BAYESIAN INFERENCE FOR SOLVING A CLASS OF HEAT CONDUCTION PROBLEMS

by

Jun-Feng LAI\textsuperscript{a}, Zai-Zai YAN\textsuperscript{a,*} and Ji-Huan HE\textsuperscript{b, c*}

\textsuperscript{a} Science College, Inner Mongolia University of Technology, Hohhot, 010051, China
\textsuperscript{b} School of Mathematics and Information Science, Henan Polytechnic University, Jiaozuo, China
\textsuperscript{c} National Engineering Laboratory for Modern Silk, College of Textile and Clothing Engineering, Soochow University, Suzhou, China

This paper considers a heat conduction problem of a common continuum-type stochastic mathematical model in an engineering field. The approximate solution is calculated with the Markov Chain Monte Carlo algorithm for the heat conduction problem. Three examples are given to illustrate the solution process of the method.

Key words: Bayesian inference, heat conduction problem, Markov Chain Monte Carlo

Introduction


Various numerical methods have been developed to estimate the parameters of stochastic differential equations, for example, the Gibbs algorithm proposed by Gemans [6], and the resampling algorithm given by Gordon et al. [7]. Eraker[8] used the Bayesian method to discuss estimation of the parameter in model with a stochastic volatility component. Golightly and Wilkinson discussed the parameter estimation of the nonlinear multivariate diffusion models based on the missing data [9]. Miguez, et al. [10] discussed the sequential Monte Carlo method in general state-space models.

Many studies have been conducted in the field of heat conduction. The interdisciplinary study has also developed rapidly and has penetrated to many disciplines. Various methods for solving the thermal conductivity of materials have been proposed, such as the spirit sensitivity method, the least squares method, the regularization method, and the conjugate gradient method. Martín-Fernández

\textsuperscript{*}Corresponding authors, e-mails: zz.yan@163.com(Z.Z. Yan) and Hejihuan@suda.edu.cm (J.H.He)
and Lanzarone used the Monte Carlo method to solve the heat conduction problem [11]. Tifkitsis and Skordos [12] developed a modified scheme based on the Markov Chain Monte Carlo for the estimation of unknown stochastic input parameters, such as the heat transfer coefficient. Zeng et al. [13] extended the approximate Bayesian computation method to the inverse heat conduction problem and developed two heat conduction solvers.

Recently, many researchers performed excellent studies in the field of the Markov Chain Monte Carlo, and some new expended approaches have been widely used in various engineering fields. Rich information on different aspects of biological mechanisms is encoded by genes, and Ko proposed an MCMC method to extract new biological information from the data[14]. Hemantha introduced an MCMC simulation method to engineering economics research [15]. Yousaf [16] considered transmuted distribution and compared priors under the squared error loss function. Begona, et al.[17] developed a new technique to estimate the SIR model parameters using the MCMC method.

Motivated by the previous ideas, our scheme utilizes Markov Chain Monte Carlo (MCMC) to solve a class of heat conduction problems.

Differential equation of heat conduction is listed as below:

\[ \frac{\partial}{\partial x} \left( k_x \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left( k_y \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z} \left( k_z \frac{\partial T}{\partial z} \right) + q_v = \rho c_p \frac{\partial T}{\partial t} \]  

(1)

where \( T \) is transient temperature, \( t \) is time of the process, \( k_x, k_y, k_z \) are thermal coefficient, \( \rho \) is density of materials, \( c_p \) is specific heat capacity at constant pressure, \( q_v \) is internal heat source strength.

We define the difference operators as:

\[ \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \]  

(2)

**Assumption 1:** When the material of an object is isotropic, it implies that:

\[ k_x = k_y = k_z = k \]  

(3)

**Assumption 2:** As there is no heat source in the object, the implication is that:

\[ \frac{q_v}{k} = 0 \]  

(4)

**Assumption 3:** Let \( \alpha_T = k (\rho c_p)^{-1} \), the object is in the steady state temperature field and it implies that:

\[ \frac{1}{\alpha_T} \frac{\partial T}{\partial t} = 0 \]  

(5)

Based on Eqs. (2) and (3), Eq. (1) can be transformed:

\[ \nabla^2 T + q_v = \frac{1}{\alpha_T} \frac{\partial T}{\partial t} \]  

(6)

Based on Eq. (4), Eq. (6) can be transformed:

\[ \nabla^2 T = \frac{1}{\alpha_T} \frac{\partial T}{\partial t} \]  

(7)

Based on Eq. (5), Eq. (6) can be transformed:
\[ \nabla^2 T + \frac{q_v}{k} = 0 \]  
with an initial condition \( T(x,0) = \rho(x) \), and some boundary conditions \( T(0,t) = \varphi(t) \) and \( T(x_0,t) = \psi(t) \), where \( \varphi(t), \psi(t), x_0 \) are known.

Let an object of length \( d \) be initialized at the uniform temperature \( x^* \)C. Suppose at \( x = 0 \) is heated to \( x^* \)C and at \( x = d \) is heated to \( x^* \)C. This problem can be modeled by Eq. (1):

\[
\begin{cases}
    a(t) \frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}, & 0 < x < d, t > 0 \\
    T(x,0) = x^* C, & 0 < x < d \\
    T(0,t) = x_2 \ C \\
    T(d,t) = x_3 \ C
\end{cases}
\]

The remaining of this paper is organized as follows. In section 2, some main results of a classical stochastic diffusion problem. In section 3, we propose MCMC method to solve a heat conduction problem. In Section 4, we present three numerical examples followed by conclusions at the end.

**Bayesian inference for Solving a Classical stochastic diffusion Problem**

Given a diffusion of physical Langevin equation

\[ dY_t = b(Y_t, \theta)dt + \sigma(Y_t, \theta)dW_t \]  
\( Y_t \) is the solution of Eq.(10). The transition density

\[ p_i(y|x) = \frac{d}{dy} P(Y_t \leq y \mid Y_0 = x) \]

further

\[ \frac{\partial}{\partial t} p_i(y|x) = b(y) \frac{\partial}{\partial x} p_i(y|x) + \frac{1}{2} \sigma^2(y) \frac{\partial^2}{\partial x^2} p_i(y|x) \]

can be obtained by Fokker-Planck equation. The time interval \([0,T]\) is subdivided equidistant points

\[ 0 = t_0 < t_1 < \ldots < t_{N-1} < t_N = T, \Delta t = \frac{T}{N}. \]

In practice, it is necessary to give the Euler approximation:

\[ Y_{t_{i+1}} - Y_{t_i} = b(Y_{t_i}, \theta)\Delta t + \sigma(Y_{t_i}, \theta)\Delta W_i \]  
\( Y_{t_i} \) is the observed at time \( t_i \) and \( \Delta W_i = W_{t_{i+1}} - W_{t_i} \sim N(0, \Delta t) \).

From Eq.(11), the transition probabilities of \( Y_{t_i} \rightarrow Y_{t_{i+1}} \) from time \( t_i \) to \( t_{i+1} \) are given by

\[ p(Y_{t_{i+1}} \mid Y_{t_i}, \theta) \approx N(Y_{t_i} + b(Y_t, \theta)\Delta t, \sigma(Y_t, \theta)\Delta W_i) \]

the posterior distribution of \( \theta \) is given by

\[ \pi(\theta \mid Y_{t_1}, Y_{t_2}, \ldots Y_{t_p}) \propto \pi(\theta) \prod_{i=1}^{p} p(Y_{t_i} \mid Y_{t_{i+1}}, \theta) \]

where \( \pi(\theta) \) is a prior distribution of \( \theta \).

**Markov Chain Monte Carlo for Solving a Classical of Heat Conduction Problem**
Many heat conduction problems of complex boundary are unable to solve with analytical methods. Instead, related numerical methods have developed rapidly [18-25]. Let’s consider

\[
\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0, 0 \leq x \leq A, 0 \leq y \leq B
\]

\[
T(0, y) = e^y - \cos y, T(x, 0) = \cos x - e^x
\]

\[
T(A, y) = e^y \cos A - e^4 \cos y
\]

\[
T(x, B) = e^B \cos x - e^4 \cos B
\]

The domain \([0, A] \times [0, B]\) is divided into an \(MN\times MN\) mesh with the step size \(h = AM^{-1}\) in the X-direction and the step size \(\tau = BN^{-1}\) in the Y-direction. By Taylor’s expansion:

\[
\begin{align*}
\frac{\partial^2 T}{\partial x^2}_{i,j} &= \frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{(\Delta x)^2} + o(\Delta x) \\
\frac{\partial^2 T}{\partial y^2}_{i,j} &= \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{(\Delta y)^2} + o(\Delta y)
\end{align*}
\]

Hence \(T_{i+1,j} + T_{i+1,j} = 2T_{i,j} + \frac{\partial^2 T}{\partial x^2} (\Delta x)^2 + \ldots\), we have

\[
\frac{\partial^2 T}{\partial x^2}_{i,j} = \frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{(\Delta x)^2} + o(\Delta x)
\]

Similarly,

\[
\frac{\partial^2 T}{\partial y^2}_{i,j} = \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{(\Delta y)^2} + o(\Delta y)
\]

Let’s consider Eq.(12) at point \((i, j)\)

\[
\frac{T_{i+1,j} - 2T_{i,j} + T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{(\Delta x)^2} = 0
\]

The Eq. (15) can be transformed:

\[
T_{i,j} = \frac{1}{2 + 2\alpha} T_{i+1,j} + \frac{1}{2 + 2\alpha} T_{i-1,j} + \frac{\alpha}{2 + 2\alpha} T_{i,j+1} + \frac{\alpha}{2 + 2\alpha} T_{i,j-1}
\]

\[
= p(T_{i+1,j} | T_{i,j}) T_{i+1,j} + p(T_{i-1,j} | T_{i,j}) T_{i-1,j} + p(T_{i,j+1} | T_{i,j}) T_{i,j+1} + p(T_{i,j-1} | T_{i,j}) T_{i,j-1}
\]

where \(\alpha = (\Delta x)^2 (\Delta y)^{-2}\), \(p(T_{i+1,j} | T_{i,j}) = (2 + 2\alpha)^{-1}\), \(p(T_{i,j+1} | T_{i,j}) = p(T_{i-1,j} | T_{i,j}) = (2 + 2\alpha)^{-1}\).

Consider a Markov chain

\[
T_{i,j} \rightarrow T_{i+1,j} \rightarrow T_{i+2,j} \rightarrow \cdots; T_{i,j} \rightarrow T_{i,j+1} \rightarrow T_{i,j+2} \rightarrow \cdots.
\]

Let \(p(T_{i,j}) = p_j\), and we get and transition matrix \(P = (p_{ij})\).

Based on Eqs.(13)-(16), a MCMC algorithm is implemented to estimate transition probability.

**Numerical results**

In this section, numerical tests for the proposed methods are demonstrated.
Example 1. Consider the problem [23]

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0, 0 \leq x \leq A, 0 \leq y \leq B$$

with boundary conditions $T(0, y) = e^y - \cos y$, $T(A, y) = e^y \cos A - e^y \cos y$, $T(x, 0) = \cos x - e^x$ and $T(x, B) = e^\theta \cos x - e^\theta \cos B$. Results obtained for $T(x, y)$ are presented under $A = 4, B = 5$ in Table 1.

**Table 1. Results with $A = 4, B = 5$ for example 1**

<table>
<thead>
<tr>
<th>point $(x, y)$</th>
<th>Average number of iterations</th>
<th>numerical result of $T(x, y)$</th>
<th>standard deviation</th>
<th>confidence interval at $\alpha = 0.05$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(0.5,0.5)$</td>
<td>7.476</td>
<td>-0.0678</td>
<td>0.1880</td>
<td>(-0.3012,0.1656)</td>
</tr>
<tr>
<td>$(0.5,1.0)$</td>
<td>8.490</td>
<td>1.7058</td>
<td>0.1572</td>
<td>(1.1572,1.9010)</td>
</tr>
<tr>
<td>$(1.0,0.5)$</td>
<td>8.618</td>
<td>-1.4581</td>
<td>0.2049</td>
<td>(-1.7125,-1.2037)</td>
</tr>
<tr>
<td>$(1.0,1.0)$</td>
<td>8.240</td>
<td>0.0129</td>
<td>0.2123</td>
<td>(-0.2506,0.2765)</td>
</tr>
<tr>
<td>$(1.5,1.5)$</td>
<td>8.507</td>
<td>-0.0545</td>
<td>0.2176</td>
<td>(-0.3246,0.2165)</td>
</tr>
</tbody>
</table>

Example 2. Consider the problem [23]

$$\frac{\partial T}{\partial t} = a \frac{\partial^2 T}{\partial x^2}, x \in [0,1]$$

with an initial condition $T(x,0) = \sin \pi x$, and two boundary conditions $T(0,t) = T(1,t) = 0 \degree \text{C}$. When $a$ is set to 1, the exact solution of the above equation is $T(x,t) = \exp(-\pi^2 t) \sin \pi x$. The obtained results are presented in Tables 2 and 3.

**Table 2. Numerical results and exact solution with $t = 0.0005, N = 50$ for example 2**

<table>
<thead>
<tr>
<th>Point(x)</th>
<th>exact solution</th>
<th>numerical solution</th>
<th>numerical result at $[20]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.3105</td>
<td>0.3085</td>
<td>0.2930</td>
</tr>
<tr>
<td>0.2</td>
<td>0.5907</td>
<td>0.5868</td>
<td>0.5621</td>
</tr>
<tr>
<td>0.3</td>
<td>0.8130</td>
<td>0.8079</td>
<td>0.7799</td>
</tr>
<tr>
<td>0.4</td>
<td>0.9558</td>
<td>0.9595</td>
<td>0.9280</td>
</tr>
<tr>
<td>0.5</td>
<td>1.0049</td>
<td>0.9986</td>
<td>0.9931</td>
</tr>
</tbody>
</table>

**Table 3. Numerical results and exact solution with $t = 0.0005, N = 100$ for example 2**

<table>
<thead>
<tr>
<th>Point(x)</th>
<th>Value of accurate</th>
<th>numerical result</th>
<th>numerical result[20]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.3105</td>
<td>0.3086</td>
<td>0.2941</td>
</tr>
<tr>
<td>0.2</td>
<td>0.5907</td>
<td>0.5870</td>
<td>0.5627</td>
</tr>
<tr>
<td>0.3</td>
<td>0.8130</td>
<td>0.8078</td>
<td>0.7798</td>
</tr>
<tr>
<td>0.4</td>
<td>0.9558</td>
<td>0.9496</td>
<td>0.9278</td>
</tr>
<tr>
<td>0.5</td>
<td>1.0049</td>
<td>0.9985</td>
<td>0.9930</td>
</tr>
</tbody>
</table>
Example 3. Consider the problem [26]

\[ dY_t = (\theta_1 + \theta_2 Y_t) dt + \theta_3 \sqrt{Y_t} dW_t, Y_0 = 10 \]

here \( \theta_3 = 1 \) is known. The explicit estimators for \( \theta_1, \theta_2 \) are presented in [23]

\[ \hat{\theta}_1 = \frac{\left( \sum_{i=1}^{n} Y_{i-1} \right)^2}{2(n \sum_{i=1}^{n} Y_{i-1}^2 - (\sum_{i=1}^{n} Y_{i-1})^2)} \]

and

\[ \hat{\theta}_2 = \frac{-n \sum_{i=1}^{n} Y_{i-1}}{2(n \sum_{i=1}^{n} Y_{i-1}^2 - (\sum_{i=1}^{n} Y_{i-1})^2)}. \]

Some results for comparison are shown in Table 4.

Table 4. Results with \((\hat{\theta}_1, \hat{\theta}_2)\) obtained for estimators

<table>
<thead>
<tr>
<th>Value of accurate</th>
<th>Numerical result</th>
<th>Value of accurate</th>
<th>Numerical result</th>
</tr>
</thead>
<tbody>
<tr>
<td>(7,-2)</td>
<td>(7.1534,-1.9652)</td>
<td>(7,-2)</td>
<td>(6.9512,-1.9458)</td>
</tr>
<tr>
<td>(8,-2)</td>
<td>(8.7679,-2.1174)</td>
<td>(8,-2)</td>
<td>(8.2297,-2.0267)</td>
</tr>
<tr>
<td>(7,-1.9)</td>
<td>(7.4036,-1.9326)</td>
<td>(7,-1.9)</td>
<td>(7.0544,-1.8760)</td>
</tr>
</tbody>
</table>
Conclusions

This paper proposed different method based on the Bayesian approach to solve a heat conduction problem. The differential equation of the heat conduction is discretized by Taylor's expansion method, and MCMC method is applied to estimate the transition probability. Three listed numerical validated the suitability and efficiency of the proposed numerical method in solving the heat conduction problem.

Acknowledgment

This work is supported by National Natural Science Foundation of China (Grant No. 11861049), Natural Science Foundation of Inner Mongolia (Grant No. 2017MS0101, 2018MS01027), Scientific Research Project of Inner Mongolia University of Technology (Grant No. BS201930).

Reference


Paper submitted: December 26, 2019
Paper revised: May 10, 2020
Accepted: May 10, 2020