GOOD CONGRUENCES ON WEAKLY AMPLE SEMIGROUPS

by

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The concept of normal congruence on a weakly ample semigroup, S, is introduced and the maximum and minimum admissible congruences whose trace is the normal congruence, π, on a weakly ample semigroup, S, are characterized in this paper. Some results about congruences on ample semigroups are generalized to weakly ample semigroups.

Key words: *weakly ample semigroups, admissible congruences, normal congruences*

Introduction

The study of regular semigroups is an important topic in the semigroup theory, which has wide applications in mathematics and thermal science. In recent years, some generalized regular semigroups have attracted great attention from various communities in mathematics, chemistry, physics, computer science, and material science, and it is an effective tool to deep insight into big data. A general way to generalizing regular semigroup is to generalize the Green's relations on semigroups. Lawson [1] defined the (~)-green relation which is a generalization of the usual Green's relation. Let S be a semigroup, a, $b \in S$:

$$\tilde{\mathcal{L}} = \{(a,b) \in S \times S : (\forall e \in E(S))ae = a \Leftrightarrow be = b\}$$
$$\tilde{\mathcal{R}} = \{(a,b) \in S \times S : (\forall e \in E(S))ea = a \Leftrightarrow eb = b\}$$
$$\tilde{\mathcal{H}} = \tilde{\mathcal{L}} \wedge \tilde{\mathcal{R}}$$

where $E(S)$ is the set of all idempotents of S.

A semigroup S is said to be semiabundant if each $\tilde{\mathcal{L}}$ and each $\tilde{\mathcal{R}}$ class contain at least one idempotent. We say a semiabundant semigroup to be semiadequate semigroup (quasi-semi-adequate) if its set of idempotent is a semilattice (if its set of idempotents forms a band).

Congruences play an important role in the investigation of properties of inverse semigroups and there are now deep and well-developed theories for congruences on inverse semigroups. It seems natural, therefore, to extend the results concerning congruences on inverse semigroups to weakly ample semigroup. El-Qallali [6] extended some results of Preston [2], Howie [3], Petrich [4], and Fountain [5] to ample semigroups. We will adopt the trace and kernel approach to investigate the congruences on weakly ample semigroups. Some results about congruences on ample semigroups are extended to semi-abundant semigroups.

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Preliminaries

Lemma 1 [1] Let S be a semigroup with the set of idempotents $E(S)$. If $a \in S$, $e \in E(S)$ then the conditions are equivalent:

– $a\tilde{\mathcal{L}}e$,
– $ae = a$ and for $f \in E(S)$, $af = a$ implies $ef = e$.

A semiadequate semigroup S that satisfies the congruence condition and in which for all $a \in S$ and $e \in E(S)$:

$$ea = a(ea)^*, \ \ ae = (ae)^+ a$$

is said to be weakly ample [8].

Lemma 2. Let S be a weakly ample semigroup $a, b \in S$. Then the conditions are equivalent:

– $a\tilde{\mathcal{L}}b$ if and only if $a^* = b^*$ and $a\tilde{\mathcal{R}}b$ if and only if $a^+ = b^+$,
– $(ab)^* = (a^*b)^*$ and $(ab)^+ = (ab^+)^+$, and
– $(ab)^*b^* = (ab)^*$ and $a + (ab)^+ = (ab)^+$.

where a^*, b^* are, respectively the idempotent of the $\tilde{\mathcal{L}}$-class and the $\tilde{\mathcal{R}}$-class.

A congruence on the weakly ample S is called an admissible congruence if for all element a of S and e of $E(S)$:

$$a\rho e\rho = a\rho \Rightarrow a^*\rho e\rho = a^*\rho, \ \ e\rho a\rho = a\rho \Rightarrow e\rho a^+\rho = a^+\rho$$

Theorem 1. Let ρ is an admissible congruence on the weakly ample S and if $a, b \in S$ and $a\rho b$, then $a^*\rho b^*$ and $a^+\rho b^+$.

Proof. Since ρ is a congruence, we have $aa^*\rho ba^*$, $ab^*\rho bb^*$ that is, $a\rho ab^*$, $b\rho ab^*$. Whence $b^*\rho a^*b^*$ since ρ is admissible. Similar, $a^*\rho b^*a^*$ and $a^*b^* = b^*a^*$ as we have $a^*\rho b^*$ as required. Then argument for $a^+\rho b^+$ is similar.

Lemma 3. If ρ is an admissible congruence on the weakly ample S, then S/ρ is a weakly ample semigroup:

$$(a\rho)^* = a^*\rho, \ \ (a\rho)^+ = a^+\rho$$

Lemma 4. If ρ is an admissible congruence on the weakly ample semigroup S, $x\rho$ is an idempotent in S/ρ, then there exists an idempotent e in S such that $(x, e) \in \rho$.

Proof. Let $x \in S$. Since S is a weakly ample semigroup, there exist idempotents $e, f \in E(S)$ such that $e\tilde{\mathcal{L}}x\tilde{\mathcal{R}}f$. If $x\rho$ is an idempotent in S/ρ, then $x\tilde{\mathcal{L}}e$ implies that $x\rho\tilde{\mathcal{L}}e\rho$ since ρ is an admissible congruence, also, $\tilde{\mathcal{L}}$ is right congruence, we have:

$$x\rho f\rho\tilde{\mathcal{L}}e\rho f\rho \Rightarrow (xf)\rho\tilde{\mathcal{L}}(ef)\rho$$

As $x\tilde{\mathcal{R}}f$, ρ is an admissible congruence, $x\rho\tilde{\mathcal{R}}f\rho$ and so $(xf)\rho = f\rho$. Hence $f\rho\tilde{\mathcal{L}}(ef)\rho$ and $f\rho = (ef)\rho$.

Dually, we can obtain that $e\rho = (ef)\rho$. Hence $e\rho = f\rho$:

$$ex\rho e \Rightarrow e\rho x\rho = e\rho \Rightarrow f\rho x\rho = e\rho \Rightarrow x\rho = e\rho$$

For any congruence ρ on the weakly ample S, we have the restriction ρ/E of ρ on E which is called the trace of ρ, denoted by $tr\rho$. Clearly $tr\rho$ is a congruence on E. Further, if $e, f \in E$ with $e\rho f$ and $a \in S$, then $(ea, fa) \in \rho$ and $(ae, af) \in \rho$. If ρ is an admissible congruence, then $(ea)^*\rho(fa)^*$ and $(ae)^+\rho(af)^+$.

A congruence π on E is said to be normal if for any $e, f \in E$ and $a \in S$:

$$e\pi f \Rightarrow (ea)^*\pi(fa)^* \text{ and } (ae)^+\pi(af)^+$$

Theorem 2. If π is a normal congruence on E, then for any elements $a, b \in S$ the conditions are equivalent:
– (i) $a^*\pi b^*$, $ae = be$ for some $e \in E$, $e\pi a^*$, and
– (ii) $a^+\pi b^+$, $ea = eb$ for some $f \in E$, $f\pi a^+$.

Proof. Let $a, b \in S$ and suppose (i) holds. Since eπa* and πis a normal, then for any $a \in S$, $(ae)^+\pi(aa^*)^+$, that is, $(ae)^+\pi a^+$. Similary, we have $(be)^+\pi b^+$. Since S is weakly ample, $ea = eb$ implies $(ae)^+ = (be)^+$ and $(ae)^+a = (be)^+b$.

It follows that $a^+\pi b^+$ and (ii) holds. The argument for (ii) implies (i) is similar.

Main results

Let π be a normal congruence on E. Define σ_π on S by the rule:

$$\sigma_\pi = \{(a,b) \in S \times S : a^* \pi b^*, \exists e \in E, e\pi a^*, ae = be\}$$

Theorem 3. Let π be a normal congruence on E, $e, f \in E$. Then $(e, f) \in \pi$ if and only if $tr\sigma_\pi = \pi$ and $(e, f) \in \sigma_\pi$.

Proof. It is obvious that $(e, f) \in \sigma_\pi$ implies $(e, f) \in \pi$. On the other side, as $(e, f) \in \pi$, then $(ee, ef) \in \pi$, that is $(e, ef) \in \pi$. Let $g = ef$. Since $ef = fe$, we have $e(ef) = f(ef)$. Then $(e, f) \in \pi$ implies $(e, f) \in \sigma_\pi$. Hence the result holds.

Theorem 4. Let S be a weakly ample semigroup. Then σ_π is the minimum admissible congruence on S whose restriction E is π.

Proof. Clearly σ_π is reflexive and symmetric. Now we prove that σ_π is transitive. Let $a, b, c \in S$, and $(a, b) \in \sigma_\pi$, $(b, c) \in \sigma_\pi$. Then $a^*\pi b^*$, $b^*\pi c^*$, $e^*\pi a^*$, $f^*\pi b^*$, for some $e, f \in E$, $ae = be$, $bf = cf$. Form the transitive of π,$a^*\pi c^*$. Since $ef \in E$, we conclude that $aef = bef = bfe = cfe = cef$. Since $e\pi a^*$, $f^*\pi b^*$, $a^*\pi b^*$, $ef\pi a^*b^*\pi a^*a^*$, that is, $ef\pi a^*$. Hence σ_π is an equivalence relation.

Let $(a, b) \in \sigma_\pi$, then $a^*\pi b^*$, $ae = be$ for some $e \in E$, $e\pi a^*$and $cae = cbe$. Thus $(cae)^* = (cbe)^*$, that is, $(ca)^*e = (cb)^*e$. Now $(ca)^*e\pi(ca)^*a^*$, $(cb)^*e\pi(cb)^*b^*$, by $(ca)^*a^* = (ca)^*$, $(cb)^*b^* = (cb)^*$, we have:

$$(ca)^* \pi (cb)^* \text{ and } (ca)(ca)^* e = (cb)(cb)^* e = (cb)(ca)^*$$

where $(ca)^*e\pi(ca)^*$. Hence $(ca, cb) \in \sigma_\pi$. On the other side:

$$ae = be \Rightarrow aec = bec \Rightarrow ac(ec)^* = bc(ec)^*$$

By the normality of π, $e\pi a^*$ implies $(ec)^*\pi(a^*c)^*$. As $(a^*c)^* = (ac)^*$, so that $(ec)^*\pi(ac)^*$. Similar $(ec)^*\pi(ba)^*$. Therefore, $(ac)^*\pi(ba)^*$. As $(ac)(ec)^* = (bc)(ec)^*$, $(ec)^*\pi(ac)^*$, we have $(ac, bc) \in \sigma_\pi$. Hence, σ_π is congruence.

To prove σ_π is admissible. Let $a \in S$, $f \in E$ such that $(af, a) \in \sigma_\pi$. Then $(af)^*\pi a^*$, $afe = ae$, for some $e \in E$, $e\pi(af)^*$. Since $(af)^* = (a^*f)^*$, we have $(a^*f)^*\pi a^*$, $e\pi(a^*f)^*$ for some $e \in E$:

$$afe = ae \Rightarrow (afe)^* = (ae)^* \Rightarrow (af)^* e = a^* e \Rightarrow a^* fe = a^* e$$

then $(a^*f, a^*) \in \sigma_\pi$. The argument for $(fa, a) \in \sigma_\pi$ implies $(fa^+, a^+) \in \sigma_\pi$ is similar.

Hence, σ_π is admissible.

It remains to prove that σ_π contained in any admissible congruence on S whose restrict to E is π. Now suppose τ is an admissible congruence on S such that $tr\tau = \pi$, and $(a, b) \in \sigma_\pi$. Then $a^*\pi b^*$, $ae = be$ for some $e \in E$, $e\pi a^*$ and thus $(a^*, e) \in \tau$, $(b^*, e) \in \tau$:

$$a\tau = (aa^*)\tau = a\tau a^*\tau = a\tau e\tau = (ae)\tau = (be)\tau = b\tau e\tau = b\tau b^*\tau = (bb^*)\tau = b\tau$$

that is $(a, b) \in \tau$. Hence $\sigma_\pi \subseteq \tau$ and σ_π is the minimum admissible congruence on S whose restrict to E is π.

Corollary 1. The relation σ_π has also the form:

$$\sigma_\pi = \{(a,b) \in S \times S : a^+ \pi b^+, \exists f \in E, f \pi a^+, fa = fb\}$$

Theorem 5. Let ρ be an admissible congruence on S whose trace is the normal congruence π. Then S/ρ is an idempotent-separating homomorphic image of S/σ_π.

Proof. The mapping ϕ: $S/\sigma_\pi \to S/\rho$ defined by $(s\sigma_\pi)\phi = s\rho$ is a homomorphism of S/σ_π on S/ρ and we have $E(S/\sigma_\pi) = \{e\sigma_\pi\colon e \in E\}$. Let $e\sigma_\pi$, $f\sigma_\pi$ be two idempotents in $S/\sigma_\pi(e, f \in E)$, we have:

$$(e\sigma_\pi)\phi = (f\sigma_\pi)\phi \Rightarrow e\rho = f\rho \Rightarrow (e,f) \in \rho \Rightarrow (e,f) \in \pi(tr\rho = \pi) \Rightarrow e\sigma_\pi = f\sigma_\pi.$$

Therefore, ϕ is idempotent-separating.

Let E be the set of weakly ample semigroup S, a, $b \in S$. Define μ on S by the rule:

$$(a,b) \in \mu \Leftrightarrow \forall e \in E, (ea)^* = (eb)^*, (ae)^+ = (be)^+$$

Let π be a normal congruence on E. Define μ_π on S by the rule:

$$(a,b) \in \mu_\pi \Leftrightarrow (\forall e \in E)(ea)^* \pi (eb)^*, (ae)^+ \pi (be)^+$$

Theorem 6. Let S be a weakly ample semigroup, π be a normal congruence on E. Then for any elements a, b of S, the statements are equivalent:

– (i) $(a, b) \in \mu_\pi$,
– (ii) $(ae)^*\pi(bf)^*$ and $(ae)^+\pi(bf)^+$ for any $e, f \in E$ with $e\pi f$, and
– (iii) $(a\sigma_\pi, b\sigma_\pi) \in \mu(S/\sigma_\pi)$.

Proof. (i)$\Rightarrow$(ii) For any $b \in S$, e, $f \in E$ with $e\pi f$, we have $(eb)^*\pi(fb)^*$. If $(a, b) \in \mu_\pi$, then$(ea)^*\pi(eb)^*$, so that $(ea)^*\pi(fb)^*$. Similarly, $(ae)^+\pi(bf)^+$.

It is clear that (i) is an immediate consequence of (ii).

The (i) $\Leftrightarrow$ (iii) For any a, $b \in S$, we have,

$$(a,b) \in \mu_\pi \Leftrightarrow (ea)^* \pi (eb)^* \text{ and } (ae)^+ \pi (be)^+, \text{ for all } e \in E$$
$$\Leftrightarrow (ea)^* \sigma_\pi = (eb)^* \sigma_\pi \text{ and } (ae)^+ \sigma_\pi = (be)^+ \sigma_\pi, \text{ for all } e \in E\,(tr\sigma_\pi = \pi)$$
$$\Leftrightarrow (e\sigma_\pi a\sigma_\pi)^* = (e\sigma_\pi b\sigma_\pi)^* \text{ and } (a\sigma_\pi e\sigma_\pi)^+ = (b\sigma_\pi e\sigma_\pi)^+, \text{ for all } e \in E$$
$$\Leftrightarrow (a\sigma_\pi, b\sigma_\pi) \in \mu(S/\sigma_\pi)$$

Theorem 7. Let S is weakly ample semigroup. Then μ_π is the maximum admissible congruence on S whose restriction E is π.

Proof. It is clear that μ_π is an equivalence relation. Let $a, b, c \in S$ with $(a, b) \in \mu_\pi$ and $e \in E$. Then $(ae)^*\pi(eb)^*$ and by the normality of π, it follows that $[(ea)^*c]^*\pi[(eb)^*c]^*$, that is $(eac)^*\pi(ebc)^*$. Since $(ec)^+ \in E$, we have $[a(ce)^+c]^+\pi[b(ce)^+]^+$, that is, $(ace)^+\pi(bce)^+$. Therefore, $(ac, bc) \in \mu_\pi$. Similarly, $(ca, cb) \in \mu_\pi$. Hence μ_π is congruence.

It is obvious that $\pi \subseteq \mu_\pi$. Let f, $g \in E$ with $f\mu_\pi g$. Then for any $e \in E$, $ef\pi eg$. Take in turn $e = f$ and $e = g$ to get $f\pi fg$ and $gf\pi g$. As $fg = gf$, so $f\pi g$. Thus $tr\mu_\pi = \pi$. To prove that μ_π is admissible. Let $a \in S$ and $f \in E$, with $(af, a) \in \mu_\pi$:

$$(af, a) \in \mu_\pi \Rightarrow (af\sigma_\pi, a\sigma_\pi) \in \mu(S/\sigma_\pi)$$
$$\Rightarrow [(af)^*\sigma_\pi, a^*\sigma_\pi] \in \mu(S/\sigma_\pi) \;\; (S/\sigma_\pi \text{ and } \sigma_\pi \text{ are admissible})$$
$$\Rightarrow [(a^* f)\sigma_\pi, a^*\sigma_\pi] \in \mu(S/\sigma_\pi)$$
$$\Rightarrow (a^* f, a^*) \in \mu_\pi$$

Similarly, $(fa, a) \in \mu_\pi$ implies that $(fa^+, a^+) \in \mu_\pi$.

It remains to prove that μ_π contains any admissible congruence on S whose restrict to E is π. Let ρ be an admissible congruence on S such that $\rho|_E = \pi$ and $(a, b) \in \rho$ for some $a, b \in S$. Then for any $e \in E$, $(ea, eb) \in \rho$ and $(ae, be) \in \rho$. In particular, we have $[(ea)^*, (eb)^*] \in \rho$, $[(ae)^+, (be)^+] \in \rho$. Thus $[(ea)^*, (ba)^*] \in \pi$, $[(ae)^+, (be)^+] \in \pi$ and $(a, b) \in \mu_\pi$. Therefore, $\rho \subseteq \mu_\pi$.

It follows the results, that for any admissible congruence ρ on S, then trρ is a normal congruence on E and $\sigma_{tr\rho}$, $\mu_{tr\rho}$ are, respectively, the minimum and the maximum admissible congruence on S such that:

$$tr\sigma_{tr\rho} = tr\rho = tr\mu_{tr\rho}$$

where

$$\sigma_{tr\rho} = \{(a,b) \in S\times S : a^*\rho b^*,\ ae = be \text{ for some } e \in a^*\rho \cap E,\ e\pi a^*\}$$

$$= \{(a,b) \in S\times S : a^+\rho b^+,\ fa = fb \text{ for some } e \in a^+\rho \cap E, e\pi a^*\}$$

$$\mu_{tr\rho} = \left\{(a,b) \in S\times S : (ea)^*\rho(eb)^* \text{ and } (ae)^+\rho(be)^+ \text{ for all } e \in E)\right\}$$

We may $\sigma_{tr\rho}$ denote and $\mu_{tr\rho}$ by σ_ρ and μ_ρ, respectively.

Corollary 2. For any admissible congruence ρ on a weakly ample semigroup S, $\sigma_\rho \subseteq \rho \subseteq \mu_\rho$, $tr\sigma_\rho = tr\rho = tr\mu_\rho$.

Let S be any semigroup. If ρ and τ are congruences on S and $\tau \subseteq \rho$, then ρ/τ is congruence on S/τ defined:

$$a\tau\rho / \tau b\tau \Leftrightarrow a\rho b,\ (a,b \in S)$$

Theorem 8. Let ρ and τ are congruences on a weakly ample semigroup S. Then the statements are equivalent:

– (i) $tr\rho = tr\tau$,
– (ii) $\rho \subseteq \mu_\tau$, $\mu_{\tau/\rho} = \mu(S/\rho)$,
– (iii) $a\rho\mu(S/\rho)b\rho \Leftrightarrow a\tau\mu(S/\tau)b\tau(a, b \in S)$,
– (iv) $a\rho\widetilde{\mathcal{H}}\mu(S/\rho)b\rho \Leftrightarrow a\tau\widetilde{\mathcal{H}}(S/\tau)b\tau(a, b \in S)$, and
– (v) $\rho/\rho \cap \tau$ and $\tau/\rho \cap \tau$ are congruences contained in $\widetilde{\mathcal{H}}(S/\rho \cap \tau)$.

Proof. The (i) ⇒ (ii), as $\mu_{tr\rho} = \mu_\rho$, $\mu_{tr\tau} = \mu_\tau$, and $tr\rho = tr\tau$, then $\mu_\rho = \mu_\tau$, so that $\rho \subseteq \mu_\tau$. For any $a, b \in S$, we have:

$$a\rho\mu_{\tau/\rho}b\rho \Leftrightarrow a\rho\mu_{\rho/\rho}b\rho$$

$$\Leftrightarrow a\mu_\rho b$$

$$\Leftrightarrow (ea)^*\rho(eb)^* \text{ and } (ae)^+\rho(be)^+, \text{ for al}l\ e \in E$$

$$\Leftrightarrow (a\rho, b\rho) \in (S / \rho)$$

The (ii) ⇒ (i), observe $tr\rho \subseteq tr\mu_\tau, \subseteq tr\tau = tr\rho$. Further, for any $e, f \in E$, we have:

$$e\tau f \Rightarrow e\mu_\tau f \Rightarrow e\rho\mu_{\tau/\rho} f\rho \Rightarrow e\rho\mu(S / \rho) f\rho \Rightarrow e\rho = f\rho \Rightarrow e\rho f$$

The (i) ⇒ (iii), for any $a, b \in S$, we have:

$$a\rho\ \mu(S / \rho)\ b\rho \Leftrightarrow \forall e \in E, (ea)^*\rho = (eb)^*\rho, (ae)^+\rho = (be)^+\rho$$

$$\Leftrightarrow \forall e \in E, (ea)^*\tau = (eb)^*\tau, (ae)^+\tau = (be)^+\tau$$

$$\Leftrightarrow a\tau\ \mu(S / \tau)\ b\tau$$

The (iii) ⇒ (i), For any $e, f \in E$, we have:

$$e\rho f \Leftrightarrow e\rho = f\rho \Leftrightarrow e\rho\mu(S/\rho)f\rho \Leftrightarrow e\tau\mu(S/\tau)f\tau \Leftrightarrow e\tau = f\tau \Leftrightarrow e\tau f$$

The (i) ⇒ (iv) Let a, $b \in S$ and assume $a\rho\tilde{\mathcal{H}}(S/\rho)b\rho$. Thus $a^*\rho = b^*\rho$ and $a^+\rho = b^+\rho$. The hypothesis implies $a^*\tau = b^*\tau$ and $a^+\tau = b^+\tau$, which imply $a\tau\tilde{\mathcal{H}}(S/\tau)b\tau$. Similarly, we have $a\tau\tilde{\mathcal{H}}(S/\tau)b\tau$ implies that $a\rho\tilde{\mathcal{H}}(S/\rho)b\rho$.

The (iv) ⇒ (i) Let $e, f \in E$, and assume $e\rho f$, then $e\rho\tilde{\mathcal{H}}(S/\rho)f\rho$ so that by hypothesis, $e\tau\tilde{\mathcal{H}}(S/\tau)f\tau$ and hence $e\tau f$. Similarly, we have $e\tau f$ implies $e\rho f$.

The (i) ⇒ (v) For any a, $b \in S$, we have:

$$a(\rho\cap\tau)\rho/_{\rho\cap\tau}\, b(\rho\cap\tau) \Rightarrow a\rho b \Rightarrow a^*\rho b^*, a^+\rho b^+$$

$$\Rightarrow a^*(\rho\cap\tau)\rho/_{\rho\cap\tau}\, b^*(\rho\cap\tau), a^+(\rho\cap\tau)\rho/_{\rho\cap\tau}\, b^+(\rho\cap\tau)$$

$$\Rightarrow a(\rho\cap\tau)\tilde{\mathfrak{L}}(S/_{\rho\cap\tau})b(\rho\cap\tau), a(\rho\cap\tau)\tilde{\mathfrak{R}}(S/_{\rho\cap\tau})b(\rho\cap\tau)$$

$$\Rightarrow a(\rho\cap\tau)\tilde{\mathfrak{H}}(S/_{\rho\cap\tau})b(\rho\cap\tau)$$

Therefore, $\rho/_{\rho\cap\tau} \subseteq \tilde{\mathcal{H}}(S/_{\rho\cap\tau})$. Similarly, $\tau/_{\rho\cap\tau} \subseteq \tilde{\mathcal{H}}(S/_{\rho\cap\tau})$.

The (v) ⇒ (i) For any $e, f \in E$, we have:

$$e\rho f \Rightarrow e(\rho\cap\tau)\rho/_{\rho\cap\tau}\, f(\rho\cap\tau)$$

$$\Rightarrow e(\rho\cap\tau)\tilde{\mathfrak{H}}(S/_{\rho\cap\tau})f(\rho\cap\tau)$$

$$\Rightarrow e(\rho\cap\tau) = f(\rho\cap\tau)$$

$$\Rightarrow e\tau f$$

And similarly, eτf implies $e\rho f$.

Conclusion

As the semigroup theory becomes a hot topic in computer science, especially in machine learning and big data technology [8], the present theory sheds a bright light on applications of semigroup theory to data mining algorithms. Additionally the semigroup theory is an alternative approach to discontinuous problems using fractional calculus and fractal calculus [9-11].

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