\xi)}{d_0+d_1\varphi(\xi)},\ \varphi(\xi)=\frac{1}{1+da^{\xi}} \tag{7}$$

Substituting eq. (7) into eq. (6) and collecting all the terms with the same power of $a^{\xi}$ together, and equating each coefficient to zero, we obtain a set of algebraic equations. Solving algebraic equations with the aid of Wu's method [21], we have four sets of solutions:

*Case 1*:

$$c_0=0,\ \ c_1=d_0+d_1,\ \ -3+2cv\ln(a)=0,\ \ 2v^2-9=0$$

and the following solution is obtained:

$$u(\xi)=\frac{d_0+d_1}{d_0+a^{\xi}dd_0+d_1}$$

*Case 2*:

$$c_0=-d_0,\ \ c_1=d_0,\ \ 3+2cv\ln(a)=0,\ \ 2v^2-9=0$$

and the following solution is obtained:

$$u(\xi)=-\frac{a^{\xi}dd_0}{d_0+a^{\xi}dd_0+d_1}$$

*Case 3*:

$$c_0 = d_0, \quad c_1 = -d_0, \quad 3 + 2cv\ln(a) = 0, \quad 2v^2 - 9 = 0$$

and the following solution is obtained:

$$u(\xi) = \frac{a^{\xi} dd_0}{d_0 + a^{\xi} dd_0 + d_1}$$

*Case 4*:

$$c_0 = 0, \quad c_1 = -d_0 - d_1, \quad -3 + 2cv\ln(a) = 0, \quad 2v^2 - 9 = 0$$

and the following solution is obtained:

$$u(\xi) = -\frac{d_0 + d_1}{d_0 + a^{\xi} dd_0 + d_1}$$

## Exact solutions of the time fractional Phi-4 equation

In this section, we consider the time fractional Phi-4 equation [19]:

$$D_t^{2\alpha} u - u_{xx} + m^2 u + wu^3 = 0, \qquad 0 \le \alpha \le 1 \tag{8}$$

with the aid of transformation (2), eq. (8) can be converted to an ODE:

$$c^2 v^2 U'' - c^2 U'' + m^2 U + wU^3 = 0 \tag{9}$$

By the homogeneous balance between the highest order derivative and non-linear term in eq. (6), we have $N = 1$, then eq. (9) has the following solution:

$$u(\xi) = \frac{c_0 + c_1\varphi(\xi)}{d_0 + d_1\varphi(\xi)}, \quad \varphi(\xi) = \frac{1}{1 + da^{\xi}} \tag{10}$$

Substituting eq. (10) into eq. (9) and collecting all the terms with the same power of $a^{\xi}$ together, equating each coefficient to zero, we obtain a set of algebraic equations. Solving the algebraic equations with the aid of Wu's method [21], we have two sets of solutions:

*Case 1:*

$$c_0 = \frac{-2d_0 d_1 m^2 - d_1^2 m^2 - c_1^2 w}{2c_1 w}, \quad c_1 = \pm\frac{i(2d_0 + d_1)m}{\sqrt{w}}, \quad -2m^2 + c^2(-1 + v^2)\ln^2(a) = 0$$

and the following solution is obtained:

$$u(\xi) = \pm\frac{i[(-1 + a^{\xi} d)d_0 - d_1]m}{(d_0 + a^{\xi} dd_0 + d_1)\sqrt{w}}$$

*Case 2*:

$$c_0 = \pm\frac{id_1 m}{2\sqrt{w}}, \quad c_1 = 0, \quad d_0 = -\frac{d_1}{2}, \quad -2m^2 + c^2(-1 + v^2)\ln^2(a) = 0$$

and the following solution is obtained:

$$u(\xi) = \mp\frac{i(1 + a^{\xi} d)m}{(-1 + a^{\xi} d)\sqrt{w}}$$

## Conclusion

In this paper, we use the novel exponential rational function method combined with Wu's method to solve the time fractional Cahn-Allen equation and the time fractional Phi-4 equation, the solution process can be reduced to solve a system of algebraic equations, which is solved by Wu's method. The results show the effectiveness of this method, and the solutions we obtain in this paper are different from the results in literature [19]. The present method can be easily extended to other fractional differential equations with different definitions for fractional derivative, especially He's fractional derivative [28-31], and fractal calculus [32-34]. Additionally Lie symmetry and conservation laws for FPDE [35, 36] and quenching phenomenon [37, 38] will be the research frontier in future.

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