STOCHASTIC BIFURCATION ANALYSIS OF A BISTABLE DUFFING OSCILLATOR WITH FRACTIONAL DAMPING UNDER MULTIPLICATIVE NOISE EXCITATION

by

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The stochastic P-bifurcation behavior of bi-stability in a Duffing oscillator with fractional damping under multiplicative noise excitation is investigated. Firstly, in order to consider the influence of Duffing term, the non-linear stiffness can be equivalent to a linear stiffness which is a function of the system amplitude, and then, using the principle of minimal mean square error, the fractional derivative term can be equivalent to a linear combination of damping and restoring forces, thus, the original system is simplified to an equivalent integer order Duffing system. Secondly, the system amplitude's stationary probability density function is obtained by stochastic averaging, and then according to the singularity theory, the critical parametric conditions for the system amplitude's stochastic P-bifurcation are found. Finally, the types of the system's stationary probability density function curves of amplitude are qualitatively analyzed by choosing the corresponding parameters in each area divided by the transition set curves. The consistency between the analytical results and the numerical results obtained from Monte-Carlo simulation verifies the theoretical analysis, and the method used in this paper can directly guide the design of the fractional order controller to adjust the behaviors of the system.

Key words: *stochastic P-bifurcation, fractional damping, transition set, multiplicative noise excitation, Monte-Carlo simulation*

Introduction

Fractional calculus is a generalization of integer-order calculus, which has a history of more than 300 years. Integer-order derivative can not express the memory characteristics of the viscoelastic substances, while the definition of fractional derivative contains convolution, which can express a memory effect and can show a cumulative effect over time. Therefore, the fractional derivative is a more suitable mathematical tool to describing memory characteristics [1-3] and has become a powerful mathematical tool for the study in the research fields such as

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thermal science, anomalous diffusion, non-Newtonian fluid mechanics, viscoelastic mechanics, and soft matter physics.

Comparing with the integer-order calculus, the fractional derivative can describe various reaction processes more accurately [4-7], thus it is necessary and significant to study the mechanical characteristics and the fractional order parametric influences on systems.

Recently, many scholars have studied the dynamic behavior of non-linear multi-stable systems under different noise excitations and achieved fruitful results. Liu *et al.* [8] studied the response of a strongly non-linear vibro-impact system with Coulomb friction ex-cited by real noise, and analyzed the *P*-bifurcation by a qualitative change of the friction amplitude and the restitution coefficient on the stationary probability distribution. He and Ain [9] elucidated the basic properties of fractal calculus and revealed the relationship between the fractal calculus and traditional calculus using the two-scale transform. Some researchers [10-12] studied the van der Pol-Duffing oscillators under Levy noise, colored noise, combined harmonic, and random noise, respectively. The stochastic *P*-bifurcation behaviors of the noise oscillators are discussed by analyzing changes in the system's stationary probability density function (PDF), and the analytical results of the bimodal stationary PDF are obtained, showing that the system parameters and noise intensity can each induce stochastic *P*-bifurcation of the systems. Chen and Zu [13] studied the response of a Duffing system with fractional damping under the combined white noise and harmonic excitations, and showed that variation of the fractional derivative's order can arouse the system's stochastic *P*-bifurcation. Li *et al.* [14] studied the bi-stable stochastic *P*-bifurcation behavior of a van der Pol-Duffing system with the fractional derivative under additive and multiplicative colored noise excitations and found that changes in the linear damping coefficient, the fractional derivative's order and the noise intensity can each lead to stochastic *P*-bifurcation in the system. Liu *et al.* [15] investigated a Duffing oscillator system with fractional damping under combined harmonic and Poisson white noise parametric excitation, and then the asymptotic Lyapunov stability with probability one of the original system was analyzed based on the largest Lyapunov exponent. Li *et al.* [16] studied the stochastic *P*-bifurcation problem for an axially moving bistable viscoelastic beam with fractional derivatives of high order non-linear terms under colored noise excitation and obtained the stationary PDF of the system amplitude by the stochastic averaging method and the singularity theory. Chen *et al.* [17] proposed a stochastic averaging technique which can be used to study the randomly ex-cited strongly non-linear system with delayed feedback fractional order proportional derivative controller, and obtained the stationary PDF of the system.

Due to complexity of the fractional derivative, the parametric vibration characteristics of the fractional system can only be analyzed qualitatively, while the critical conditions of the parametric influences can not be obtained. In practice, the critical conditions of the parametric influences play a vital role for the analysis and design of the fractional order systems. Additionally, the stochastic *P*-bifurcation for the bistable Duffing system with the fractional damping has not been reported in the open literature. In this paper, taking a Duffing system with a fractional damping ex-cited by multiplicative Gaussian white noise excitation as the example, non-linear vibration of this kind of fractional order systems are studied through the fractional derivative. The transition set curves and critical parametric conditions for the system's stochastic *P*-bifurcation are obtained by the singularity method. The types of the system's stationary PDF curves in each area of the parametric plane are analyzed. We also compare the numerical results from Monte-Carlo simulation with analytical solutions obtained by stochastic averaging. The comparison shows that the numerical results are in good agreement with the analytical solutions, verifying our theoretical analysis.

Derivation of the equivalent system

The initial condition of the Riemann-Liouville derivative has no physical meaning, while the initial condition of the system described by the Caputo derivative has not only clear physical meaning but also forms the same initial condition with the integer-order differential equation. Therefore, in this paper we adopt the Caputo fractional derivative:

$$ {}_{a}^{C}D^{p}[x(t)] = \frac{1}{\Gamma(m-p)} \int_{a}^{t} \frac{x^{(m)}(u)}{(t-u)^{1+p-m}} \mathrm{d}u \tag{1} $$

where $m - 1 < p \le m$, $m \in N$, $t \in [a, b]$, $x^{(m)}(t)$ is the m-order derivative of $x(t)$ and $\Gamma(m)$ – the Gamma function.

For a given physical system, the initial moment of oscillators is $t = 0$ and the Caputo derivative is usually expressed:

$$ {}_{0}^{C}D^{p}[x(t)] = \frac{1}{\Gamma(m-p)} \int_{0}^{t} \frac{x^{(m)}(u)}{(t-u)^{1+p-m}} \mathrm{d}u \tag{2} $$

where $m - 1 < p \le m$, $m \in N$.

In this paper, we study the bistable Duffing system with the fractional damping ex-cit-ed by multiplicative Gaussian white noise excitation:

$$ \ddot{x}(t) + c\dot{x}(t) + w^2 x(t) + K_1\, {}_{0}^{C}D^{p}[x(t)] + \alpha x^3(t) = x(t)\xi(t) \tag{3} $$

where w is the system's natural frequency, ${}_{0}^{C}D^{p}[x(t)]$ – the p $(0 \le p \le 1)$ order Caputo derivative of $x(t)$, which is defined by eq. (2), and $\xi(t)$ – the Gaussian white noise excitation:

$$ E[\xi(t)],\ \ E[\xi(t)\xi(t-\tau)] = 2D\delta(\tau) \tag{4} $$

where D is the intensity of Gaussian white noises $\xi(t)$ and $\delta(\tau)$ – the Dirac function.

The fractional derivative has the contributions of damping and restoring forces [18], hence, we introduce the equivalent system:

$$ \ddot{x}(t) + c\dot{x}(t) + w^2 x(t) + K_1[C(p)\dot{x}(t) + K(p)x(t)] + \alpha x^3(t) = x(t)\xi(t) \tag{5} $$

where $C(p)$ and $K(p)$ are the coefficients of the equivalent damping and restoring forces of the fractional derivative ${}_{0}^{C}D^{p}[x(t)]$, respectively.

Applying the equivalent methods mentioned in the references [15, 19, 20], we get the ultimate forms of $C(p)$ and $K(p)$:

$$ C(p) = w^{p-1} \sin\left(\frac{p\pi}{2}\right),\ \ K(p) = w^{p} \cos\left(\frac{p\pi}{2}\right) \tag{6} $$

Therefore, the equivalent Duffing oscillator associated with system eq. (5) can be written:

$$ \ddot{x}(t) + \gamma\dot{x}(t) + w_0^2 x(t) + \alpha x^3(t) = x(t)\xi(t) \tag{7} $$

where

$$ \begin{aligned} \gamma &= c + K_1 w^{p-1} \sin\left(\frac{p\pi}{2}\right) \\ w_0^2 &= w^2 + K_1 w^{p} \cos\left(\frac{p\pi}{2}\right) \end{aligned} \tag{8} $$

Stationary PDF of the system amplitude

Linearizing the cubic stiffness terms and taking the undetermined damping and stiffness coefficients as functions of the system amplitude, the vibrational structure of the equivalent system can be written [21]:

$$\ddot{x}(t)+\gamma\dot{x}(t)+w_0^2x(t)+\alpha[C(a)\dot{x}(t)+K(a)x(t)]=x(t)\xi(t) \tag{9}$$

To determine the coefficients $C(a)$ and $K(a)$ in eq. (9), the error between system eq. (7) and system in eq. (9) is defined:

$$e=\alpha x^3(t)-\alpha[C(a)\dot{x}(t)+K(a)x(t)] \tag{10}$$

Assuming that the system eq. (9) has the solution:

$$\begin{gathered} x(t)=a(t)\cos\varphi(t) \\ \varphi(t)=w_0t+\theta(t) \end{gathered} \tag{11}$$

where $w_0^2 = w^2 + K_1w^p\cos(p\pi/2)$, using the generalized harmonic balance technique and making the error eq. (10) minimized in the mean square sense, the undetermined coefficients $C(a)$ and $K(a)$ can be obtained [21]:

$$\begin{gathered} C(a)=-\frac{1}{aw_0\pi}\int_0^{2\pi}[\alpha x^3(t)-\alpha K(a)x(t)]\sin\varphi\,\mathrm{d}\varphi=0 \\ K(a)=-\frac{1}{a\pi}\int_0^{2\pi}[\alpha x^3(t)-\alpha C(a)\dot{x}(t)]\cos\varphi\,\mathrm{d}\varphi=\frac{3}{4}a^2 \end{gathered} \tag{12}$$

Substituting eq. (12) into eq. (9) gives the equivalent system:

$$\ddot{x}(t)+\Omega^2x(t)+\gamma\dot{x}(t)=x(t)\xi(t) \tag{13}$$

where

$$\Omega^2=w_0^2+\alpha\frac{3}{4}a^2$$

Assuming that system eq. (13) has the solution of the periodic form, we introduce the transformation [22]:

$$\begin{gathered} X=x(t)=a(t)\cos\Phi(t) \\ Y=\dot{x}(t)=-a(t)w_0\sin\Phi(t) \\ \Phi(t)=\Omega t+\theta(t) \end{gathered} \tag{14}$$

where Ω is the natural frequency of the aforementioned equivalent system eq. (13), $a(t)$ and $\theta(t)$ are the amplitude and phase processes of the system's response, respectively, and they are both random processes.

Substituting eq. (14) into eq. (13), we obtain:

$$\begin{gathered} \frac{\mathrm{d}a}{\mathrm{d}t}=F_{11}(a,\theta)+G_{11}(a,\theta)\xi(t) \\ \frac{\mathrm{d}\theta}{\mathrm{d}t}=F_{21}(a,\theta)+G_{21}(a,\theta)\xi(t) \end{gathered} \tag{15}$$

in which

$$
\begin{aligned}
F_{11}(a,\theta) &= -\gamma a \sin^2 \Phi \\
F_{21}(a,\theta) &= -\gamma \sin\Phi \cos\Phi \\
G_{11} &= -\frac{a \sin\Phi \cos\Phi}{\Omega} \\
G_{21} &= -\frac{\cos^2 \Phi}{\Omega}
\end{aligned}
\tag{16}
$$

Equation (15) can be treated as the Stratonovich stochastic differential equation, and by adding the relevant Wong-Zakai correction term, we transform it into the corresponding Ito stochastic differential equation:

$$
\begin{aligned}
\mathrm{d}a &= \left[F_{11}(a,\theta) + F_{12}(a,\theta)\right]\mathrm{d}t + \sqrt{2D}\,\mathrm{G}_{11}(a,\theta)\mathrm{d}B(t) \\
\mathrm{d}\theta &= \left[F_{21}(a,\theta) + F_{22}(a,\theta)\right]\mathrm{d}t + \sqrt{2D}G_{21}(a,\theta)\mathrm{d}B(t)
\end{aligned}
\tag{17}
$$

where $B(t)$ is the normalized Wiener process:

$$
\begin{aligned}
F_{12}(a,\theta) &= D\frac{\partial G_{11}}{\partial a}G_{11} + D\frac{\partial G_{11}}{\partial \theta}G_{21} \\
F_{22}(a,\theta) &= D\frac{\partial G_{21}}{\partial a}G_{11} + D\frac{\partial G_{21}}{\partial \theta}G_{21}
\end{aligned}
\tag{18}
$$

By stochastic averaging [23] of eq. (17) over Φ, we obtain the following averaged Ito equation:

$$
\begin{aligned}
\mathrm{d}a &= m_1(a)\mathrm{d}t + \sigma_{11}(a)\mathrm{d}B(t) \\
\mathrm{d}\theta &= m_2(a)\mathrm{d}t + \sigma_{21}(a)\mathrm{d}B(t)
\end{aligned}
\tag{19}
$$

where

$$
\begin{aligned}
m_1(a) &= -\frac{a}{2}\gamma + \frac{3Da}{8\Omega^2} \\
\sigma_{11}^2(a) &= \frac{Da^2}{4\Omega^2} \\
m_2(a) &= 0 \\
\sigma_{21}^2(a) &= \frac{3D}{4\Omega^2}
\end{aligned}
\tag{20}
$$

Equations (19) and (20) show that da is the not depend on θ, the averaged Ito equation of $a(t)$ is independent of $\theta(t)$ and that the random process $a(t)$ is a 1-D diffusion process. Thus, the correspondingly Fokker-Planck-Kolmogorov (FPK) equation of $a(t)$ can be written:

$$
\frac{\partial\left[p(a,t)\right]}{\partial t} = -\frac{\partial}{\partial a}\left[m_1(a)p(a)\right] + \frac{1}{2}\frac{\partial^2}{\partial a^2}[\sigma_{11}^2(a)p(a)]
\tag{21}
$$

The boundary conditions:

$$
\begin{aligned}
&p(a) = c,\ c \in (-\infty, +\infty) \quad \text{as } a = 0 \\
&p(a) \to 0,\ \partial p / \partial a \to 0 \quad \text{as } a \to \infty
\end{aligned}
\tag{22}
$$

Based on the boundary conditions given in eq. (22), the amplitude's stationary PDF can be obtained:

$$p(a)=\frac{C}{\sigma_{11}^2(a)}\exp\left[\int_0^a \frac{2m_1(u)}{\sigma_{11}^2(u)}\mathrm{d}u\right] \tag{23}$$

where C is the normalized constant that satisfies:

$$C=\left[\int_0^\infty\left(\frac{1}{\sigma_{11}^2(a)}\exp\left[\int_0^a \frac{2m_1(u)}{\sigma_{11}^2(u)}\mathrm{d}u\right]\right)\mathrm{d}a\right]^{-1} \tag{24}$$

Substituting eq. (20) into eq. (23), we get the explicit expression of stationary PDF of the system amplitude a:

$$p(a)=\frac{4C\Omega^2}{D}a^{-\frac{\Delta_1}{D}}\exp\left(-\frac{\Delta_2}{2D}\right) \tag{25}$$

where

$$\Delta_1=2K_1{}^2w^{2p\ \ 1}\sin(p\pi)+4cK_1w^p\cos\left(\frac{p\pi}{2}\right)+4cw^2+4K_1w^{p+1}\sin\left(\frac{p\pi}{2}\right)-D$$

$$\Delta_2=3\ \ \left[K_1w\ \ ^1\sin\left(\frac{\pi}{\quad}\right)+c\right]a^2 \tag{26}$$

$$\Omega^2=w^2+K\ w\ \ \cos\left(\frac{\quad}{2}\right)+\ \ \frac{\quad}{4}a^2$$

Stochastic *P*-bifurcation of the system amplitude

Stochastic *P*-bifurcation means that the changes in number of the stationary PDF curve's peaks. To obtain the critical parametric conditions for stochastic *P*-bifurcation, we analyze the influences of parameters on the system's stochastic *P*-bifurcation by using the singularity theory in this section.

For the sake of convenience, $p(a)$ is expressed:

$$p(a)=4CR(a,D,c,K_1,w,p,\alpha)\exp[Q(a,D,c,K_1,w,p,\alpha)] \tag{27}$$

in which

$$R(a,D,c,K_1,w,p,\alpha)=\frac{\Omega^2}{D}a^{-\frac{2K_1{}^2w^{2p-1}\sin(p\pi)+4cK_1w^p\cos(p\pi/2)+4cw^2+4K_1w^{p+1}\sin(p\pi/2)-D}{D}}$$

$$Q(a,D,c,K_1,w,p,\alpha)=-\frac{3\alpha\left[K_1w^{p-1}\sin\left(\frac{p\pi}{2}\right)+c\right]a^2}{2D} \tag{28}$$

$$\Omega^2=w^2+K_1w^p\cos\left(\frac{p\pi}{2}\right)+\alpha\frac{3}{4}a^2$$

Based on the singularity theory [24], the stationary PDF of the system amplitude needs to satisfy:

$$\frac{\partial p(a)}{\partial a}=0,\ \frac{\partial^2 p(a)}{\partial a^2}=0 \tag{29}$$

Substituting eq. (27) into eq. (29), we obtain [15]:

$$H = \left\{ R' + RQ' = 0, \quad R'' + 2R'Q' + RQ'' + RQ'^2 = 0 \right\} \tag{30}$$

where H is the condition for the changes in number of the PDF curve's peaks.

The influence of p and D on the system

In this part, the influences of p and D on the system are investigated. Without loss of generality, we choose the parameters $c = 0.2$, $\alpha = 1$, $K_1 = 0.2$, and $w = 1$ as the example for illustration. According to eqs. (28) and (30), we obtain the transition set for the system's stochastic P-bifurcation with the unfolding parameters p and D shown in fig. 1.

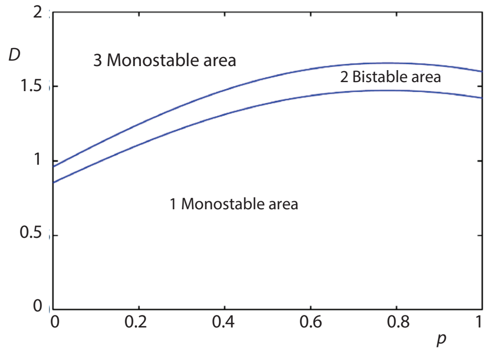

Figure 1. Transition set curves (taking p and D as the unfolding parameters)

We analyze the characteristics of stationary PDF $p(a)$ for a point (p, D) in each of the three sub-areas of fig. 1, and then compare the analytical solutions with the numerical results obtained by Monte-Carlo simulation from original system (3) using the numerical method for fractional derivative [19]. The corresponding results are shown in fig. 2.

It can be seen from fig. 1 that when the parameter (p, D) is taken as $p = 0.1$, $D = 0.3$ in Area 1, fig. 2(a), the PDF $p(a)$ appears in the form of Dirac function, and the steady-state

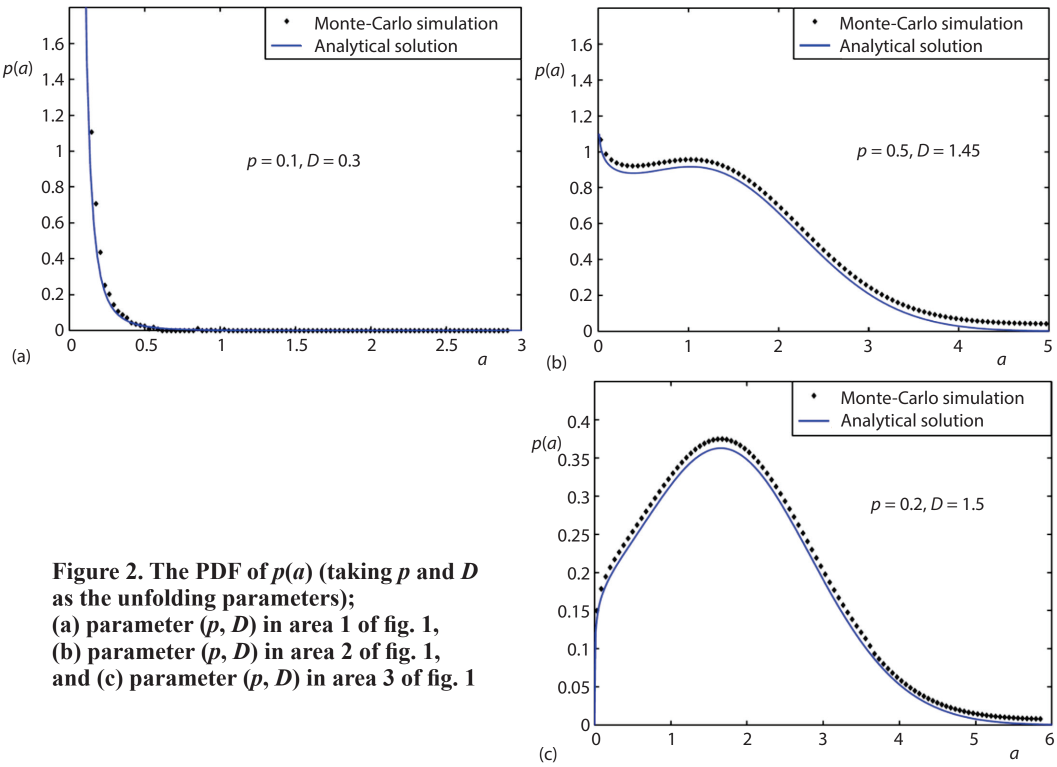

Figure 2. The PDF of $p(a)$ (taking p and D as the unfolding parameters); (a) parameter (p, D) in area 1 of fig. 1, (b) parameter (p, D) in area 2 of fig. 1, and (c) parameter (p, D) in area 3 of fig. 1

response amplitude of the system is constant at 0, similar to the stable equilibrium in the deterministic system at this time. The randomness of the system is suppressed. When the parameter (p, D) is taken as $p = 0.5$, $D = 1.45$ in Area 2, fig. 2(b), the PDF $p(a)$ has a stable limit cycle, and the probability is not zero near the origin, showing that equilibrium coexists with the limit cycle in the system at the moment. When the parameter (p, D) is taken as $p = 0.2$, $D = 1.5$ in Area 3, fig. 2(c), the PDF $p(a)$ has an obvious peak far away from the origin and the system only has a stable limit cycle, the system behavior appears large vibration with a high probability.

Apparently, the stationary PDF $p(a)$ in any two adjacent areas in fig. 1 are very qualitatively different. Regardless of the exact values of the unfolding parameters, if they cross any line in fig. 1, the system will demonstrate stochastic *P*-bifurcation behavior. Therefore, the transition set curves are just the critical parametric conditions of the system's stochastic *P*-bifurcation. The analytic results shown in fig. 2 are well consistent with those numerical results obtained by Monte-Carlo simulation from the original system (3), further verifying the theoretical analysis and showing that it is feasible to use the methods presented in this paper to analyze the stochastic *P*-bifurcation behaviors of the fractional order non-linear systems.

Compared with the integral-order controllers, the fractional-order controllers have the better dynamic performances and robustness [17]. In the past several years, various fractional-order controllers have been developed [25-27]. In the analysis, we obtained the areas where the stochastic *P*-bifurcation occurs in system (3), which can make the system switch between monostable and bistable states by selecting the corresponding unfolding parameters. This could provide theoretical guidance for the analysis and design of the fractional order controllers.

Conclusion

We studied the stochastic *P*-bifurcation of a bistable Duffing system with the fractional derivative element ex-cited by the multiplicative Gaussian white noise excitation in this paper. According to the principle of minimal mean square error and the equivalent linear method, we can transform the original system into an integer-order system with an equivalent stiffness whose coefficient is a function of the system amplitude, and we obtained the system amplitude's stationary PDF using the stochastic averaging method. In addition, the critical parametric conditions for the system's stochastic *P*-bifurcation are obtained by using the singularity theory. According to this, we can maintain the system response at the small amplitude near the equilibrium or the monostability by selecting the corresponding unfolding parameters, which can provide the theoretical guidance for system design and avoid the damage and instability caused by the system's non-linear jump phenomenon or large amplitude vibration. The consistency between the numerical results obtained from Monte-Carlo simulation and the analytical results can also verify the theoretical analysis. It shows that the fractional order, p, and noise intensity, D, can each cause the system's stochastic *P*-bifurcation, and number of peaks of the system's stationary PDF curves can change from two to one by selecting the corresponding unfolding parameters.

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