VARIATIONAL THEORY FOR (2+1)-DIMENSIONAL FRACTIONAL DISPERSIVE LONG WAVE EQUATIONS

by

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This paper extends the (2+1)-dimensional Eckhaus-type dispersive long wave equations in continuous medium to their fractional partner, which is a model of non-linear waves in fractal porous media. The derivation is shown briefly using He's fractional derivative. Using the semi-inverse method, the variational principles are established for the fractional system, which up to now are not discovered. The obtained fractal variational principles are proved correct by minimizing the functional with the calculus of variations, and might find potential applications in numerical modeling.

Key words: *variational principle, (2+1)-dimensional fractional equations, semi-inverse method, fractal dimension*

Introduction

Partial differential equations are usually used to model different phenomena in non-linear sciences, ranging from physics to mechanics, plasma, materials, biology, chemistry, ocean, meteorology, and so on [1, 2]. Investigating solutions to such non-linear PDE is an important research area, and numerous mathematical techniques have been developed to explore the approximate or exact solutions [3-11]. Because variational principles are theoretical bases for many methods to solve or analyze the non-linear problem, such as Ritz technique [10], variational iteration method [11-15] and variational approximation method [16-18], it is a very important but difficult task to seek explicit variational formulations for non-linear PDE. The semi-inverse method [19-27] is widely used to establish variational principles from the governing equations directly, which was firstly proposed by the famous Chinese mathematician, He [19-25, 28-30]. Because it is not necessary to introduce Lagrange multipliers, the Lagrange crisis frequently encountered can be overcome [19-27]. The semi-inverse method has been used to search for variational formulations in plasma, mechanics, materials, fluid dynamics, solitary theory and so on [19-32]. It is Wang *et al*. [26] who established a variational formulation in a fractal space for wave travelling. Wang and He [27] extended Wang *et al.*'s variational principle to a fractal time/space. In this paper, we will apply the He's semi-inverse method [19-27] to establish some variational principles for (2+1)-dimensional Eckhaus-type dispersive long wave equations with fractal derivatives.

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At first, we consider the following (2+1)-dimensional Eckhaus-type dispersive long-wave equation [33, 34] in continuum mechanics.

$$\begin{cases} \dfrac{\partial U}{\partial T}+\dfrac{\partial H}{\partial X}+U\dfrac{\partial U}{\partial X}=0 \\ \dfrac{\partial^2 H}{\partial T\partial X}+\dfrac{\partial^2 (UH+U+U_{XX})}{\partial X^2}+\dfrac{\partial^2 U}{\partial Y^2}=0 \end{cases} \tag{1}$$

where $U=U(X, Y, T)$ and $H=H(X, Y, T)$ denotes the wave horizontal velocity and wave height, respectively, and both of them are dimensionless variables [33, 34]. Eckhaus firstly obtained the coupled non-linear eq. (1) from the basic equations of fluid dynamics by some proper approximations. It has been proved that there are only finite dimensional symmetry groups in the system (1), which does not own Painleve property [33]. There are lots of solitary wave solutions, rational solutions, triangular solutions, hyperbolic function solutions, Jacobi elliptic function solutions contained in the eq. (1) [34].

The fractional partner

Usually, we can view physical motions and phenomena from two distinctly different scales [35, 36]. One is the large-scale, where Newton's calculus is approximately valid and the traditional mechanics can be roughly applied. The other scale is a much smaller one, a scale of molecule size. Under such a small scale, the media becomes discontinuous, and the fractal calculus [29-31, 37-43] has to be adopted. Equation (1) is a very useful model to describe many kinds of waves in continuous media, however an unsmooth boundary will greatly affect the properties of non-linear waves. Therefore, the smooth space (X, Y, T) should be replaced by a fractal space (X^β, Y^γ, T^α), where, β, γ, and α are, respectively, fractal dimensions in space and time. In the fractal space, eq. (1) becomes:

$$\begin{cases} \dfrac{\partial U}{\partial T^\alpha}+\dfrac{\partial H}{\partial X^\beta}+U\dfrac{\partial U}{\partial X^\beta}=0 \\ \dfrac{\partial^2 H}{\partial T^\alpha\partial X^\beta}+\dfrac{\partial^2 (UH+U)}{\partial X^{2\beta}}+\dfrac{\partial^4 U}{\partial X^{4\beta}}+\dfrac{\partial^2 U}{\partial Y^{2\gamma}}=0 \end{cases} \tag{2}$$

where the He's fractal derivatives are defined [37-43]:

$$\frac{\partial U}{\partial T^\alpha}(X,Y,T_0)=\Gamma(1+\alpha)\lim_{\substack{T-T_0\to\Delta T\\ \Delta T\neq 0}}\frac{U(X,Y,T)-U(X,Y,T_0)}{(T-T_0)^\alpha} \tag{3}$$

$$\frac{\partial U}{\partial X^\beta}(X_0,Y,T)=\Gamma(1+\beta)\lim_{\substack{X-X_0\to\Delta X\\ \Delta X\neq 0}}\frac{U(X,Y,T)-U(X_0,Y,T)}{(X-X_0)^\beta} \tag{4}$$

$$\frac{\partial U}{\partial Y^\gamma}(X,Y_0,T)=\Gamma(1+\gamma)\lim_{\substack{Y-Y_0\to\Delta Y\\ \Delta Y\neq 0}}\frac{U(X,Y,T)-U(X,Y_0,T)}{(Y-Y_0)^\gamma} \tag{5}$$

The similar definitions of eqs. (3)-(5) can also be given for $H(X, Y, T)$ in eq. (2). For the fractal derivative, we have the following chain rules:

$$\frac{\partial^2}{\partial T^\alpha\partial X^\beta}=\frac{\partial}{\partial T^\alpha}\frac{\partial}{\partial X^\beta} \tag{6}$$

$$\frac{\partial^2}{\partial X^{2\beta}}=\frac{\partial}{\partial X^\beta}\frac{\partial}{\partial X^\beta} \tag{7}$$

$$\frac{\partial^4}{\partial X^{4\beta}} = \frac{\partial}{\partial X^{\beta}}\frac{\partial}{\partial X^{\beta}}\frac{\partial}{\partial X^{\beta}}\frac{\partial}{\partial X^{\beta}} \tag{8}$$

$$\frac{\partial^2}{\partial Y^{2\gamma}} = \frac{\partial}{\partial Y^{\gamma}}\frac{\partial}{\partial Y^{\gamma}} \tag{9}$$

In the definitions given in eqs. (3)-(5), ΔX, ΔY, and ΔT are, respectively, the smallest spatial and temporal scale for watching the physical phenomena. For example, when the spatial scale is larger than ΔX, the boundary is considered as a smooth one, and traditional continuum mechanics works. However, when we observe the wave on the scale of ΔX, the boundary is discontinuous, and it is considered as a fractal curve [37-43]. In the fractal space, all variables depend upon the scales used for observation and the fractal dimensions of the discontinuous boundary. The fractal derivatives are widely used in applications for discontinuous media [29-31, 37-43].

Variational principle

The semi-inverse method [19] is widely used to establish variational principles directly from the governing equations. It can be used to search for variational formulations in plasma, economics and Lane-Emden equation [19-27]. It is Wang *et al.* [26] who established a variational formulation in a fractal space for wave travelling. Wang and He extended Wang *et al.*'s variational principle to a fractal time/space [27]. In this paper, we will apply the He's semi-inverse method [19-25] to establish needed variational principles for (2+1)-dimensional Eckhaus-type dispersive long wave equation with fractal derivatives eq. (2). In a fractal space, the physical laws should also be followed. According to the basic properties given previously, the following time and space scale transforms [35-37] can be introduced:

$$t = T^{\alpha} \tag{10}$$

$$x = X^{\beta} \tag{11}$$

$$y = Y^{\gamma} \tag{12}$$

Equation (2) becomes:

$$\begin{cases} U_t + H_x + UU_x = 0 \\ H_{tx} + (U + UH + U_{xx})_{xx} + U_{yy} = 0 \end{cases} \tag{13}$$

In order to use the semi-inverse method [19-27] to establish a variational formulation for Eq. (13), we can rewrite Eq. (13) in the conservative forms:

$$\begin{cases} U_t + \left(H + \dfrac{U^2}{2}\right)_x = 0 \\ [H_t + (UH + U + U_{xx})_x]_x + U_{yy} = 0 \end{cases} \tag{14}$$

According to the second equation in (14), two potential functions Φ and Ψ can be introduced:

$$\begin{cases} \Phi_x = H \\ \Phi_t = -(UH + U + U_{xx}) - \Psi_{yy} \\ \Psi_{xx} = U \end{cases} \tag{15}$$

so that the second equation in (14) is automatically satisfied.

The objective in this paper is to establish variational principles for the (2+1)-dimensional Eckhaus-type dispersive long wave equations (2) by He's semi-inverse method [19-25]. At first, a trial-functional can be constructed:

$$J(U,H,\Phi,\Psi)=\iiint L\mathrm{d}x\mathrm{d}y\mathrm{d}t \tag{16}$$

where L is the trial-Lagrangian. By the semi-inverse method [19-25], we assume that the trial-Lagrange function can be written:

$$L=U\Phi_t+\left(H+\frac{U^2}{2}\right)\Phi_x+F(U,H,\Psi) \tag{17}$$

and F is an unknown function of U, H, and Ψ and their derivatives. There exist alternative ways to the construction of the trial-functional, see [19-25]. The advantage of the aforementioned trial-Lagrange lies on the fact that the stationary condition with respect to Φ results in the following Euler-Lagrange equation:

$$\frac{\partial L}{\partial \Phi}-\frac{\partial}{\partial x}\left(\frac{\partial L}{\partial \Phi_x}\right)-\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \Phi_t}\right)=0 \tag{18}$$

or

$$-U_t-\left(H+\frac{U^2}{2}\right)_x=0$$

which is identical to the first equation in eq. (14).

Now calculating the stationary condition of the aforementioned trial-Lagrange L with respect to U and H, respectively:

$$\frac{\partial L}{\partial U}+\frac{\delta F}{\delta U}=0 \tag{19}$$

$$\frac{\partial L}{\partial H}+\frac{\delta F}{\delta H}=0 \tag{20}$$

where $\delta F/\delta U$ and $\delta F/\delta H$ is called the variational derivative [19] with respect to U and H. In view of eqs. (17), (19), and (20) can be re-written:

$$\Phi_t+U\Phi_x+\frac{\delta F}{\delta U}=0 \tag{21}$$

$$\Phi_x+\frac{\delta F}{\delta H}=0 \tag{22}$$

In view of the first and second equations in eq. (15), we have:

$$\frac{\delta F}{\delta U}=U+U_{xx}+\Psi_{yy} \tag{23}$$

$$\frac{\delta F}{\delta H}=-H \tag{24}$$

From the previous eqs. (23) and (24), F can be identified easily:

$$F=\frac{U^2-H^2-U_x^2}{2}+U\Psi_{yy}+F_1 \tag{25}$$

and

$$F=\frac{U^2-H^2+UU_{xx}}{2}+U\Psi_{yy}+F_1 \tag{26}$$

where F_1 is free of U, H, and Φ and their derivatives. Substituting (25) or (26) into (17), the trial-Lagrangian function can be updated:

$$L=U\Phi_t+\left(\frac{H+U^2}{2}\right)\Phi_x+\left(\frac{U^2-H^2-U_x^2}{2}\right)+U\Psi_{yy}+F_1(\Psi) \tag{27}$$

$$L=U\Phi_t+\left(\frac{H+U^2}{2}\right)\Phi_x+\left(\frac{U^2-H^2+UU_{xx}}{2}\right)+U\Psi_{yy}+F_1(\Psi) \tag{28}$$

and F_1 is an unknown function only of Ψ and/or its derivatives. Now calculating the stationary condition of the aforementioned trial-Lagrange L with respect to Ψ only:

$$U_{yy}+\frac{\delta F_1}{\delta\Psi}=0 \tag{29}$$

In view of the third equation in eq. (15):

$$\frac{\delta F_1}{\delta\Psi}=-U_{yy}=-\Psi_{xxyy} \tag{30}$$

From the previous equation, F_1 can be identified as two various forms:

$$F_1=\frac{-\Psi_{xy}^2}{2} \tag{31}$$

or

$$F_1=\frac{-\Psi_{xx}\Psi_{yy}}{2} \tag{32}$$

Substituting (31) and (32) into (27) or (28) leads to four different trial-Lagrange functionals：

$$L=U\Phi_t+\left(\frac{H+U^2}{2}\right)\Phi_x+\left(\frac{U^2-H^2-U_x^2}{2}\right)+U\Psi_{yy}-\frac{\Psi_{xy}^2}{2} \tag{33}$$

and

$$L=U\Phi_t+\left(\frac{H+U^2}{2}\right)\Phi_x+\left(\frac{U^2-H^2-U_x^2}{2}\right)+U\Psi_{yy}-\frac{\Psi_{xx}\Psi_{yy}}{2} \tag{34}$$

and

$$L=U\Phi_t+\left(\frac{H+U^2}{2}\right)\Phi_x+\left(\frac{U^2-H^2+UU_{xx}}{2}\right)+U\Psi_{yy}-\frac{\Psi_{xy}^2}{2} \tag{35}$$

and

$$L=U\Phi_t+\left(\frac{H+U^2}{2}\right)\Phi_x+\left(\frac{U^2-H^2+UU_{xx}}{2}\right)+U\Psi_{yy}-\frac{\Psi_{xx}\Psi_{yy}}{2} \tag{36}$$

Finally, we obtain the variational formulations for the (2+1)-dimensional Eckhaus-type dispersive long-wave equation with fractal derivatives eq. (14):

$$J(U,H,\Phi,\Psi)=\iiint\left[U\Phi_t+\left(\frac{H+U^2}{2}\right)\Phi_x+\left(\frac{U^2-H^2-U_x^2}{2}\right)+U\Psi_{yy}-\frac{\Psi_{xy}^2}{2}\right]\Bigg\}\mathrm{d}x\mathrm{d}y\mathrm{d}t \tag{37}$$

and

$$J(U,H,\Phi,\Psi)=\iiint\left[U\Phi_t+\left(\frac{H+U^2}{2}\right)\Phi_x+\left(\frac{U^2-H^2-U_x^2}{2}\right)+U\Psi_{yy}-\frac{\Psi_{xx}\Psi_{yy}}{2}\right]\Bigg\}\mathrm{d}x\mathrm{d}y\mathrm{d}t \tag{38}$$

and

$$J(U,H,\Phi,\Psi)=\iiint\left[U\Phi_t+\left(\frac{H+U^2}{2}\right)\Phi_x+\left(\frac{U^2-H^2+UU_{xx}}{2}\right)+U\Psi_{yy}-\frac{\Psi_{xy}^2}{2}\right]\Bigg\}\mathrm{d}x\mathrm{d}y\mathrm{d}t \tag{39}$$

and

$$J(U,H,\Phi,\Psi)=\iiint\left[U\Phi_t+\left(\frac{H+U^2}{2}\right)\Phi_x+\left(\frac{U^2-H^2+UU_{xx}}{2}\right)+U\Psi_{yy}-\frac{\Psi_{xx}\Psi_{yy}}{2}\right]\Bigg\}\mathrm{d}x\mathrm{d}y\mathrm{d}t \tag{40}$$

Proof. Making any of the aforementioned functionals, eqs. (37)-(40), stationary with respect to Φ, Ψ, U, and H, respectively, we obtain the following Euler-Lagrange equations:

$$\delta\Phi: -U_t-\left(\frac{H+U^2}{2}\right)_x=0 \tag{41}$$

$$\delta\Psi: U_{yy}+\Psi_{xxyy}=0 \tag{42}$$

$$\delta U: \Phi_t+U\Phi_x+U+U_{xx}+\Psi_{yy}=0 \tag{43}$$

$$\delta H: \Phi_x-H=0 \tag{44}$$

in which $\delta\Phi$, $\delta\Psi$, δU, and δH is the first-order variation for Φ, Ψ, U, and H, respectively. Obviously, the eq. (41) is equivalent to the first equation in eq. (14). From the eq. (42), we have $\Psi_{xx}=-U$, which is identical to the third equation in eq. (15). In view of the eq. (44), we have $\Phi x = H$, which is the first equation in eq. (15). Submitting $\Phi_x = H$ into eq. (43) yields, $\Phi_t = -UH - U - U_{xx} - \Psi_{yy}$, which is completely equivalent to the second equation in eq. (15). Successfully, we prove the variational principles (37)-(40) correct. The variational formulation can also be written in the fractal space (X^β, Y^γ, T^α):

$$\begin{aligned}J(U,H,\Phi,\Psi)=\iiint\Bigg\{&U\frac{\partial\Phi}{\partial T^\alpha}+\left(H+\frac{U^2}{2}\right)\frac{\partial\Phi}{\partial X^\beta}+\frac{1}{2}\left[U^2-H^2-\left(\frac{\partial U}{\partial X^\beta}\right)^2\right]+\\&+U\frac{\partial^2\Psi}{\partial Y^{2\gamma}}-\frac{1}{2}\left(\frac{\partial^2\Psi}{\partial X^\beta\partial Y^\gamma}\right)^2\Bigg\}\mathrm{d}X^\beta\mathrm{d}Y^\gamma\mathrm{d}T^\alpha\end{aligned} \tag{45}$$

and

$$\begin{aligned}J(U,H,\Phi,\Psi)=\iiint\Bigg\{&U\frac{\partial\Phi}{\partial T^\alpha}+\left(H+\frac{U^2}{2}\right)\frac{\partial\Phi}{\partial X^\beta}+\frac{1}{2}\left[U^2-H^2-\left(\frac{\partial U}{\partial X^\beta}\right)^2\right]+\\&+U\frac{\partial^2\Psi}{\partial Y^{2\gamma}}-\frac{1}{2}\frac{\partial^2\Psi}{\partial X^{2\beta}}\frac{\partial^2\Psi}{\partial Y^{2\gamma}}\Bigg\}\mathrm{d}X^\beta\mathrm{d}Y^\gamma\mathrm{d}T^\alpha\end{aligned} \tag{46}$$

and

$$J(U,H,\Phi,\Psi)=\iiint\{U\frac{\partial\Phi}{\partial T^{\alpha}}+\left(H+\frac{U^{2}}{2}\right)\frac{\partial\Phi}{\partial X^{\beta}}+\frac{1}{2}\left[U^{2}-H^{2}+U\frac{\partial^{2}U}{\partial X^{2\beta}}\right]+ +U\frac{\partial^{2}\Psi}{\partial Y^{2\gamma}}-\frac{1}{2}\left(\frac{\partial^{2}\Psi}{\partial X^{\beta}\partial Y^{\gamma}}\right)^{2}\right\}\mathrm{d}X^{\beta}\mathrm{d}Y^{\gamma}\mathrm{d}T^{\alpha} \tag{47}$$

and

$$J(U,H,\Phi,\Psi)=\iiint\left\{U\frac{\partial\Phi}{\partial T^{\alpha}}+\left(H+\frac{U^{2}}{2}\right)\frac{\partial\Phi}{\partial X^{\beta}}+\frac{1}{2}\left[U^{2}-H^{2}+U\frac{\partial^{2}U}{\partial X^{2\beta}}\right]+ +U\frac{\partial^{2}\Psi}{\partial Y^{2\gamma}}-\frac{1}{2}\frac{\partial^{2}\Psi}{\partial X^{2\beta}}\frac{\partial^{2}\Psi}{\partial Y^{2\gamma}}\right\}\mathrm{d}X^{\beta}\mathrm{d}Y^{\gamma}\mathrm{d}T^{\alpha} \tag{48}$$

The aforementioned fractal variational integral formulations provide conservation laws in an energy form for the considered system, and can help us to understand deeply the physical relations and interactions among variables U, H, Φ, and Ψ in the fractal space. They also provide hints for designing numerical algorithms and attaining possible solution structures for analytical methods for the discussed problem.

Discussion and conclusion

In this paper, different groups of variational principles have been successfully constructed for the (2+1)-dimensional Eckhaus-type dispersive long wave equation with fractal derivatives, by the semi-inverse method and designing skillfully trial-Lagrange functionals. Then, the obtained fractal variational principles have proved correct by minimizing the corresponding functionals. From the results of analysis, it is concluded that the variational principle for the fractional non-linear equations studied in this paper is not unique, but has many different integral formulations. According to the obtained fractal variational principles, on one hand, we can study possible solution structures for solitary waves, and understand deeply the physical relations and interactions between horizontal velocity and wave height. On the other hand, according to the obtained variational principle, we can study the laws of motion for related solitary waves and the discussed problems can also be solved numerically by variational methods. The procedure also reveals that the semi-inverse method is straightforward and powerful.

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