THE SECOND ELLIPTIC EQUATION METHOD FOR NON-LINEAR EQUATIONS WITH VARIABLE COEFFICIENTS

by

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The second elliptic equation method is a more general form of Jacobi elliptic function expansion method, which can obtain more kinds of solutions of a non-linear evolution equation. In this paper, the method is used to solve the Kdv-Burgers-Kuramoto (Benny) equation with variable coefficients, and its extremely rich solution properties are elucidated, among which the biperiodic solutions, solitary wave solutions and trigonometric periodic solutions are analyzed graphically.

Key words: *Benny equation with variable coefficient, double periodic solution, the second elliptic equation method,*

Introduction

In this paper, we will study the following Benny equation [1, 2]

$$u_t + uu_x + \alpha u_{xx} + \beta u_{xxx} + \delta u_{xxxx} = 0 \tag{1}$$

where β is the frequency coefficient, α and δ are the dissipation and instability. Equtation (1) is also called as KdV-Burgers -Kuramoto (KBK) equation, and has been widely studied, many solution properties have been revealed [3-5]. However, much literature focused on constant coefficient, and its partner with variable coefficients was rarely studied:

$$u_t + uu_x + \alpha(t)u_{xx} + \beta(t)u_{xxx} + \delta(t)u_{xxxx} = 0 \tag{2}$$

This equation can model many real problems arising in plasma physics, fluid dynamics and thermodynamics, and this paper aims at solving eq. (2) exactly by the second elliptic equation method [6, 7].

Second elliptic equation method

The second elliptic equation method is a more general form of Jacobi elliptic function expansion method [6, 7] and the exp-function method [8-10], (G/G ‘)-expansion method [11], and the auxiliary equation method [12]. To illustrate the solution process, we consider the following non-linear evolution equation:

$$H\left(u, u_t, u_x, u_{xx}, u_{tx}, \cdots\right) = 0 \tag{3}$$

and assume that its solution can be expressed in the form:

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$$u(x,t)=u(\xi)=a_0+\sum_{i=1}^{n}\left[a_i(t)\phi^i(\xi)+b_i\phi^{-i}(\xi)\right] \tag{4}$$

where $\xi = \kappa x + \omega t$, and ϕ is the solution:

$$\phi'^2 = A\phi + B\phi^2 + C\phi^3 \tag{5}$$

where A, B, and C are all constants:

Case 1. If $A = 4$, $B = -4(1 + m^2)$, $C = 4m^2$. The eq. (5) has a solution $\phi(\xi) = sn^2\xi$, $cd^2\xi$.

Case 2. If $A = 4(1 - m^2)$, $B = -4(2m^2 - 1)$, $C = -4m^2$. The eq. (5) has a solution $\phi(\xi) = cn^2\xi$.

Case 3. If $A = 4(m^2 - 1)$, $B = 4(2 - m^2)$, $C = -4$. The eq. (5) has a solution $\phi(\xi) = dn^2\xi$.

Case 4. If $A = 4(1 - m^2)$, $B = 4(2 - m^2)$, $C = 4$. The eq. (5) has a solution $\phi(\xi) = \mathrm{c}s^2\xi$.

Case 5. If If $A = -4m^2(1 - m^2)$, $B = 4(2m^2 - 1)$, $C = 4$. The eq. (5) has a solution $\phi(\xi) = \mathrm{d}s^2\xi$.

Case 6. If $A = C = m^2 - 1$, $B = 2(1 + m^2)$. The eq. (5) has a solution

$$\phi(\xi)=\frac{dn^2\xi}{(1\pm msn\xi)^2}$$

Case 7. If $A = C = 1 - m^2$, $B = 2(1 + m^2)$. The (5) has a solution

$$\phi(\xi)=\frac{cn^2\xi}{(1\pm sn\xi)^2}$$

Case 8. If $A = C = 1$, $B = 2(1 + m^2)$. The eq. (5) has a solution

$$\phi(\xi)=\frac{\mathrm{s}n^2\xi}{(1\pm cn\xi)^2}$$

Case 9. If $A = 1$, $B = 2(m^2 - 2)$, $C = m^2$. The eq. (5) has a solution

$$\phi(\xi)=\frac{\mathrm{s}n^2\xi}{(1\pm dn\xi)^2}$$

Case 10. If $A = -(1 - m^2)^2$, $B = 2(1 + m^2)$, $C = -1$. The eq. (5) has a solution

$$\phi(\xi)=(mcn\xi\pm dn\xi)^2$$

Case 11. If $A = 1$, $B = 2(m^2 - 2)$, $C = m^4$. The eq. (5) has a solution

$$\phi(\xi)=\frac{\mathrm{c}n^2\xi}{\left(\sqrt{1-m^2}\pm dn\xi\right)^2}$$

Case 12. If $A = 1$, $B = 2(m^2 - 2)$, $C = m^4$. The (5) has a solution

$$\phi(\xi)=\frac{\mathrm{s}n^2\xi}{(dn\xi\pm cn\xi)^2}$$

Case 13. If $A = C = 1$, $B = 2(1 - m^2)$. The eq. (5) has a solution

$$\phi(\xi)=\frac{\mathrm{c}n^2\xi}{\left(\sqrt{1-m^2}\,sn\xi\pm dn\xi\right)^2}$$

Case 14. If $A = B = 0$, $C > 0$. The eq. (5) has a solution

$$\phi(\xi) = \frac{4}{\left(\sqrt{C}\xi \pm c_0\right)^2}$$

The solution process follows the following *Steps*:

Step 1. The value of n in eq. (4) is determined by the highest derivative of the linear term and the non-linear term of the equilibrium eq. (3);

Step 2. Put eq. (4) into eq. (3), and collect coefficients of same powers of ϕ and its derivatives, set the coefficients to be zero to obtain a system of algebraic equations for a_0, a_i, b_i ($i = 0, 1, 2,... n$) and κ, ω;

Step 3. Solve the unknown a_0, a_i, b_i ($i = 0, 1, 2,... n$) and κ, ω from the aforementioned algebraic equations;

Step 4. Exact solutions are obtained.

The double periodic solution Benny's equation

Now let's consider again the exact solution of the variable coefficient Benny equation given in eq. (2), where $\alpha(t)$, $\beta(t)$, and $\delta(t)$ are functions of time. By homogeneous equilibrium method, the highest derivative term and non-linear term of the equilibrium eq. (2) can be known as $n = 3$, so the equation is assumed to have a solution:

$$u(x,t) = a_0 + a_1(\omega) + a_2(\omega)^2 + a_3(\omega)^3 + \frac{b_1}{\phi(\omega)} + \frac{b_2}{\phi(\omega)^2} + \frac{b_3}{\phi(\omega)^3} \tag{6}$$

where ω is defined:

$$\omega = p(t)x + q(t) \tag{7}$$

Putting eqs. (6) and (5) into eq. (2), and setting the coefficients of the same power of ϕ and its derivatives to zero, we obtain non-linear ordinary differential equations:

$$\frac{1}{2}Ap^2(t)\alpha(t)a_1 + \frac{1}{2}Cp^2(t)b_1\alpha(t) + \frac{1}{2}ABp^4(t)a_1\delta(t) +$$
$$+\frac{3}{2}A^2p^4(t)a_2\delta(t) + \frac{1}{2}BCp^4(t)b_1\delta(t) + \frac{3}{2}C^2P^4(t)b_2\delta(t) = 0$$

$$189A^2p^4(t)b_3\delta(t) = 0$$

$$\frac{21}{2}Ap^2(t)b_3\alpha(t) + \frac{105}{2}A^2P^4(t)b_2\delta(t) + \frac{525}{2}ABp^4(t)b_3\delta(t) = 0$$

$$5Ap^2(t)b_2\alpha(t) + 9Bp^2(t)b_3\delta(t) + \frac{15}{2}A^2p^4(t)b_1\delta(t) +$$
$$+65ABp^4(t)b_2\delta(t) + 81B^2p^4(t)b_3\delta(t) + \frac{369}{2}ACP^4(t)b_3\delta(t) = 0$$

$$\frac{3}{2}Ap^2(t)\alpha(t)b_1 + 4Bp^2(t)b_2\alpha(t) + \frac{15}{2}Cp^2(t)b_3\alpha(t) + \frac{15}{2}ABp^4(t)b_1\delta(t) + 16B^2p^4(t)b_2\delta(t) +$$
$$+42ACP^4(t)b_2\delta(t) + \frac{195}{2}BCp^4(t)b_3\delta(t) = 0$$

$$Bp^2(t)\alpha(t)b_1+3Cp^2(t)b_2\alpha(t)+B^2p^4(t)b_1\delta(t)+$$
$$+\frac{9}{2}ACp^4(t)b_1\delta(t)+15BCp^4(t)b_2\delta(t)+\frac{45}{2}C^2p^4(t)b_3\delta(t)=0$$
$$Bp^2(t)\alpha(t)a_1+3Ap^2(t)a_2\alpha(t)+B^2p^4(t)a_1\delta(t)+$$
$$+\frac{9}{2}ACp^4(t)a_1\delta(t)+15ABp^4(t)a_2\delta(t)+\frac{45}{2}A^2p^4(t)a_3\delta(t)=0$$
$$\frac{3}{2}Cp^2(t)\alpha(t)a_1+4Bp^2(t)a_2\alpha(t)+\frac{15}{2}Ap^2(t)a_3\alpha(t)+\frac{15}{2}BCp^4(t)a_1\delta(t)+16B^2p^4(t)a_2\delta(t)+$$
$$+42ACP^4(t)a_2\delta(t)+\frac{195}{2}ABp^4(t)a_3\delta(t)=0$$
$$5Cp^2(t)\alpha(t)a_2+9Bp^2(t)a_3\alpha(t)+\frac{15}{2}C^2p^4(t)a_1\delta(t)+65BCp^4(t)a_2\delta(t)+$$
$$+81B^2p^4(t)a_3\delta(t)+\frac{369}{2}ACp^4(t)a_3\delta(t)=0$$
$$\frac{21}{2}Cp^2(t)a_3\alpha(t)+\frac{105}{2}C^2P^4(t)a_2\delta(t)+\frac{525}{2}BCp^4(t)a_3\delta(t)=0 \tag{8}$$
$$189C^2p^4(t)a_3\delta(t)=0$$
$$-3p(t)b_3^2=0$$
$$-5p(t)b_2b_3=0$$
$$-2p(t)b_2{}^2\alpha(t)-4P(t)b_1b_3\delta(t)-42Ap^3(t)a_3\beta(t)=0$$
$$3p(t)\left[-b_1b_2-a_0b_3-5Ap^2(t)b_2\beta(t)-9Bp^2(t)b_3\beta(t)\right]-3xb_3p'(t)-3b_3q'(t)=0$$
$$-p(t)b_1^2-2p(t)a_0b_2-2p(t)a_1b_3-3Ap^3(t)b_1\beta(t)-8Bp^3(t)b_2\beta(t)-15Cp^3(t)b_3\beta(t)-$$
$$-2xb_2p'(t)-2b_2q'(t)=0$$
$$p(t)\left[-a_0b_1-a_1b_2-a_2b_3-Bp^2(t)b_1\beta(t)-3Cp^2(t)b_2\beta(t)\right]-xb_1p'(t)-b_1q'(t)=0$$
$$p(t)\left[a_0a_1+a_2b_1+a_3b_2+Bp^2(t)a_1\beta(t)+3Ap^2(t)a_2\beta(t)\right]+xa_1p'(t)+a_1q'(t)=0$$
$$p(t)a_1^2+2p(t)a_0b_2+2p(t)a_3b_1+3Cp^3(t)a_1\beta(t)+8Bp^3(t)a_2\beta(t)+15Ap^3(t)a_3\beta(t)+$$
$$+2xa_2p'(t)+2a_2q'(t)=0$$
$$3p(t)\left[a_1a_2+a_0a_3+5Cp^2(t)a_2\beta(t)+9Bp^2(t)a_3\beta(t)\right]+3xa_3p'(t)+3a_3q'(t)=0$$
$$2p(t)a_2{}^2+4P(t)a_1a_3+42Cp^3(t)a_3\beta(t)=0$$
$$5p(t)a_2a_3=0$$
$$3p(t)a_3{}^2=0$$

Solving the previous algebraic linear equations:

$$a_2 = a_3 = b_2 = b_3 = 0$$
$$a_0 = -\frac{Bp^3(t)\beta(t) + xp'(t) + q'(t)}{p(t)}$$
$$a_1 = -3Cp^2(t)\beta(t) \tag{9}$$
$$b_1 = -3Ap^2(t)\beta(t)$$

So the double periodic solution of the Benny equation with variable coefficient:

$$u_1 = \frac{4(1+m^2)p^3(t)\beta(t) + xp'(t) + q'(t)}{p(t)} - 12m^2p^2(t)\beta(t)sn^2\xi - \frac{12m^2p^2(t)\beta(t)}{sn^2\xi} \tag{10}$$

$$u_2 = -\frac{4(2m^2-1)p^3(t)\beta(t) + xp'(t) + q'(t)}{p(t)} + 12m^2p^2(t)\beta(t)cn^2\xi - \frac{12(1-m^2)p^2(t)\beta(t)}{cn^2\xi} \tag{11}$$

$$u_3 = -\frac{4(2-m^2)p^3(t)\beta(t) + xp'(t) + q'(t)}{p(t)} + 12p^2(t)\beta(t)dn^2\xi - \frac{12(m^2-1)p^2(t)\beta(t)}{dn^2\xi} \tag{12}$$

$$u_4 = -\frac{4(2-m^2)p^3(t)\beta(t) + xp'(t) + q'(t)}{p(t)} - 12p^2(t)\beta(t)cs^2\xi - \frac{12(1-m^2)p^2(t)\beta(t)}{cs^2\xi} \tag{13}$$

$$u_5 = -\frac{4(2m^2-1)p^3(t)\beta(t) + xp'(t) + q'(t)}{p(t)} - 12p^2(t)\beta(t)ds^2\xi - \frac{12m^2(1-m^2)p^2(t)\beta(t)}{ds^2\xi} \tag{14}$$

$$u_6 = -\frac{2(m^2+1)p^3(t)\beta(t) + xp'(t) + q'(t)}{p(t)} - 3(m^2-1)p^2(t)\beta(t)\frac{dn^2\xi}{(1\pm msn\xi)^2} - \\ -3(m^2-1)p^2(t)\beta(t)\frac{(1\pm msn\xi)^2}{dn^2\xi} \tag{15}$$

$$u_7 = -\frac{2(m^2+1)p^3(t)\beta(t) + xp'(t) + q'(t)}{p(t)} - 3(1-m^2)p^2(t)\beta(t)\frac{cn^2\xi}{(1\pm sn\xi)^2} - \\ -3(m^2-1)p^2(t)\beta(t)\frac{(1\pm sn\xi)^2}{cn^2\xi} \tag{16}$$

$$u_8 = -\frac{2(1-m^2)p^3(t)\beta(t) + xp'(t) + q'(t)}{p(t)} - 3p^2(t)\beta(t)\frac{sn^2\xi}{(1\pm cn\xi)^2} - 3p^2(t)\beta(t)\frac{(1\pm cn\xi)^2}{sn^2\xi} \tag{17}$$

$$u_9 = -\frac{2(m^2-2)p^3(t)\beta(t) + xp'(t) + q'(t)}{p(t)} - \\ -3m^2p^2(t)\beta(t)\frac{sn^2\xi}{(1\pm dn\xi)^2} - 3p^2(t)\beta(t)\frac{(1\pm dn\xi)^2}{sn^2\xi} \tag{18}$$

$$u_{10}=-\frac{2\left(1+m^2\right)p^3(t)\beta(t)+xp'(t)+q'(t)}{p(t)}+3p^2(t)\beta(t)\left(mcn\xi\pm dn\xi\right)^2+$$
$$+3\left(1-m^2\right)p^2(t)\beta(t)\frac{1}{\left(mcn\xi\pm dn\xi\right)^2} \tag{19}$$

$$u_{11}=-\frac{2\left(m^2-2\right)p^3(t)\beta(t)+xp'(t)+q'(t)}{p(t)}-3m^4p^2(t)\beta(t)\frac{cn^2\xi}{\left(\sqrt{1-m^2}\pm dn\xi\right)^2}-$$
$$-3p^2(t)\beta(t)\frac{\left(\sqrt{1-m^2}\pm dn\xi\right)^2}{cn^2\xi} \tag{20}$$

$$u_{12}=-\frac{2\left(1+m^2\right)p^3(t)\beta(t)+xp'(t)+q'(t)}{p(t)}-3\left(1-m^2\right)p^2(t)\beta(t)\frac{sn^2\xi}{\left(dn\xi\pm cn\xi\right)^2}-$$
$$-3p^2(t)\beta(t)\frac{\left(dn\xi\pm cn\xi\right)^2}{sn^2\xi} \tag{21}$$

$$u_{13}=-\frac{2\left(1-2m^2\right)p^3(t)\beta(t)+xp'(t)+q'(t)}{p(t)}-3p^2(t)\beta(t)\frac{cn^2\xi}{\left(\sqrt{1-m^2}sn\xi\pm dn\xi\right)^2}-$$
$$-3p^2(t)\beta(t)\frac{\left(\sqrt{1-m^2}sn\xi\pm dn\xi\right)^2}{cn^2\xi} \tag{22}$$

$$u_{14}=-\frac{xp'(t)+q'(t)}{p(t)}-3Cp^2(t)\beta(t)\frac{4}{\left(\sqrt{C}\xi\pm c_0\right)^2} \tag{23}$$

where $\xi = p(t)x + q(t)$.

– When $m \to 1$, $sn\xi \to \tanh\xi$, $cn\xi$, $dn\xi \leftarrow \text{sech}\xi$. Therefore, the soliton solution of eq. (2) can be obtained:

$$u_{15}=\frac{8p^3(t)\beta(t)+xp'(t)+q'(t)}{p(t)}-12p^2(t)\beta(t)\tanh^2\xi-\frac{12p^2(t)\beta(t)}{\tanh^2\xi} \tag{24}$$

$$u_{16}=-\frac{4p^3(t)\beta(t)+xp'(t)+q'(t)}{p(t)}+12p^2(t)\beta(t)\text{sech}^2\xi \tag{25}$$

$$u_{17}=-\frac{4p^3(t)\beta(t)+xp'(t)+q'(t)}{p(t)}-12p^2(t)\beta(t)\left(\frac{\text{sech}\,\xi}{\tanh\xi}\right)^2 \tag{26}$$

$$u_{18}=-\frac{4p^3(t)\beta(t)+xp'(t)+q'(t)}{p(t)} \tag{27}$$

$$u_{19}=-3p^2(t)\beta(t)\frac{\tanh^2\xi}{\left(1\pm\text{sech}\,\xi\right)^2}-3p^2(t)\beta(t)\frac{\left(1\pm\text{sech}\,\xi\right)^2}{\tanh^2\xi} \tag{28}$$

$$u_{20} = \frac{2p^3(t)\beta(t) + xp'(t) + q'(t)}{p(t)} - 3p^2(t)\beta(t)\frac{\tanh^2\xi}{(1 \pm \operatorname{sech}\xi)^2} - 3p^2(t)\beta(t)\frac{(1 \pm \operatorname{sech}\xi)^2}{\tanh^2\xi} \tag{29}$$

$$u_{21} = -\frac{4p^3(t)\beta(t) + xp'(t) + q'(t)}{p(t)} + 3p^2(t)\beta(t)(\operatorname{sech}\xi \pm \operatorname{sech}\xi)^2 \tag{30}$$

$$u_{22} = \frac{2p^3(t)\beta(t) + xp'(t) + q'(t)}{p(t)} - 6p^2(t)\beta(t) \tag{31}$$

$$u_{23} = -\frac{8p^3(t)\beta(t) + xp'(t) + q'(t)}{p(t)} - 3p^2(t)\beta(t)\frac{(\operatorname{sech}\xi \pm sch\xi)^2}{\tanh^2\xi} \tag{32}$$

$$u_{24} = -\frac{xp'(t) + q'(t)}{p(t)} - 3Cp^2(t)\beta(t)\frac{4}{\left(\sqrt{C}\xi \pm c_0\right)^2} \tag{33}$$

The soliton solution is shown in figs. 1-4.

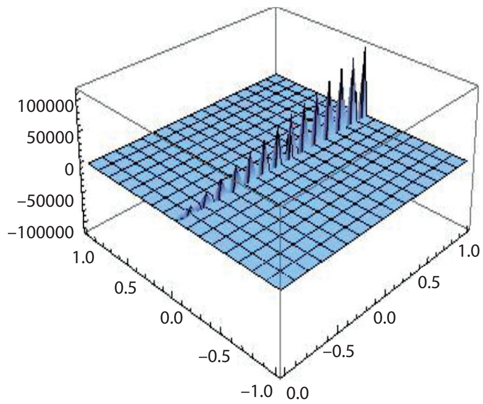

Figure 1. Soliton solution (u_{15})

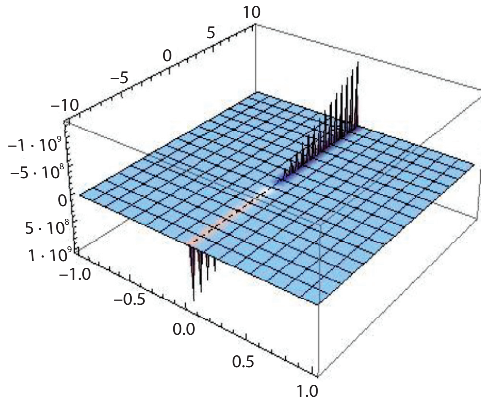

Figure 2. Soliton solution (u_{16})

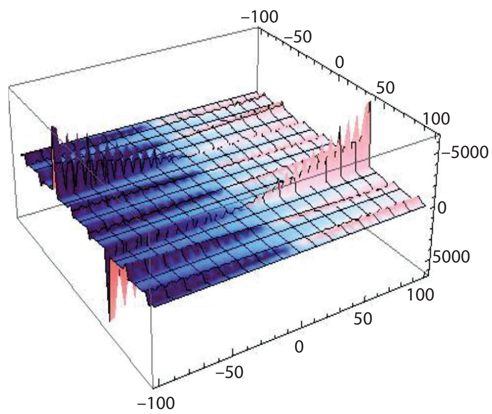

Figure 3. Soliton solution (u_{20})

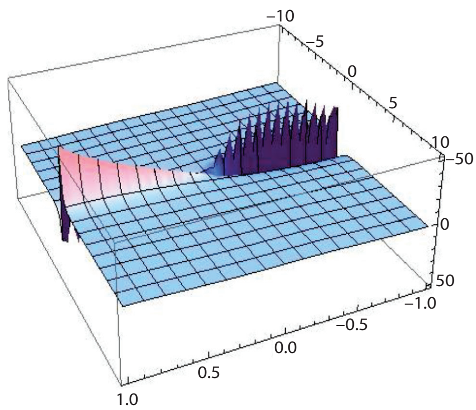

Figure 4. Soliton solution (u_{21})

It can be seen from the aforementioned four images that when x and t take different values, the solitary wave propagation direction and soliton morphology are varied greatly. The two peaks in figs. 2-4 are all anti-symmetric, and the upper and lower sides are symmetrical, with significant periodic changes.

– When $m \to 0$, $sn\xi \to \sin\xi$, $cn\xi \to \cos\xi$, $dn\xi \to 1$. Therefore, the periodic solution of the eq. (6) can be obtained:

$$u_{25} = \frac{4p^3(t)\beta(t)+xp'(t)+q'(t)}{p(t)} - \frac{12p^2(t)\beta(t)}{\sin^2\xi} \tag{34}$$

$$u_{26} = \frac{4p^3(t)\beta(t)+xp'(t)+q'(t)}{p(t)} - \frac{12p^2(t)\beta(t)}{\cos^2\xi} \tag{35}$$

$$u_{27} = -\frac{8p^3(t)\beta(t)+xp'(t)+q'(t)}{p(t)} + 24p^2(t)\beta(t) \tag{36}$$

$$u_{28} = \frac{-8(1+m^2)p^3(t)\beta(t)+xp'(t)+q'(t)}{p(t)} - 12p^2(t)\beta(t)\left(\frac{\cos\xi}{\sin\xi}\right)^2 - \frac{12p^2(t)\beta(t)}{\left(\dfrac{\cos\xi}{\sin\xi}\right)^2} \tag{37}$$

$$u_{29} = \frac{4p^3(t)\beta(t)+xp'(t)+q'(t)}{p(t)} - 12p^2(t)\beta(t)\left(\frac{1}{\sin\xi}\right)^2 \tag{38}$$

$$u_{30} = -\frac{2p^3(t)\beta(t)+xp'(t)+q'(t)}{p(t)} + 6p^2(t)\beta(t) \tag{39}$$

$$u_{31} = -\frac{2p^3(t)\beta(t)+xp'(t)+q'(t)}{p(t)} - 3p^2(t)\beta(t)\left(\frac{\cos\xi}{1\pm\sin\xi}\right)^2 - 3p^2(t)\beta(t)\left(\frac{1\pm\sin\xi}{\cos\xi}\right)^2 \tag{40}$$

$$u_{32} = -\frac{2p^3(t)\beta(t)+xp'(t)+q'(t)}{p(t)} - 3p^2(t)\beta(t)\left(\frac{\sin\xi}{1\pm\cos\xi}\right)^2 - 3p^2(t)\beta(t)\left(\frac{1\pm\cos\xi}{\sin\xi}\right)^2 \tag{41}$$

$$u_{33} = \frac{4p^3(t)\beta(t)+xp'(t)+q'(t)}{p(t)} - 3p^2(t)\beta(t)\left(\frac{1\pm1}{\sin\xi}\right)^2 \tag{42}$$

$$u_{34} = \frac{4p^3(t)\beta(t)+xp'(t)+q'(t)}{p(t)} - 3p^2(t)\beta(t)\left(\frac{1\pm1}{\cos\xi}\right)^2 \tag{43}$$

The graph part of the periodic solution of the trigonometric function of eq. (2) is shown in figs. 5 and 6.

As can be seen from that figures, the waveform of each image moves along an obvious track. Figure 5 shows a semicircular track, and fig. 6 gives a broken line track. The solution morphology sees both maximum and minimum. From the aforementioned analysis, it can be seen that most of the images have a certain regularity, which is the basic feature of periodic solutions and fully demonstrates the diversification of images of periodic solutions.

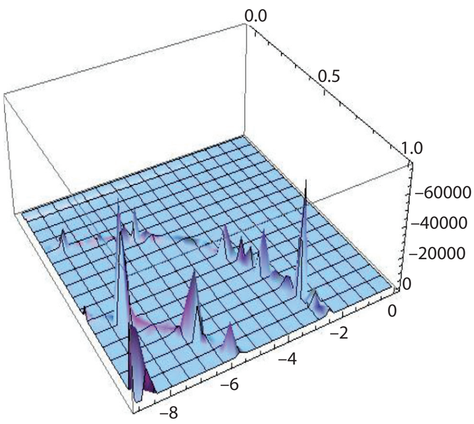

Figure 5. Periodic solution for trig functions (u_{28})

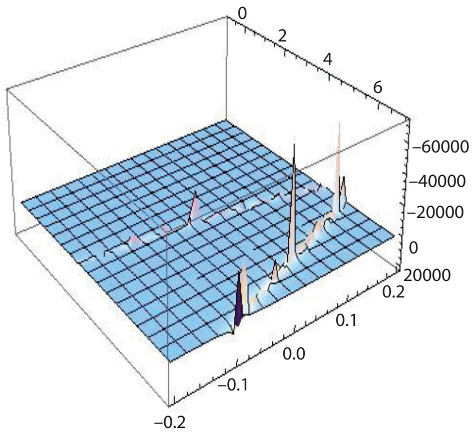

Figure 6. Periodic solution for trig functions (u_{31})

Conclusions and discussion

In short, the Benny equation has extremely rich in the solution properties, such as double periodic solutions, and soliton solutions of triangle function periodic solutions. These new analytical solutions can be degraded into corresponding soliton solutions and triangle function periodic solutions, respectively, when $m \to 1$, $m \to 0$. The solution process shows the present method is extremly effective to non-linear equations with variable coefficients, and it can be extended easily to fractional calculus and fractal calculus [13-20].

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