MULTI-COMPLEXITON SOLUTIONS OF THE (2+1)-DIMENSIONAL ASYMMENTRICAL NIZHNIK-NOVIKOV-VESELOV EQUATION

by

Pin-Xia WU, Wei-Wei LING

a School of Mathematics and Statistics, Central South University, Changsha, Hunan, 410083, China
b College of Social Management, Jiangxi College of Applied Technology, Ganzhou, Jiangxi, 341000, China

In this paper, the (2+1)-dimensional asymmetrical Nizhnik-Novikov-Veselov equation (ANNV) is investigated to acquire the complexiton solutions by the Hirota direct method. It is essential to transform the equation into Hirota bilinear form and to build N-complexiton solutions by pairs of conjugate wave variables.

Key words: the (2+1)-dimensional Asymmetrical Nizhnik-Novikov-Veselov equation (ANNV), complexiton solution, the Hirota bilinear form, the pairs of conjugate wave variables

Introduction

Nonlinear differential equations (NLDEs) represent numerous phenomena in many fields of mathematics and physics [1-6]. Therefore, it has always attracted attentions of mathematicians and physicists to find exact solutions. The Hirota direct method is considered an effective method to find multiple soliton solutions [7-11].

Complexiton solution was first introduced in Ref. [12] which means a combination of exponential waves and trigonometric waves and corresponds to complex eigenvalues of associated characteristic problems. Several methods for complexitons have been developed [13-16], among which the Hirota direct method was proved a promising one[17-21].

In this paper, we investigate the (2+1)-dimensional asymmetrical Nizhnik-Novikov-Veselov (ANNV) equation

\[ u_t + u_{xxx} + 3(u u_x)_{xx} = 0 \]  \hspace{1cm} (1)

The ANNV equation describes an incompressible fluid, which was first proposed by Boiti et al. [22]. Hu obtained the variable separation solutions by Darboux transformations of the Eq. (1) [23]. This equation has been extensively studied, for examples, Chen obtained a series of double periodic solutions through the rational elliptic function expansion method [24], Dai constructed separation solutions by the extended tanh-function method [25], Fan derived the quasi-periodic wave solutions and established the relations between the quasi-periodic wave solutions and soliton solutions [26], and Zhao et al. analyzed the lump soliton, mixed lump stripe and periodic lump solutions [27].

With the dependent variable transformation:

\[ u = u_0 + 2 \ln(f)_{xy}, \] \hspace{1cm} (2)

in which \( u_0 \) is a constant solution of Eq. (1). The Hirota bilinear form of ANNV is given as

\[ (D_x^3 D_y + D_y D_x + 3u_0 D_x^2) f \cdot f = 0, \] \hspace{1cm} (3)

where \( f \) is a real function of \( x, y \) and \( t \), \( D_x^3, D_x^2, D_y \) and \( D_x \) are Hirota bilinear operators.
Fundamental methods

We consider the bilinear equation:

\[ H(D_1, D_2, \ldots, D_M) f \cdot f = 0, \]  

(4)

where \( H \) is a polynomials with \( M \) variables, and it satisfies \( H(0) = 0 \) and \( H(-x) = H(x) \).

We introduce a complex wave variable defined as

\[ \xi_i = \xi_0 + \sum_{k=1}^{M} \zeta_k x_k, \]  

(5)

where \( \zeta_k \in \mathbb{C}, \ k = 0, 1, \ldots, M \).

Supposing the expansion \( f = 1 + \varepsilon f_1 + \varepsilon^2 f_2 + \ldots \), letting \( f_1 = e^{\hat{\xi}} \), substituting the expansion and \( f_1 \) into Eq. (4) and collecting all the coefficients about \( \varepsilon \), taking coefficient of \( \varepsilon \), we have

\[ H(\xi_1, \xi_2, \ldots, \xi_M) f_1 = 0. \]  

(6)

Consequently, if a complex function \( f = 1 + e^{\hat{\xi}} \) is a solution of Eq. (4), they are corresponding to the same dispersion relation:

\[ H(\xi_1, \xi_2, \ldots, \xi_M) = 0, \quad H(\bar{\xi}_1, \bar{\xi}_2, \ldots, \bar{\xi}_M) = 0. \]  

(7)

Through the Hirota method, we study \( N = 2 \), \( \xi_j = \xi_{j0} + \sum_{k=1}^{M} \zeta_{jk} x_k, \quad j = 1, 2 \), which corresponds to the dispersion relation and the complex function:

\[ f = 1 + e^{\hat{\xi}} + e^{\hat{\xi}_2} + \theta_{12} e^{\hat{\xi} + \hat{\xi}_2} + \theta_{12} e^{\hat{\xi} + \hat{\xi}_2} = 1 + e^{Re(\xi)} \cos(Im(\xi)) + \theta_{12} e^{2Re(\xi)}. \]  

(8)

If \( \xi_1, \xi_2 \) are reciprocal conjugation, Eq. (8) become

\[ f = 1 + e^{\hat{\xi}} + e^{\hat{\xi}_2} + \theta_{12} e^{2\hat{\xi}} = 1 + e^{Re(\xi)} \cos(Im(\xi)) + \theta_{12} e^{2Re(\xi)}. \]  

(9)

When \( N \geq 3 \), based on Ref. [17] the form of \( N \)-soliton solutions can be shown as

\[ \sum_{j=1}^{N} \exp(\sum_{k=1}^{j} \mu_j \xi_j + \sum_{j<k} \delta_{jk} \mu_j \xi_k), \]  

(10)

where \( \mu_j = 0 \) or 1 to \( j = 1, 2, \ldots, N \), \( e^{\beta_{jk}} = \theta_{jk} \) which equals to

\[ \theta_{jk} = -\frac{H(\xi_{k1} - \xi_{j1}, \ldots, \xi_{kM} - \xi_{jM})}{H(\xi_{k1} + \xi_{j1}, \ldots, \xi_{kM} + \xi_{jM})}, \quad 1 \leq j < k \leq N. \]  

(11)

Furthermore, \( H \) corresponds to the Hirota condition

\[ \sum_{j=1}^{N} H(\sum_{j=1}^{N} \sigma_j \xi_{j1}, \ldots, \sum_{j=1}^{N} \sigma_j \xi_{jM}) \prod_{k=1}^{N} H(\sigma_k \xi_{k1} - \sigma_j \xi_{j1}, \ldots, \sigma_k \xi_{kM} - \sigma_j \xi_{jM}) \sigma_k \sigma_j = 0. \]  

(12)

where \( \sigma_j = \pm 1, \ j, k = 1, 2, \ldots, N \). According to the theorem, we can derive multi-complexitons.

**Theorem [11]:** Let \( H \) be a real polynomials satisfying \( H(0) = 0, \ H(-x) = H(x) \) for
\( x \in \mathbb{R}^M, \ N \) be a positive integer. Assume that the complex wave variables \( \xi_j = \xi_{j0} + \sum_{k=1}^{M} \xi_{jk} x_k, \ j = 1, 3, ..., 2N - 1, \) satisfying the dispersion relation and the Hirota condition, and suppose \( \xi_{2j} = \overline{\xi_{2j-1}}, \ j = 1, 2, ..., N, \) then the function

\[
f = 1 + \sum_{m=1}^{2N} \sum_{n=1}^{2N} \exp(\sum_{j=1}^{2N} \mu_j \xi_j + \sum_{k,j} \delta_{jk} \mu_k \mu_j),
\]

(13)

presents a complexiton solutions to Eq. (4), where \( \mu_j \) is \( 0 \) or \( 1, \) for \( j = 1, 2, ..., 2N, \) and \( e^{\theta_k} = \theta_k, \ j, k = 1, 2, ..., 2N, \) determined by Eq. (11).

**Multi-complexiton solutions to ANNV**

In this part, we utilize \( 2N \)-soliton solutions to builds \( N \)-complexitons solutions by pairs of conjugate wave variables and Hirota direct method. One, two and \( N \)-complexiton to Eq. (3) are respectively obtained in following procedures.

Consider \( 2N = 2, \) according to Eq. (5), we take

\[
\xi' = k'x + 1'y + m't + \xi, \quad (14)
\]

where \( k', \ l', \ m' \) and \( \xi \) are parameters. Through pairs of conjugate wave variables, we suppose that \( \xi' = \xi_1^l = \xi_1 + I \xi_2, \ \xi_2 = \xi_2 - I \xi_2 \) (in which \( I = \sqrt{-1} \)), that means \( \xi_1 = \text{Re}(\xi'), \ \xi_2 = \text{Im}(\xi'). \)

If \( f = 1 + e^{\theta_k} \) is a complexiton solution of Eq. (3), if and only if the following dispersion relation is satisfied:

\[
H(k', l', m') = H(\overline{k'}, \overline{l'}, \overline{m'}) = 0. \quad (15)
\]

Making \( k = \text{Re}(k'), \ l = \text{Im}(l'), \ m = \text{Im}(m'), \) we have \( \xi_1 = k_1 x + l_1 t + m_1 + \xi_1, \ \xi_2 = k_2 x + l_2 t + m_2 + \xi_2. \) Then, Eq. (15) becomes the following form:

\[
m_1 = \frac{-k_1^3 l_2^2 - k_1^3 l_2^2 + 3k_1^2 k_2 l_2^2 + 3k_1 k_2^2 l_2^2 - 3k_1^2 l_2^2 u_0 - 6k_1 l_2 k_2 u_0 + 3k_2^2 l_2 u_0}{l_1^2 + l_2^2},
\]

(16)

\[
m_2 = \frac{k_2^3 l_2^2 + l_2^3 k_2^3 - 3k_1^2 k_2 l_2^2 - 3k_1 k_2^2 l_2^2 + 3k_1^2 l_2^2 u_0 - 6k_1 l_2 k_2 u_0 - 3k_2^2 l_2 u_0}{l_1^2 + l_2^2}.
\]

(16)

Through 2-soliton formulation, we can derive the 1-complexiton solution of Eq. (3)

\[
f = 1 + e^{\theta_1} + e^{\theta_2} + \theta_{12} e^{\theta_1 + \theta_2} = 1 + 2e^{\theta} \cos(\xi) + \theta_{12} e^{2\theta},
\]

(17)

in which

\[
\theta_{12} = \frac{H(k' - \overline{k'}, l' - \overline{l'}, m' - \overline{m'})}{H(k' + \overline{k'}, l' + \overline{l'}, m' + \overline{m'})} = \frac{H(2ik_2, 2il_2, 2im_2)}{H(2k_1, 2l_1, 2m_1)}
\]

\[
= \frac{l_1^3 l_2 + l_2^3 k_2 - u_0 l_2^2 k_2^2 + 2u_0 k_1 k_2 l_1 + l_1^3 l_2 + l_2^3 k_2 - u_0 k_2^2 l_1^2}{l_1^3 + l_2^3 k_1 + u_0 k_2 l_1 + l_1^3 + l_2^3 k_2^2 - 2u_0 k_1 k_2 l_1 + u_0 k_2^2 l_1^2}.
\]

(18)

The 1-complexiton solution of Eq. (1) is expressed as follows:
\[
 u = u_0 + \frac{2(4\theta_{12}k_1e^{2\xi_2} - 2 - k_1l_1 + k_2l_2 \cos \xi_2 + \sin \xi_2 + k_1l_2 + k_2l_1 \ e^\xi)}{1 + 2e^\xi \cos \xi_2 + \theta_{12}e^{2\xi}} \]

\[
 \frac{8 \ k_1e^\xi \ \cos \xi_2 - e^\xi k_2 \ \sin \xi_2 + \theta_{12}k_1e^{2\xi_2} \ \ln e^\xi \ \cos \xi_2 - e^\xi l_2 \ \sin \xi_2 + \theta_{12}l_1e^{2\xi_2}}{1 + 2e^\xi \ \cos \xi_2 + \theta_{12}e^{2\xi}} .
\]

where \( l_1, \ l_2, \ k_1, \ k_2, \ \xi_1, \ \xi_2, \ t \) and \( u_0 \) are arbitrary constants.

Next, in order to obtain 2-complexiton solution, we consider \( 2N = 4 \), assume the following function:

\[
 \xi'(x, y, t) = k'x + l'y + m't + \xi_\nu, \quad \xi''(x, y, t) = k''x + l''y + m''t + \xi_\nu ,
\]

where \( k', \ k'', \ l', \ l'', \ m', \ m'' \), \( \xi_\nu \) and \( \xi_\nu \) are constants.

Through pairs of conjugate wave variables, we take \( \xi' = \xi' = \xi_1 + l_1 \xi_4 , \xi_2 = \xi_2 - l_1 \xi_4 , \xi_3 = \xi_3 + l_2 \xi_4 , \xi_4 = \xi_3 - l_2 \xi_4 \), that is to say \( \xi_1 = Re(\xi') , \xi_2 = Im(\xi') , \xi_3 = Re(\xi'') , \xi_4 = Im(\xi'') \).

\[
 k_1 = Re(k'), \quad k_2 = Im(k'), \quad k_3 = Re(k''), \quad k_4 = Im(k''),
\]

\[
 l_1 = Re(l'), \quad l_2 = Im(l'), \quad l_3 = Re(l''), \quad l_4 = Im(l''),
\]

we have

\[
 \xi_1 = k_1x + l_1t + m_1 + \xi_1, \quad \xi_2 = k_2x + l_2t + m_2 + \xi_2,
\]

\[
 \xi_3 = k_3x + l_3t + m_3 + \xi_3, \quad \xi_4 = k_4x + l_4t + m_4 + \xi_4,
\]

then the dispersion relation can be converted into:

\[
 m_1 = -k_1^3l_1^2 - k_1l_1^2 + 3k_2k_2^2l_2^2 + 3k_1l_2^2k_1^2 - 3k_1^2l_1u_0 - 6k_1l_1k_2u_0 + 3k_2^2l_2u_0 ,
\]

\[
 m_2 = k_2^3l_2^2 + l_2k_2^3 - 3k_2^2k_3l_3^2 + 3k_2l_3^2k_2^2 - 3k_2^2l_2u_0 - 6k_2k_3l_3u_0 - 3l_2^2k_2^2u_0 ,
\]

\[
 m_3 = -k_3^3l_3^2 - k_3l_3^2 + 3k_4k_4^2l_4^2 + 3k_4l_4^2k_4^2 - 3k_4^2l_3u_0 - 6k_4k_4l_4u_0 + 3k_4^2l_4u_0 ,
\]

\[
 m_4 = k_4^3l_4^2 + l_4k_4^3 - 3k_4^2k_5l_5^2 + 3k_4l_5^2k_4^2 - 3k_4^2l_4u_0 - 6k_4k_5l_4u_0 - 3l_4^2k_4^2u_0 .
\]

Through 4-soliton formulation, we can derive the 2-complexiton solution of Eq. (3)

\[
 f = 1 + e^\xi + e^{2\xi} + e^{3\xi} + e^{4\xi} + \theta_{12}e^{\xi+\xi'} + \theta_{13}e^{\xi+\xi'} + \theta_{14}e^{\xi+\xi'} + \theta_{23}e^{\xi+\xi'} + \theta_{24}e^{\xi+\xi'} + \theta_{34}e^{\xi+\xi'} + \theta_{123}e^{\xi+\xi+\xi'} + \theta_{134}e^{\xi+\xi+\xi'} + \theta_{234}e^{\xi+\xi+\xi'} + \theta_{1234}e^{\xi+\xi+\xi+\xi'}
\]

\[
 = 1 + 2e^\xi \cos(\xi_2) + 2e^{2\xi} \cos(\xi_2) + \theta_{12}e^{2\xi} + \theta_{13}e^{2\xi} + \theta_{23}e^{2\xi} + 2Re(\theta_{13}e^{\xi+\xi'+\xi'+\xi'}) + \theta_{14}e^{\xi+\xi+\xi+\xi'} + \theta_{124}e^{\xi+\xi+\xi+\xi'} + \theta_{134}e^{\xi+\xi+\xi+\xi'} + \theta_{234}e^{\xi+\xi+\xi+\xi'} + \theta_{1234}e^{\xi+\xi+\xi+\xi'}
\]

where
\[ \theta_{12} = \frac{H(k' - \vec{k}, l' - \vec{l}, m' - \vec{m})}{H(k' + \vec{k}, l' + \vec{l}, m' + \vec{m})}, \quad \theta_{13} = -\frac{H(k' - k'', l' - l'', m' - \vec{m})}{H(k' + k'', l' + l'', m' + \vec{m})}, \]
\[ \theta_{14} = -\frac{H(k' - k'', l' - \vec{l}, m' - \vec{m})}{H(k' + k'', l' + \vec{l}, m' + \vec{m})}, \quad \theta_{23} = -\frac{H(k'' - \vec{k}, \vec{l} - l'', \vec{m} - m'')}{H(\vec{l} + k'', \vec{l} + l'', \vec{m} + m'')}, \]
\[ \theta_{24} = -\frac{H(k'' - \vec{k}, \vec{l} - \vec{l}, \vec{m} - \vec{m})}{H(\vec{l} + k'', \vec{l} + \vec{l}, \vec{m} + \vec{m})}, \quad \theta_{34} = -\frac{H(k'' - k, \vec{l} - l, m - \vec{m})}{H(k'' + k, \vec{l} + l, m + \vec{m})}, \]
\[ \text{and } \theta_{123} = \theta_{13} \theta_{23} \theta_{24}, \quad \theta_{134} = \theta_{13} \theta_{4} \theta_{24}, \quad \theta_{234} = \theta_{23} \theta_{34} \theta_{4}, \quad \theta_{1234} = \theta_{12} \theta_{23} \theta_{34} \theta_{4}. \]

Substituting Eq. (15) into Eq. (2), we can obtain 2-complexiton solution to Eq. (1).

Finally, we construct \( N \)-complexiton solutions by \( 2N \)-soliton solutions.

\[ \xi'(x, y, t) = k_x x + l_y y + m_t t + \xi, \quad \ldots, \quad \xi^N(x, y, t) = k^N x + l^N y + m^N t + \xi^N. \]

where \( k', k^N, l', l^N, m', m^N, \xi \) and \( \xi^N \) are constants. Through the same processing procedure, let

\[ \xi_1 = k_x x + l_y y + m_t t + \xi, \quad \xi_2 = k_x x + l_y y + m_t t + \xi, \]
\[ \ldots, \quad \ldots, \]
\[ \xi_{2N-1} = k_x x + l_y y + m_t t + \xi_{2N-1}, \quad \xi_{2N} = k_x x + l_y y + m_t t + \xi_{2N}, \]
then the dispersion relation can be converted into as following form:

\[ m_1 = \frac{-k_1 i_1^2 - k_1 l_1^3 + 3k_1 k_2 l_1^2 + 3k_2 l_1^2 k_2^2 - 3k_1 l_1^2 u_0 - 6k_1 l_1 k_2 u_0 + 3k_2^2 l_1 u_0}{l_1^2 + l_2^2}, \]
\[ m_2 = \frac{k_2 i_1^2 + l_1^2 k_2^3 - 3k_1 k_2 l_1^2 - 3k_1^2 l_1^2 k_2 + 3k_1 l_1^2 u_0 - 6k_1 k_2 l_1 u_0 - 3l_2^2 u_0}{l_1^2 + l_2^2}, \]
\[ \ldots, \]
\[ m_{2l-1} = \frac{-k_2 i_2 l_2^2 + k_2 i_2 l_2^2 + 3k_2 l_2^2 k_2^2 + 3k_2 l_2^2 k_2^2 - 3k_2 l_2^2 l_2^2 u_0}{(l_2^2 + l_2^2)}, \]
\[ m_{2l} = \frac{(k_2 i_2 l_2^2 + l_2^2 k_2^3 + 3k_2 l_2^2 k_2^2 + 3k_2 l_2^2 k_2^2 - 3k_2 l_2^2 l_2^2 u_0)}{(l_2^2 + l_2^2)}, \]
\[ -6k_2 l_2^2 k_2 l_2 u_0 - 3l_2^2 k_2^2 u_0 (l_2^2 + l_2^2), \]

with \( 2 \leq l \leq N \).

Based on a \( 2N \)-soliton formulation, we can derive the \( N \)-complexiton solution of Eq. (3),

\[ f = 1 + \sum_{\text{all}}^{2N} e^{\xi} + \sum_{k=2}^{N} \sum_{k=2}^{N} \sum_{k=2}^{N} \theta_{k l} e^{\xi} + \xi, \]

Similarly, substituting Eq. (19) into Eq. (2), we can obtain \( N \)-complexiton solution to Eq. (1).

**Conclusion**

In this paper, we construct complexiton solutions of ANNV equation by applying the Hirota direct method and the pairs of conjugate wave variables. The key is to utilize the bilinear ANNV equation. Using pairs of conjugate wave variables in \( 2N \)-soliton solutions, we obtain a series of multi-complexiton solutions. It's recommended that this method can be further used to find multi-complexiton solutions of nonlinear equations with fractal derivatives[28-35].

**Acknowledgements:**

This work was supported by the Fundamental Research Funds for the Central Universities of
Central South University.

References


[34] Shen, Y., He, J.H. Variational principle for a generalized KdV equation in a fractal space, Fractals, 2020, https://doi.org/10.1142/S0218348X20500693

Paper submitted: March 1, 2020
Paper revised: June 15, 2020
Paper accepted: June 16, 2020