THE SPACE SPECTRAL INTERPOLATION COLLOCATION METHOD FOR REACTION-DIFFUSION SYSTEMS

by

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Original scientific paper
https://doi.org/10.2298/TSCI200402022Z

A space spectral interpolation collocation method is proposed to study non-linear reaction-diffusion systems with complex dynamics characters. A detailed solution process is elucidated, and some pattern formations are given. The numerical results have a good agreement with theoretical ones. The method can be extended to fractional calculus and fractal calculus.

Key words: *numerical simulation, reaction-diffusion, fractal derivative, complex dynamics characters, turing bifurcation condition, space spectral interpolation collocation method*

Introduction

Reaction diffusion systems are a class of parabolic PDE arising in physics, chemistry, engineering, biology and ecology [1-9]. In this paper, we consider the following reaction-diffusion system [5-6]:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \delta_1 \nabla^2 u + f(u,v) \\ \dfrac{\partial v}{\partial t} = \delta_2 \nabla^2 v + g(u,v) \end{cases}, \quad (x,y) \in \Omega, t > 0 \tag{1}$$

with boundary conditions:

$$\frac{\partial u}{\partial n}\Big|_{\partial\Omega} = \frac{\partial v}{\partial n}\Big|_{\partial\Omega} = 0 \tag{2}$$

where $u(x, y, t)$ and $v(x, y, t)$ are unknown functions, δ_1 and δ_2 – the diffusion coefficients, $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$, $\partial\Omega$ – the boundary, f and g the – functions of u and v.

Many approaches are used to solve reaction-diffusion systems. These methods include the finite difference method [10], Galerkin finite element method [11], the best uniform rational approximation [12], the variational iteration method [13, 14] and the homotopy perturbation methods [15-17], the barycentric interpolation collocation method [18-20], and the reproducing kernel method [21-25], *etc.*

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Description of the space spectral interpolation collocation method

In this section, we use the n^{th} spectral differential matrix to solve system of eq. (1). From [18], the interpolation function $I_N u(x)$ of the discrete sequence $u_1, u_2,... u_N$ can be written:

$$u(x) \sim I_N u(x) = \sum_{m=1}^{N} u_m S_N(x - x_m) \tag{3}$$

where

$$S_N(x) = \frac{\sin\left(\frac{\pi x}{h}\right)}{\left(\frac{2\pi}{h}\right)\tan\left(\frac{x}{2}\right)}$$

I_N – the interpolation operator such that for any a function $u(x)$ defined on [0, 2π], $u_j = u(jh)$, $j = 1,..., N$, $x_j - x_m = (j - m)h$ the interpolation space is span$\{S_N(x - jh), j = 1, 2,..., N\}$.

It is not difficult to derive simple expressions of the n^{th} derivatives of $I_N u(x)$ at $x_j = jh$:

$$I_N u^{(n)}(x_j) = \sum_{m=1}^{N} u_m S_N^{(n)}(x_j - x_m)$$

where

$$D_N^{(n)} = [S_N^{(n)}(x_j - x_m)]_{i,j=1,2,...,N}$$

is called the n^{th} spectral differential matrix [26].

Here we consider a finite spatial domain Ω = [0, 2π] × [0, 2π]. We define equally spaced N_2 grid points over Ω:

$$(x_i, y_j) = (ih, jh), \quad i, j = 1, 2, 3, ..., N$$

where $h = 2\pi/N$ for a given finite natural number $N \in \mathbb{N}$. Using eq. (3), the interpolation function $I_N u(x, y, t)$ and $I_N v(x, y, t)$ of function $u(x, y, t)$ and $v(x, y, t)$ can be written:

$$\begin{aligned} u(x,y,t) \sim I_N u(x,y,t) &= \sum_{i=1}^{N}\sum_{j=1}^{N} S_N(x - x_i) S_N(y - y_i) u(x_i, y_j, t) \\ v(x,y,t) \sim I_N v(x,y,t) &= \sum_{i=1}^{N}\sum_{j=1}^{N} S_N(x - x_i) S_N(y - y_i) v(x_i, y_j, t) \end{aligned} \tag{4}$$

where $u_{i,j} = u(x, y, t)$, $v_{i,j} = v(x, y, t)$, $i, j = 1, 2,..., N$. Thus the following relations is hold at collocation points (x_p, y_q):

$$\begin{aligned} u(x_p, y_q, t) \sim I_N u(x_p, y_q, t) &= \sum_{i=1}^{N}\sum_{j=1}^{N} S_N(x_p - x_i) S_N(y_q - y_i) u(x_i, y_j, t) \\ v(x_p, y_q, t) \sim I_N v(x_p, y_q, t) &= \sum_{i=1}^{N}\sum_{j=1}^{N} S_N(x_p - x_i) S_N(y_q - y_i) v(x_i, y_j, t) \end{aligned} \tag{5}$$

$$u^{(2,0)}(x_p, y_q, t) \sim I_N u^{(2,0)}(x_p, y_q, t) = \frac{\partial^2 u(x_p, y_q, t)}{\partial x^2} = \sum_{i=1}^{N}\sum_{j=1}^{N} S_N^{(2)}(x_p - x_i) S_N(y_q - y_i) u(x_i, y_j, t)$$

$$u^{(0,2)}(x_p, y_q, t) \sim I_N u^{(0,2)}(x_p, y_q, t) = \frac{\partial^2 u(x_p, y_q, t)}{\partial y^2} = \sum_{i=1}^{N}\sum_{j=1}^{N} S_N(x_p - x_i) S_N^{(2)}(y_q - y_i) u(x_i, y_j, t)$$

$$v^{(2,0)}(x_p,y_q,t)\sim I_N v^{(2,0)}(x_p,y_q,t)=\frac{\partial^2 v(x_p,y_q,t)}{\partial x^2}=\sum_{i=1}^{N}\sum_{j=1}^{N}S_N^{(2)}(x_p-x_i)S_N(y_q-y_i)v(x_i,y_j,t)$$

$$v^{(0,2)}(x_p,y_q,t)\sim I_N v^{(0,2)}(x_p,y_q,t)=\frac{\partial^2 v(x_p,y_q,t)}{\partial y^2}=\sum_{i=1}^{N}\sum_{j=1}^{N}S_N(x_p-x_i)S_N^{(2)}(y_q-y_i)v(x_i,y_j,t)$$

Noting

$$\begin{gathered}u=[u_{11},u_{21},...,u_{N1},u_{12},u_{22},...,u_{N2},u_{1N},...,u_{NN}]^T\\ v=[v_{11},v_{21},...,v_{N1},v_{12},v_{22},...,v_{N2},v_{1N},...,v_{NN}]^T\end{gathered}\tag{6}$$

Therefore, the formula eq. (5) can be written as matrix forms:

$$u^{(2,0)}=D_N^{(2,0)}u,\ ,u^{(0,2)}=D_N^{(0,2)}u,\ v^{(2,0)}=D_N^{(2,0)}v,\ v^{(0,2)}=D_N^{(0,2)}v\tag{7}$$

$$D_N^{(2,0)}u=D_N^{(2)}\otimes E_N,\ D_N^{(0,2)}=E_N\otimes D_N^{(2)},\ D_N^{(0,0)}=E_N\otimes E_N\tag{8}$$

where E_N is N order unit matrix, $\otimes$ – the Kronecker product of matrix.

Employing eqs. (6) and (7), eq. (1) can be written:

$$\frac{\partial}{\partial t}\begin{bmatrix}u\\ v\end{bmatrix}=\begin{bmatrix}\delta_1 D & 0\\ 0 & \delta_2 D\end{bmatrix}\begin{bmatrix}u\\ v\end{bmatrix}=\begin{bmatrix}f_2(u,v)\\ f_3(u,v)\end{bmatrix}\tag{9}$$

here

$$[u,v]=[u_{11},...,u_{N1},u_{12},...,u_{N2},u_{1N},...,u_{NN},v_{11},...,v_{N1},v_{12},...,v_{N2},v_{1N},...,v_{NN}]$$

$$D=D_N^{(2,0)}+D_N^{(0,2)}=D_N^{(2)}\otimes E_N+E_N\otimes D_N^{(2)}$$

$$\begin{gathered}[f_1(u,v),\ f_2(u,v)]=[f_1(u_{11},v_{11}),...,\ f_1(u_{NN},v_{NN}),\ f_2(u_{11},v_{11}),...,\\ f_2(u_{NN},v_{NN}),\ f_3(u_{11},v_{11}),...,\ f_3(u_{NN},v_{NN})]\end{gathered}$$

We can get the numerical solution of system (1).

Numerical simulation

As an example, we consider [7-9]:

$$\begin{cases}\dfrac{\partial u}{\partial t}=\delta_1\Delta u+u(1-u)-\dfrac{uv}{u+\alpha}\\[2mm] \dfrac{\partial v}{\partial t}=\delta_2\Delta v+\dfrac{\beta uv}{u+\alpha}-rv\end{cases}\tag{10}$$

where α, β, and r are constants.

The equilibrium point of the non-diffusive system (10) is $E^* = (u_0, v_0)$, where $\beta > (\alpha + 1)\gamma$:

$$u_0=\frac{\alpha\gamma}{\beta-\gamma},\ v_0=\frac{\alpha\beta(\beta-\gamma-\alpha\gamma)}{(\beta-\gamma)^2}$$

The Jacobian matrix A_0 of the non-diffusive system (10) at E^* reads:

$$A_0=\begin{bmatrix}1-2u_0-\dfrac{\alpha v_0}{(u_0+\alpha)^2} & -\dfrac{u_0}{u_0+\alpha}\\ \dfrac{\alpha\beta v_0}{(u_0+\alpha)^2} & \dfrac{\beta u_0}{u_0+\alpha}-\gamma\end{bmatrix}=\begin{bmatrix}\dfrac{(1-\alpha)\beta-(1+\alpha-\alpha^2)\gamma}{\beta-\gamma} & -\dfrac{\gamma}{\beta}\\ (1-\alpha)\beta-\gamma & 0\end{bmatrix}\tag{11}$$

The Turing bifurcation conditions of eq. (10) can be written:

$$(\alpha+1)\gamma < \beta < \frac{1+\alpha-\alpha^2}{1-\alpha}\gamma$$

$$\delta_2\beta[\beta-\alpha\beta-(1+\alpha-\alpha^2)\gamma]^2 > 4\delta_1\gamma(\beta-\alpha\beta-\gamma)(\beta-\gamma)^2$$

Taking the parameters $\alpha = 0.4$, $\gamma = 0.6$, $\beta = 1$, $\delta_1 = 1$, $\delta_2 = 1$. Using the present method, numerical solution and patterns of eq. (10) with different initial conditions are showed in figs. 1 and 2.

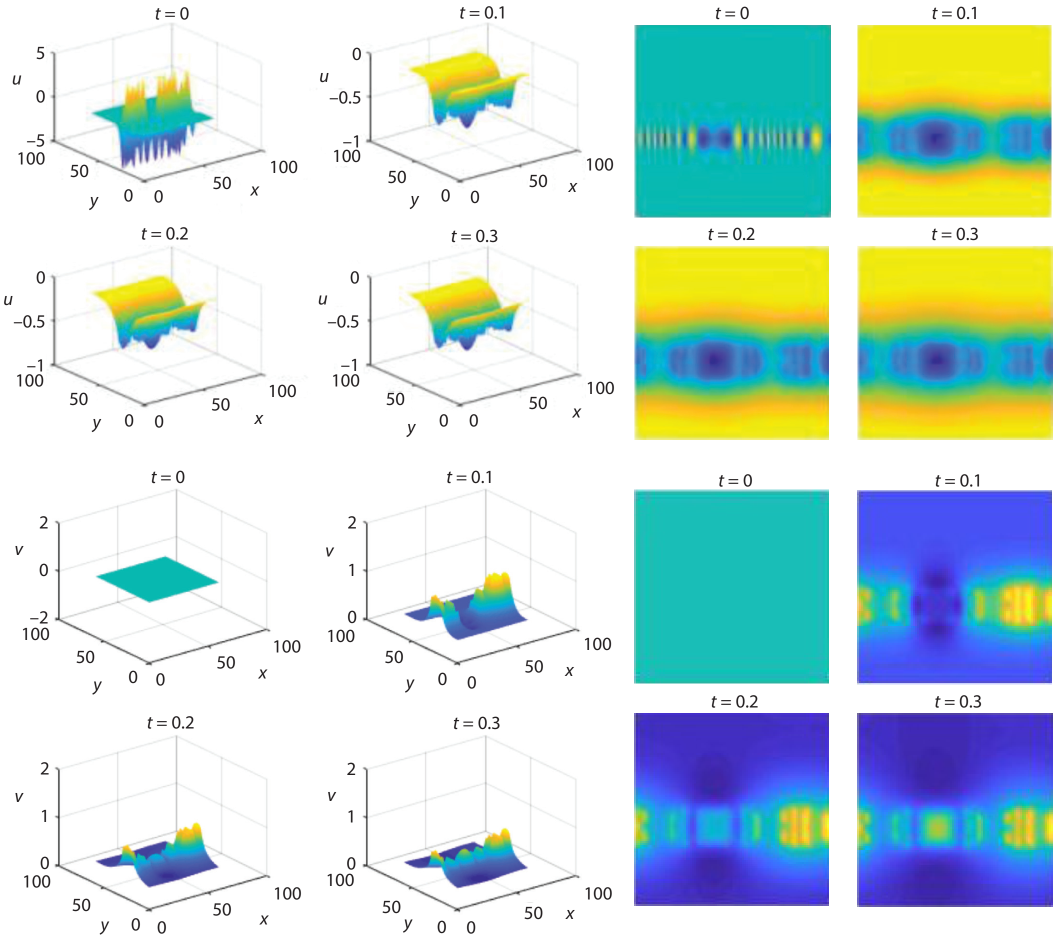

Figure 1. Numerical solution and pattern of eq. (10) with initial condition $v(x, y, 0) = 1$, $u(x, y, 0) = e^{-20[(x-0.4)^2+(y-0.4)^2]}\sin\{-20[(x-0.4)^2+(y-0.4)^2]\}$

Conclusion

This paper proposes a new numerical method called as the space spectral interpolation collocation method to investigate a class of reaction-diffusion systems with complex dynamics characters. Numerical results show some interesting pattern dynamic behaviors, and our results are consistent with theoretical results given in [7-9], showing the reliability of the numerical technique, which can be easily extended to fractal reaction-diffusion system [27-33]:

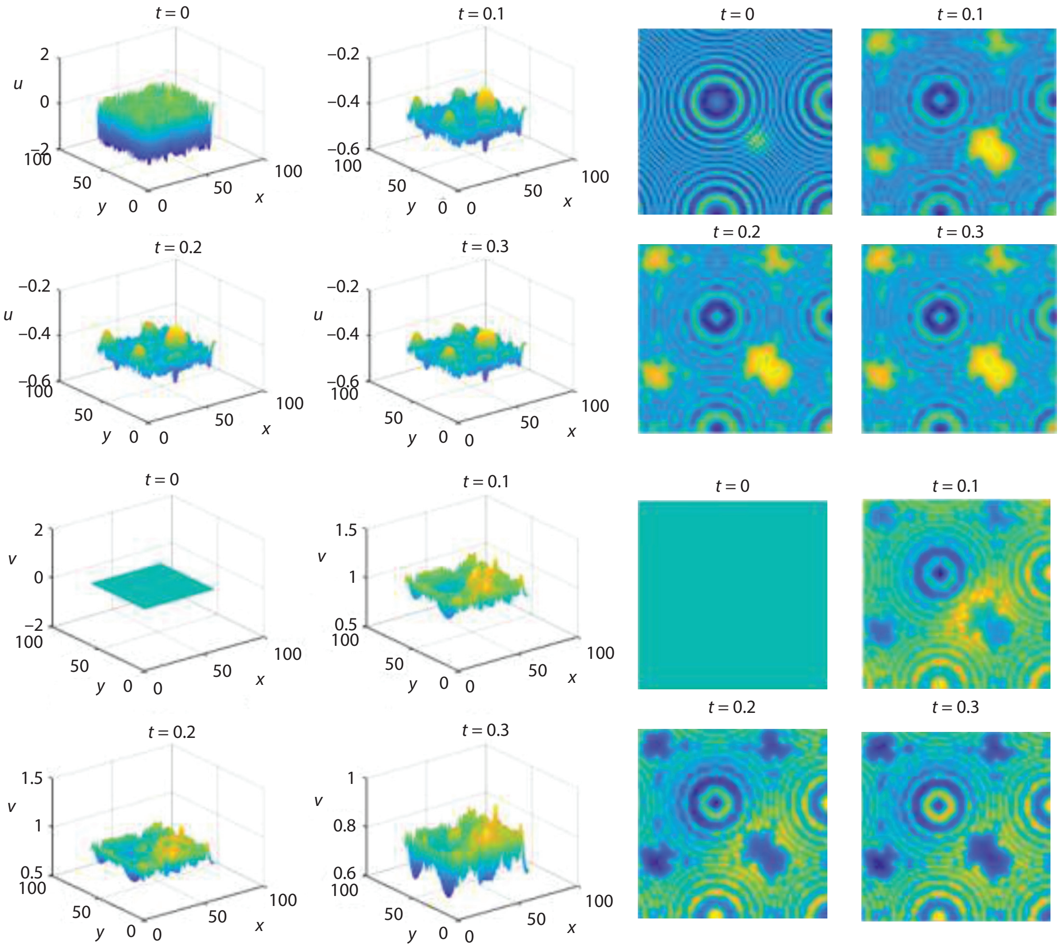

Figure 2. Numerical solution and pattern of eq. (10) with initial condition $v(x, y, 0) = 1$, $u(x, y, 0) = e^{-20[(x-0.4)^2 + (y-0.4)^2]}\sin\{-20[(x-0.4)^2 + (y-0.4)^2$

$$\begin{cases} \dfrac{\partial u}{\partial t^{\alpha}} = \delta_1 \nabla^{2\alpha} u + f(u,v) \\ \dfrac{\partial v}{\partial t^{\beta}} = \delta_2 \nabla^{2\beta} v + g(u,v) \end{cases} \tag{12}$$

where $2/\partial t^{\alpha}$ is the fractal derivative, $\nabla^{2\alpha} = \partial^2/\partial x^{2\alpha} + \partial^2/\partial y^{2\alpha}$.

Acknowledgment

The authors would like to express their thanks to the unknown referees for their careful reading and helpful comments. This paper is supported by the Natural Science Foundation of Inner Mongolia [2017MS0103].

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Paper submitted: April 2, 2020
Paper revised: June 26, 2020
Paper accepted: June 28, 2020