THE EXTENDED VARIATIONAL ITERATION METHOD FOR LOCAL FRACTIONAL DIFFERENTIAL EQUATION

by

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An extended variational iteration method within the local fractional derivative is introduced for the first time, where two Lagrange multipliers are adopted. Moreover, the sufficient conditions for convergence of the new variational iteration method are also established.

Key words: *variational iteration method, boundary value problems, Lagrange's multiply, local fractional calculus*

Introduction

The development of physical science and engineering has promoted the generation of various incompatible fractional derivatives, and it is an active research field to choose an appropriate fractional calculus according to a real physical problem. For examples, He's fractional derivative [1] was adopted to describe polar hair's thermal property [2], the fractional Caputo-Fabrizio derivative was applied to the groundwater and thermal science [3], the local fractional calculus was used to process silk cocoon hierarchy [4], He's fractal derivative was applied to explanation of snow's thermal insulation property [5], adsorption kinetics [6], biomechanism of polar hairs [7, 8], microgravity fluid [9], convection-diffusion process [10], and two-scale thermodynamics [11-13].

The local fractional calculus was firstly introduced by Yang [14], which could be used as a powerful tool to describing the motion of a fluid in a porous medium, and this calculus currently has a wide range of physical applications, such as [15-20].

The variational iteration method was first proposed by He [21], and the improvements and applications of the variational iterative method have been an active issue in solving differential equations, such as [22-33]. In this paper, we extend the variational iteration method by modifying the correction functional to make it more suitable for solving differential equations.

Local fractional operators

In this section, we introduce two definitions of the local fractional calculus theory, which shall be used in this paper [14].

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Definition 1. In fractal space, the local fractional derivative of $f(x)$ of order α at $x = x_0$ is defined:

$$D_x^{(\alpha)} f(x_0) = f^{(\alpha)}(x_0) = \frac{d^\alpha f(x)}{dx^\alpha}\bigg|_{x=x_0} = \lim_{x\to x_0} \frac{\Delta^\alpha \left[f(x) - f(x_0)\right]}{(x-x_0)^\alpha} \tag{1}$$

where

$$\Delta^\alpha \left[f(x) - f(x_0)\right] \cong \Gamma(1+\alpha)\Delta\left[f(x) - f(x_0)\right]$$

Definition 2. If a function $f(x) \in C[a, b]$, the local fractional integral of $f(x)$ of order α in the interval $[a, b]$ is defined:

$$_aI_b^{(\alpha)} f(x) = \frac{1}{\Gamma(1+\alpha)} \int_b^a f(t)(dt)^\alpha = \frac{1}{\Gamma(1+\alpha)} \lim_{\Delta t\to 0} \sum_{j=0}^{j=N-1} f(t_j)(\Delta t_j)^\alpha \tag{2}$$

where

$$\Delta t_j = t_{j+1} - t_j, \Delta t = \max\{\Delta t_1, \Delta t_2, \Delta t_j,...\}$$

and

$$[t_j, t_{j+1}], j = 0,..., N-1, t_0 = a, t_N = b$$

is a partition of the interval $[a, b]$.

The extended variational iteration method for local fractional differential equation

Consider the following local fractional differential equation:

$$\begin{gathered} \frac{d^{2\alpha} u(x)}{dx^{2\alpha}} + k_1 \frac{d^\alpha u(x)}{dx^\alpha} + k_2 u(x) + L\left[u(x)\right] + N\left[u(x)\right] = f(x), \\ u(a) = \beta_1,\ u(b) = \beta_2 \end{gathered} \tag{3}$$

where L and N are linear and non-linear operators, respectively, which depict the fractal behaviors of the physical processes, k_1 and k_2 are all real constants, β_1 and β_2 – the initial conditions, and where $f(x)$ is the inhomogeneous part.

Moreover, α is the value of fractal dimensions of the porous medium. For $\alpha = 1$, the proposed medium dose not have holes.

Based on the classic variational iteration method, we modify the classic correction functional for eq. (3):

$$\begin{aligned} u_{n+1} = u_n &+ \frac{1}{\Gamma(1+\alpha)} \int_a^x \lambda_1(t,x) f(t)(dt)^\alpha + \frac{1}{\Gamma(1+\alpha)} \int_x^b \lambda_2(t,x) f(t)(dt)^\alpha \cdot \\ &\cdot \frac{1}{\Gamma(1+\alpha)} \int_a^x \lambda_1(t,x)\left[\frac{d^{2\alpha} u_n}{dt^{2\alpha}} + k_1 \frac{\partial^\alpha u_n}{\partial t^\alpha} + k_2 u_n + L\tilde{u}_n + N\tilde{u}_n \right](dt)^\alpha + \\ &+ \frac{1}{\Gamma(1+\alpha)} \int_x^b \lambda_2(t,x)\left[\frac{\partial^{2\alpha} u_n}{\partial t^{2\alpha}} + k_1 \frac{\partial^\alpha u_n}{\partial t^\alpha} + k_2 u_n + L\tilde{u}_n + N\tilde{u}_n \right](dt)^\alpha \end{aligned} \tag{4}$$

where $\lambda_1(t, x)$ and $\lambda_2(t, x)$ are two general Lagrange's multipliers defined on the intervals $[a, x]$ and $[x, b]$, respectively, which satisfy the corresponding conditions at $x = a$ and $x = b$, respectively, and where the terms $L[\tilde{u}_n(t)]$ and $R[\tilde{u}_n(t)]$ are all considered as restricted local fractional variation, *i. e.*, $\delta^\alpha L[\tilde{u}_n(t)] = 0$ and $\delta^\alpha R[\tilde{u}_n(t)] = 0$. The selected initial term $u_0(x)$ should satisfy the given boundary conditions of eq. (3).

Next, we will derive the proper correctional functional for eq. (3) and accordingly construct the extended variation iterative scheme:

$$u_{n+1}(x) = u_n(x) + \frac{1}{\Gamma(1+\alpha)}\int_a^x \lambda_1(t,x) f(t)(\mathrm{d}t)^\alpha + \frac{1}{\Gamma(1+\alpha)}\int_x^b \lambda_2(t,x) f(t)(\mathrm{d}t)^\alpha \cdot$$
$$\cdot\frac{1}{\Gamma(1+\alpha)}\int_a^x \lambda_1(t,x)\left[\frac{\mathrm{d}^{2\alpha}u_n(t)}{\mathrm{d}t^{2\alpha}} + k_1\frac{\mathrm{d}^\alpha u_n(t)}{\mathrm{d}t^\alpha} + k_2 u_n(t) + Lu_n(t) + N\tilde{u}_n(t)\right](\mathrm{d}t)^\alpha +$$
$$+\frac{1}{\Gamma(1+\alpha)}\int_x^b \lambda_2(t,x)\left[\frac{\mathrm{d}^{2\alpha}u_n(t)}{\mathrm{d}t^{2\alpha}} + k_1\frac{\mathrm{d}^\alpha u_n(t)}{\mathrm{d}t^\alpha} + k_2 u_n(t) + Lu_n(t) + N\tilde{u}_n(t)\right](\mathrm{d}t)^\alpha \quad (5)$$

where the Lagrange's multipliers $\lambda_1(t, x)$ and $\lambda_2(t, x)$ satisfy the corresponding conditions at $x = a$ and $x = b$, respectively, *i. e.* $\lambda_1(t, x) = 0$ and $\lambda_2(t, x) = 0$.

To find the optimal values of $\lambda_1(t, x)$ and $\lambda_2(t, x)$, we proceed as follows. Integrating by parts within each of the integrals in eq. (5), we have:

$$u_{n+1}(x) = \frac{1}{\Gamma(1+\alpha)}\left[\int_a^x \lambda_1^{(2\alpha)}u_n(t)(\mathrm{d}t)^\alpha - k_1\int_a^x \lambda_1^{(\alpha)}u_n(t)(\mathrm{d}t)^\alpha + k_2\int_a^x \lambda_1 u_n(t)(\mathrm{d}t)^\alpha\right] + u_n(x) +$$
$$+\lambda_1^{(\alpha)}(x,a)u_n(a) + \lambda_1(x,x)u_n^{(\alpha)}(x) - \lambda_1(x,a)u_n^{(\alpha)}(a) + k_1\lambda_1(x,x)u_n(x) - k_1\lambda_1(x,a)u_n(a) -$$
$$-\lambda_1^{(\alpha)}(x,x)u_n(x) - \lambda_2(x,x)u_n^{(\alpha)}(x) + \lambda_2(x,b)u_n^{(\alpha)}(b) - k_1\lambda_2(x,x)u_n(x) +$$
$$+k_1\lambda_2(x,b)u_n(b) + \lambda_2^{(\alpha)}(x,x)u_n(x) - \lambda_2^{(\alpha)}(x,b)u_n(b) -$$
$$-\frac{1}{\Gamma(1+\alpha)}\left[\int_x^b \lambda_2^{(2\alpha)}u_n(t)(\mathrm{d}t)^\alpha - k_1\int_x^b \lambda_2^{(\alpha)}u_n(t)(\mathrm{d}t)^\alpha + k_2\int_x^b \lambda_2 u_n(t)(\mathrm{d}t)^\alpha\right] \quad (6)$$

Next, taking the variation with respect to $u_n(x)$ of both sides of eq. (6), we obtain

$$\delta u_{n+1}(x) = \left(1 - \lambda_1^{(\alpha)} + k_1\lambda_1 + \lambda_2^{(\alpha)} - k_1\lambda_2\big|_{t=x}\right)\delta u_n(x) +$$
$$+\frac{1}{\Gamma(1+\alpha)}\delta\left[\int_a^x \lambda_1^{(2\alpha)}u_n(t)(\mathrm{d}t)^\alpha - k_1\int_a^x \lambda_1^{(\alpha)}u_n(t)(\mathrm{d}t)^\alpha + k_2\int_a^x \lambda_1 u_n(t)(\mathrm{d}t)^\alpha\right]\cdot$$
$$\cdot\left(\lambda_{12} - \lambda_2\big|_{t=x}\right)\delta u_n'(x) + \frac{1}{\Gamma(1+\alpha)}\delta\left[\int_x^b \lambda_2^{(2\alpha)}u_n(t)(\mathrm{d}t)^\alpha - k_1\int_x^b \lambda_2^{(\alpha)}u_n(t)(\mathrm{d}t)^\alpha + k_2\int_x^b \lambda_2 u_n(t)(\mathrm{d}t)^\alpha\right] \quad (7)$$

Therefore, we obtain the following stationary conditions:

$$\begin{gathered}
\lambda_1^{(2\alpha)} - k_1\lambda_1^{(\alpha)} + k_2\lambda_1 = 0 \\
\lambda_2^{(2\alpha)} - k_1\lambda_2^{(\alpha)} + k_2\lambda_2 = 0 \\
1 - \lambda_1^{(\alpha)} + k_1\lambda_1 + \lambda_2^{(\alpha)} - k_1\lambda_2\big|_{t=x} = 0 \\
\lambda_1 - \lambda_2\big|_{t=x} = 0 \\
\lambda_1(t,b) = 0 \\
\lambda_2(t,a) = 0
\end{gathered} \quad (8)$$

By virtue of eq. (8),we can rewrite eq. (6) as:

$$u_{n+1}(x)=-\lambda_1(x,a)u_n^{(\alpha)}(a)-k_1\lambda_1(x,a)u_n(a)+\lambda_1^{(\alpha)}(x,a)u_n(a)+\lambda_2(x,b)u_n^{(\alpha)}(b)+$$
$$+k_1\lambda_2(x,b)u_n(b)-\lambda_2^{(\alpha)}(x,b)u_n(b)+\frac{1}{\Gamma(1+\alpha)}\int_a^x\lambda_1(t,x)f(t)(\mathrm{d}t)^\alpha+\frac{1}{\Gamma(1+\alpha)}\int_x^b\lambda_1(t,x)f(t)(\mathrm{d}t)^\alpha+$$
$$+\frac{1}{\Gamma(1+\alpha)}\int_a^x\lambda_2(t,x)Nu_n(t)(\mathrm{d}t)^\alpha+\frac{1}{\Gamma(1+\alpha)}\int_x^b\lambda_2(t,x)Nu_n(t)(\mathrm{d}t)^\alpha \tag{9}$$

The eq. (9) does provide an effective tool for constructing the following equivalent integral equation of eq. (3):

$$u(x)=-\lambda_1(x,a)u^{(\alpha)}(a)-k_1\lambda_1(x,a)u(a)+\lambda_1^{(\alpha)}(x,a)u(a)+\lambda_2(x,b)u^{(\alpha)}(b)+k_1\lambda_2(x,b)u(b)-$$
$$-\lambda_2^{(\alpha)}(x,b)u(b)+\frac{1}{\Gamma(1+\alpha)}\int_a^x\lambda_1(t,x)f(t)(\mathrm{d}t)^\alpha+\frac{1}{\Gamma(1+\alpha)}\int_x^b\lambda_2(t,x)f(t)(\mathrm{d}t)^\alpha+$$
$$+\frac{1}{\Gamma(1+\alpha)}\int_a^x\lambda_1(t,x)\left[Lu(t)+Nu(t)\right](\mathrm{d}t)^\alpha+\frac{1}{\Gamma(1+\alpha)}\int_x^b\lambda_2(t,x)\left[Lu(t)+Nu(t)\right](\mathrm{d}t)^\alpha \tag{10}$$

The solutions $u(x)$ of eq. (3) can be obtained by the following recursive relation:

$$v_0(x)=\lambda_1^{(\alpha)}(x,a)u(a)-\lambda_2^{(\alpha)}(x,b)u(b)$$
$$v_{n+1}(x)=v_0(x)+\frac{1}{\Gamma(1+\alpha)}\int_a^x\lambda_1(t,x)f(t)(\mathrm{d}t)^\alpha+\frac{1}{\Gamma(1+\alpha)}\int_x^b\lambda_2(t,x)f(t)(\mathrm{d}t)^\alpha\cdot$$
$$\cdot\frac{1}{\Gamma(1+\alpha)}\int_a^x\lambda_1(t,x)\left[Lv_n(t)+Nv_n(t)\right](\mathrm{d}t)^\alpha+\frac{1}{\Gamma(1+\alpha)}\int_x^b\lambda_2(t,x)\left[Lv_n(t)+Nv_n(t)\right](\mathrm{d}t)^\alpha \tag{11}$$

and

$$u(x)=\lim_{n\to\infty}v_n(x) \tag{12}$$

Since there are many choices of the Lagrange multiplier based on eq. (8), many corresponding equivalent integral equations of eq. (3) can be derived.

Convergence analysis

The distance function between function $u(x)$ and function $v(x)$ is defined in the following form:

$$\mathrm{d}\left[u(x),v(x)\right]=\max_{a\le x\le b}\left|u(x)-v(x)\right|$$

Theorem 1. If the linear and non-linear local fractional operators $L[u(x)]$ and $N[v(x)]$ satisfy the following Lipschitz conditions, respectively:

$$\mathrm{d}\left\{L\left[u(x)\right],L\left[v(x)\right]\right\}\le M_1d\left[u(x),v(x)\right],M_1>0,d\left\{N\left[u(x)\right],N\left[u(x)\right]\right\}\le$$
$$\le M_2d\left[u(x),v(x)\right],\ M_2>0 \tag{13}$$

where M_1 and M_2 are all positive constants:

$$m=(M_1+M_2)\frac{2}{\Gamma^2(1+\alpha)}\left(\int_a^x\lambda_1(t,x)(\mathrm{d}t)^\alpha+\int_x^b\lambda_2(t,x)(\mathrm{d}t)^\alpha\right),\ 0<m<1 \tag{14}$$

then eq. (11) has a unique solution.

Proof. Let $u(x)$ and $\dot{u}(x)$ be two different solutions for eq. (3), then:

$$d\left[u(x),\dot{u}(x)\right]\le\frac{1}{\Gamma^2(1+\alpha)}d\left[\int_a^x\lambda_1(t,x)Lu(t)(\mathrm{d}t)^\alpha,\int_a^x\lambda_1(t,x)L\dot{u}(t)(\mathrm{d}t)^\alpha\right]+$$
$$+\frac{1}{\Gamma^2(1+\alpha)}d\left[\int_a^x\lambda_1(t,x)Nu(t)(dt)^\alpha,\int_a^x\lambda_1(t,x)N\dot{u}(t)(dt)^\alpha\right]+$$
$$+\frac{1}{\Gamma^2(1+\alpha)}d\left[\int_x^b\lambda_1(t,x)Lu(t)(\mathrm{d}t)^\alpha,\int_x^b\lambda_1(t,x)L\dot{u}(t)(\mathrm{d}t)^\alpha\right]+$$
$$+\frac{1}{\Gamma^2(1+\alpha)}d\left[\int_x^b\lambda_1(t,x)Nu(t)(\mathrm{d}t)^\alpha,\int_x^b\lambda_1(t,x)N\dot{u}(t)(\mathrm{d}t)^\alpha\right]\le$$
$$\le(M_1+M_2)\frac{1}{\Gamma^2(1+\alpha)}\left(\int_a^x\lambda_1(t,x)(\mathrm{d}t)^\alpha+\int_x^b\lambda_2(t,x)(\mathrm{d}t)^\alpha\right)d\left[u(x),\,\dot{u}(x)\right] \tag{15}$$

Now, we get

$$d\left[u(x),\dot{u}(x)\right]\le md\left[u(x),\dot{u}(x)\right]$$

Since $0 < m < 1$, then $d[u(x),\ \dot{u}(x)] \to 0$ implies $u(x) = \dot{u}(x)$ and this completes the proof.

Theorem 2. If the conditions shown in *Theorem 1* are satisfied, the solution $v_n(x)$ obtained from (11) converges to the exact solution $u(x)$ of eq. (3).

Proof.

$$d\left[v_{n+1}(x),u(x)\right]\le\frac{1}{\Gamma^2(1+\alpha)}d\left[\int_a^x\lambda_1(t,x)Lv_n(t)(\mathrm{d}t)^\alpha,\int_a^x\lambda_1(t,x)Lu(t)(\mathrm{d}t)^\alpha\right]+$$
$$+\frac{1}{\Gamma^2(1+\alpha)}d\left[\int_a^x\lambda_1(t,x)Nv_n(t)(\mathrm{d}t)^\alpha,\int_a^x\lambda_1(t,x)Nu(t)(dt)^\alpha\right]+$$
$$+\frac{1}{\Gamma^2(1+\alpha)}d\left[\int_x^b\lambda_1(t,x)Lv_n(t)(\mathrm{d}t)^\alpha,\int_x^b\lambda_1(t,x)Lu(t)(\mathrm{d}t)^\alpha\right]+$$
$$+\frac{1}{\Gamma^2(1+\alpha)}d\left[\int_x^b\lambda_1(t,x)Nv_n(t)(\mathrm{d}t)^\alpha,\int_x^b\lambda_1(t,x)Nu(t)(\mathrm{d}t)^\alpha\right]\le$$
$$\le\frac{1}{\Gamma^2(1+\alpha)}(M_1+M_2)\left(\int_a^x\lambda_1(t,x)(\mathrm{d}t)^\alpha+\int_x^b\lambda_2(t,x)(\mathrm{d}t)^\alpha\right)d\left[v_n(x),\,u(x)\right] \tag{16}$$

Since $0 < m < 1$, then $d[v_n(x),\ u(x)] \to 0$ as $n \to \infty$. Therefore, $v_n(x) \to u(x)$ as $n \to \infty$. This completes the proof.

Applications

In this section, to demonstrate the applicability of this method, we apply it to the following local fractional differential equations.

Case 1:

$$\frac{\mathrm{d}^{2\alpha}u}{\mathrm{d}x^{2\alpha}}=A,\ u(0)=0,\ u(1)=A \tag{17}$$

Upon eqs. (8) and (17), we construct the following Lagrange's multipliers:

$$\lambda_1 = -xt + x,\ 0 \le t < x,\quad \lambda_2 = (1-x)t,\ x < t \le 1 \tag{18}$$

Consequently, based on eqs. (11) and (18), the equivalent integral equation of eq. (17) can be established:

$$u(x,y) = u_0(x,y) + \frac{1}{\Gamma(1+\alpha)}\int_0^x (-xt+x)A(\mathrm{d}t)^\alpha + \frac{1}{\Gamma(1+\alpha)}\int_x^1 \left[(1-x)t\right]A(\mathrm{d}t)^\alpha \tag{19}$$

where

$$u_0(x,y) = \lambda_1^{(\alpha)}(x,a)u(a) - \lambda_2^{(\alpha)}(x,b)u(b)$$

The required solution $u(x)$ for eq. (19) and hence for eq. (17) can be obtained from the recurrence relation:

$$v_0(x) = u_0(x) = -Ax$$

$$v_{n+1}(x) = u_0(x) + \frac{1}{\Gamma(1+\alpha)}\int_0^x (-xt+x)A(\mathrm{d}t)^\alpha + $$
$$+ \frac{1}{\Gamma(1+\alpha)}\int_x^1 \left[(1-x)t\right]A(\mathrm{d}t)^\alpha = -\frac{1}{2}xA + \frac{x^2 A}{2} \tag{20}$$

Thence

$$u(x) = \lim_{n\to\infty} v_n(x) = -\frac{3}{2}xA + \frac{x^2 A}{2} \tag{21}$$

In the following, let $M_2 > 0$, I only prove that the following equation satisfy the conditions of *Theorems 1 and 2*.

Case 2:

$$\frac{\mathrm{d}^{2\alpha}u}{\mathrm{d}x^{2\alpha}} + u + M_2 \sin_\alpha u = 0,\quad u(0) = \beta_1, u\left(\frac{\pi}{2}\right) = \beta_2 \tag{22}$$

Upon the eqs. (8) and (22), we have the following Lagrange's multipliers:

$$\begin{aligned} \lambda_1 &= \sin_\alpha(t)^\alpha \cos_\alpha(x)^\alpha,\ \ a \le t < x \\ \lambda_2 &= \cos_\alpha(t)^\alpha \sin_\alpha(x)^\alpha,\ \ x < t \le b \end{aligned} \tag{23}$$

Consequently, based on eqs. (11) and (23), the equivalent integral equation of eq. (22) can be established:

$$\begin{aligned} u(x) &= u_0(x) + \\ &+ \frac{M_2}{\Gamma(1+\alpha)}\int_0^x \sin_\alpha(t)^\alpha \cos_\alpha(x)^\alpha \sin_\alpha u(\mathrm{d}t)^\alpha + \\ &+ \frac{M_2}{\Gamma(1+\alpha)}\int_x^{\pi/2} \cos_\alpha(t)^\alpha \sin_\alpha(x)^\alpha \sin_\alpha u(\mathrm{d}t)^\alpha \end{aligned} \tag{24}$$

where

$$v_0(x) = \lambda_1^{(\alpha)}(x,0)\beta_1 - \lambda_2^{(\alpha)}\left(x, \frac{\pi}{2}\right)\beta_2 = \cos_\alpha(x)^\alpha \beta_1 + \sin_\alpha(x)^\alpha \beta_2 \tag{25}$$

Obviously:

$$\begin{aligned}
&\frac{M_2}{\Gamma^2(1+\alpha)}\left[\int_0^x \sin_\alpha(t)^\alpha \cos_\alpha(x)^\alpha (\mathrm{d}t)^\alpha + \int_x^{\pi/2} \cos_\alpha(t)^\alpha \sin_\alpha(x)^\alpha (\mathrm{d}t)^\alpha\right] = \\
&= \frac{M_2}{\Gamma^2(1+\alpha)}\left\{\cos_\alpha(x)^\alpha\left[1-\cos_\alpha(x)^\alpha\right] + \sin_\alpha(x)^\alpha\left[1-\sin_\alpha(x)^\alpha\right]\right\} = \\
&= \frac{M_2}{\Gamma^2(1+\alpha)}\left[\cos_\alpha(x)^\alpha + \sin_\alpha(x)^\alpha - 1\right] = \\
&= \frac{M_2}{\Gamma^2(1+\alpha)}\left[\sqrt{2}\sin_\alpha\left(x+\frac{\pi}{4}\right)^\alpha - 1\right] \le \frac{M_2}{\Gamma^2(1+\alpha)}\left(\sqrt{2}-1\right)
\end{aligned} \tag{26}$$

Conclusion

In this paper, an extended variational iteration method is introduced and the sufficient conditions for this method to converge are established. Several examples are given to confirm the validity of this method. As this method greatly enriches the content of variational iterative method, I believe that in the near future, more people will pay attention or discuss this method.

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