ABUNDANT EXACT ANALYTICAL SOLUTIONS AND NOVEL INTERACTION PHENOMENA OF THE GENERALIZED (3+1)-DIMENSIONAL SHALLOW WATER EQUATION

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This paper reveals abundant exact analytical solutions to the generalized (3+1)-dimensional shallow water equation. The generalized bilinear method is used in the solution process and the obtained solutions include the high-order lump-type solutions, the three-wave solutions, the breather solutions. The interaction between the high-order lump-type solutions and the soliton solutions is also elucidated. These solutions have greatly enriched the generalized (3+1)-dimensional shallow water equation in open literature.

Key words: Exact analytical solution, high-order lump-type solution, interaction solution, generalized bilinear method, generalized (3+1)-dimensional Shallow Water equation

1. Introduction

It is well-known that nonlinear evolution equations (NLEEs) have a wide applications in areas of mathematics, physics, atmospheric science, fluid mechanics, plasma, optical fiber communication, biologic nerve propagation, atmospheric science, marine science and thermal science. Therefore, the exact solutions of NLEEs play an important role in understanding the nonlinear phenomena of nonlinear science. To find exact solutions of nonlinear systems is a difficult and tedious but very important and meaningful work. So far, several effective methods have been established by mathematicians and physicists to obtain exact solutions of NLEEs [1-8]. By using these methods, researchers constructed the exact solutions of NLEEs, such as soliton [9], rogue wave [10], breathers [11], periodic wave [12], three-wave solution [13], rational solutions [14], lump solution [15] and interaction solutions [16]-[18], etc.

As one of the three branches of nonlinear science, the theory of solitons has become an important research field of nonlinear science. It has a wide and important role in the fields of nonlinear wave theory and elementary particle theory. Recently, the research of breather waves [11], rational solutions [14] become a hot research topic. In general, breather waves, which have a periodic outline in one direction, can degenerate into the rogue wave solutions in the limiting case. The rational solutions have appeared in many nonlinear fields, such as nonlinear optic fibers, Bose-Einstein condensates, biophysics and economics. As one of critical exact solutions, rational solutions can be used to describe natural phenomena well. In contrast to soliton solutions, lump solutions are a kind of rational function solutions, localized in all directions in the space. In soliton theory, lump solutions have received increasing attention recently [15]. In particular, collisions between lump solutions and
other forms of soliton solutions have been studied [16]-[18].

2. Exact analytical solutions of the generalized (3+1)-dimensional Shallow Water equation

We consider the generalized (3+1)-dimensional shallow water equation [14]

\[ u_{xxy} + 3u_x u_y + 3u_t u_{xy} - u_{yy} - u_{zz} = 0. \] (1)

Researchers studied the rational solutions and the lump solutions (only when \( z=\xi \)) of the generalized (3+1)-dimensional Shallow Water equation. In the following, we will study the general exact analytical solutions of Eq. (1).

**Step 1.**

By using the Cole-Hopf transformation

\[ u(x, y, z, t) = 2[\ln f(x, y, z, t)], \] (2)

Eq. (1) becomes the generalized Hirota bilinear equation

\[ GB_{GSW}(f) := (D_{p,x}^3 D_{p,y} - D_{p,y} D_{p,t} - D_{p,z} D_{p,z}) f \cdot f = 0, \] (3)

where \( p \) is an arbitrarily natural number. \( D \) is generalized bilinear differential operator [3]. We note that when \( p=2 \), the generalized bilinear form is transformed into Hirota bilinear form.

When taking \( p=3 \), we can obtain the generalized bilinear Shallow Water equation

\[ GB_{GSW}(f) := (D_{3,x}^3 D_{3,y} - D_{3,y} D_{3,t} - D_{3,z} D_{3,z}) f \cdot f = 2(3f_{xx} f_{xy} + f_{yx} f_{y} - f_{yy} f_{t} + f_{xz} f_{z} - f_{yz} f_{z}) = 0. \] (4)

**Step 2.**

We suppose that the generalized bilinear equation (4) has the following solution:

\[ f = a_0 + \sum_{i=1}^{N} \xi_i^{2 \alpha_i} + \sum_{j=1}^{M} m_j g_j(\eta_j), \] (5)

\[ \xi_i = a_{i0} + a_{i1} x + a_{i2} y + a_{i3} z + a_{i4} t, \eta_j = b_{j0} + b_{j1} x + b_{j2} y + b_{j3} z + b_{j4} t, \] (6)

where \( a_{ik}, m_j, b_{jk} (i=1, \ldots, N; j=1, \ldots, M; k=0,1,2,3,4) \) are arbitrary real constants.

To search for the high-order lump-type solutions, three-wave solutions, breather solutions and interaction solutions between the high-order lump-type solution and other function solutions, we suppose

\[ N = 3, M = 4, n_1 = 2, n_2 = 1, n_3 = 1, \]

\[ g_1(\eta_1) = e^{\eta_1}, g_2(\eta_2) = e^{\eta_2}, g_3(\eta_3) = \cos \eta_3, g_4(\eta_4) = \cosh \eta_4. \] (7)

The exact analytical solutions of generalized bilinear equation (4) is written the following form

\[ f = a_0 + \xi_1^{2 \alpha_1} + \xi_2^{2 \alpha_2} + \xi_3^{2 \alpha_3} + m_1 e^{\eta_1} + m_2 e^{\eta_2} + m_3 \cos \eta_3 + m_4 \cosh \eta_4. \] (8)

**Step 3.**
By substituting (8) into Eq.(4) and collecting all terms with the same order of \(x, y, z, t, e^{n}, e^{m}, \sin \eta, \cos \eta, \sinh \eta, \cosh \eta\) together, the left-hand side of Eq.(4) is converted into another polynomial in \(x, y, z, t, e^{n}, e^{m}, \sin \eta, \cos \eta, \sinh \eta, \cosh \eta\). Equating each coefficient of this different power terms to zero yields a set of nonlinear algebraic equations for \(a_{0}, a_{1}, b_{0}, b_{1}, m_{i}\). Solving the algebraic equations by symbolic computation Maple, yields the following sets of solutions. According to different parameter values, we can obtain abundant exact analytical solutions of the generalized (3+1)-dimensional Shallow Water equation (1).

2.1. High-order lump-type solutions, three-wave solutions and Breather solutions

I. High-order lump-type solutions and lump solutions:

When \(m_{j}=0\) \((j=1,2,3,4)\) in (8), the solution (8) represents the high-order lump-type solutions \(f=a_{0}+\zeta_{1}^{4}+\zeta_{2}^{2}+\zeta_{3}^{2}\).

Case 1.1:

\[
a_{11} = 0, a_{13} = -\frac{a_{12}a_{31}}{a_{31}}, a_{14} = 0, a_{21} = 0, a_{22} = -\frac{a_{23}a_{31}}{a_{34}}, a_{24} = 0, a_{32} = 0, a_{33} = 0, a_{34} \neq 0.
\]

Case 1.2:

\[
a_{11} = 0, a_{12} = \frac{-a_{13}a_{32}a_{33}}{a_{33}}, a_{14} = 0, a_{22} = -\frac{a_{23}a_{32}}{a_{21}}, a_{23} = -\frac{a_{23}a_{33}}{a_{21}}, a_{24} = -\frac{a_{23}a_{33}}{a_{32}},
\]

\[
a_{30} = \frac{a_{30}a_{31}}{a_{21}}, a_{34} = -\frac{a_{30}a_{33}}{a_{32}}, a_{21}a_{32}a_{33} \neq 0.
\]

Case 1.3:

\[
a_{11} = 0, a_{14} = 0, a_{22} = -\frac{a_{13}a_{32}}{a_{21}}, a_{23} = -\frac{a_{13}a_{31}a_{32}}{a_{12}a_{21}}, a_{24} = -\frac{a_{13}a_{31}a_{32}}{a_{12}a_{21}}, a_{33} = \frac{a_{13}a_{32}}{a_{12}},
\]

\[
a_{34} = -\frac{a_{13}a_{31}}{a_{12}}, a_{12}a_{21} \neq 0,
\]

where other parameters are arbitrary real constants.

When \(m_{i}=0, a_{1}=0\) \((j=1,2,3,4)\) in (8), we obtain the lump solutions \(f=a_{0}+\zeta_{2}^{2}+\zeta_{3}^{2}\), \(\bar{a}_{0}=a_{0}+a_{10}^{4}\) which are different from those lump solutions given in [14].

Case 1.4:

\[
a_{21} = 0, a_{23} = -\frac{a_{22}(a_{10}^{4}a_{24} + 3a_{31}a_{34} + a_{6}a_{24})}{3a_{31}^{3}}, a_{32} = \frac{a_{22}a_{24}(a_{10}^{4} + a_{0})}{3a_{31}^{3}},
\]

\[
a_{33} = -\frac{a_{22}a_{34}(a_{10}^{4}a_{34} - 3a_{31}^{3} + a_{6}a_{34})}{3a_{31}^{4}}, a_{31} \neq 0.
\]

Case 1.5:

\[
a_{22} = \frac{a_{22}a_{34}(a_{10}^{4} + a_{0})}{3a_{31}^{4}}, a_{23} = -\frac{a_{32}a_{34}(a_{10}^{4}a_{24} - 3a_{31}^{3} + a_{6}a_{34})}{3a_{21}^{4}}, a_{30} = 0, a_{31} = 0,
\]

\[
a_{33} = -\frac{a_{32}a_{34}^{2} + 3a_{31}^{2}a_{34} + a_{6}a_{34}^{2}}{3a_{21}^{4}}, a_{21} \neq 0.
\]

Case 1.6:

\[
a_{23} = \frac{a_{22}a_{33}}{a_{32}} + \frac{3(a_{22}^{2} + a_{32}^{2})(a_{21}^{2} + a_{31}^{2})(a_{21}a_{22} + a_{31}a_{32})}{a_{22}^{2}(a_{10}^{4} + a_{0})(a_{21}a_{22} - a_{31}a_{32})},
\]
\[ a_{24} = -\frac{a_{21}a_{32}}{a_{32}} - \frac{3(a_{2}^2 + a_{3}^2)(a_{24}a_{32} + a_{33})^2}{a_{22}a_{32}^2 + a_{21}a_{32}(a_{21}a_{32} - a_{22}a_{31})}, \quad a_{30} = \frac{a_{20}a_{31}}{a_{21}}, \]
\[ a_{34} = -\frac{a_{21}a_{33}}{a_{32}} + \frac{3(a_{2}^2 + a_{3}^2)(a_{24}a_{33} + a_{31})^2}{a_{22}a_{33}^2 + a_{21}a_{33}(a_{21}a_{33} - a_{22}a_{32})}, \quad a_{32} = \frac{a_{22}a_{33} + a_{24}a_{33}}{a_{31}^2 + a_{32}^2} \neq 0. \]

**Case 1.7:**
\[ a_{0} = -a_{10}^4 + \frac{3a_{22}a_{32}^2}{a_{33}}a_{21}^2 + a_{31}^2, \quad a_{30} = \frac{a_{20}a_{31}}{a_{21}}, \quad a_{32} = 0, a_{22}a_{31}a_{33} \neq 0. \]

**Case 1.8:**
\[ a_{0} = -a_{10}^4 + \frac{3(a_{2}^2 + a_{3}^2)a_{21}^2}{a_{22}a_{32}^2 + a_{21}a_{32}(a_{21}a_{32} - a_{22}a_{31})}, \quad (a_{21}a_{34} - a_{24}a_{31})(a_{21}a_{32} - a_{22}a_{31}) \neq 0, \]
\[ a_{23} = -\frac{a_{21}(a_{22}a_{34} - a_{24}a_{32}) + a_{31}(a_{22}a_{34} + a_{24}a_{32})}{a_{21}^2 + a_{31}^2}, \]
\[ a_{33} = -\frac{a_{21}(a_{22}a_{34} + a_{24}a_{32}) + a_{31}(a_{22}a_{34} - a_{24}a_{32})}{a_{21}^2 + a_{31}^2}, \quad a_{21}^2 + a_{31}^2 \neq 0. \]

**Case 1.9:**
\[ a_{22} = -\frac{a_{21}a_{32}}{a_{21}}, \quad a_{23} = \frac{a_{22}a_{32}}{a_{21}}, \quad a_{33} = -\frac{a_{22}a_{32}}{a_{21}}, \quad a_{33} = \frac{a_{22}a_{32}}{a_{21}}, \quad a_{21} \neq 0, \]

where other parameters are arbitrary real constants.

**II. Three-wave solutions:**

When \( a_{k}=0(i=1,2,3; k=1,2,3,4), \eta_2=-\eta_1, m_2=m_1 \) in (8), the solution (8) represents the three-wave solutions \( f=a_{0}+m_{1}e^{\eta_{1}}+m_{2}e^{-\eta_{1}}+m_{3}\cosh\eta_{3}+m_{4}\cosh\eta_{4}. \)

**Case 2.1:**
\[ b_{12} = b_{13} = b_{31} = b_{34} = b_{42} = b_{43} = 0, b_{32} = -\frac{b_{1}b_{32}}{b_{4}}, \quad b_{41} = b_{1}b_{44}, \quad b_{14} \neq 0. \]

**Case 2.2:**
\[ b_{12} = b_{13} = b_{31} = b_{34} = b_{41} = b_{43} = 0, b_{32} = -\frac{b_{1}b_{32}}{b_{4}}, \quad b_{42} = b_{1}b_{43}, \quad b_{14} \neq 0. \]

**Case 2.3:**
\[ b_{12} = b_{13} = b_{31} = b_{34} = b_{41} = b_{44} = 0, b_{32} = -\frac{b_{1}b_{32}}{b_{12}}, \quad b_{42} = b_{1}b_{43}, \quad b_{14} \neq 0. \]

**Case 2.4:**
\[ b_{12} = b_{13} = b_{31} = b_{34} = b_{41} = b_{44} = 0, b_{32} = -\frac{b_{1}b_{32}}{b_{12}}, \quad b_{42} = b_{1}b_{43}, \quad b_{13}, b_{14} \neq 0. \]

**Case 2.5:**
\[ b_{12} = b_{13} = b_{31} = b_{34} = b_{42} = b_{43} = 0, b_{33} = -\frac{b_{1}b_{33}}{b_{2}}, \quad b_{43} = -\frac{b_{1}b_{43}}{b_{2}}, \quad b_{12} \neq 0. \]

**Case 2.6:**
\[ b_{12} = b_{13} = b_{31} = b_{34} = b_{41} = b_{44} = 0, b_{33} = -\frac{b_{1}b_{33}}{b_{1}}, \quad b_{43} = -\frac{b_{1}b_{43}}{b_{1}}, \quad b_{11} \neq 0. \]

**Case 2.7:**
\[ b_{12} = b_{13} = b_{31} = b_{34} = b_{42} = b_{43} = 0, b_{33} = -\frac{b_{1}b_{33}}{b_{1}}, \quad b_{43} = -\frac{b_{1}b_{43}}{b_{1}}, \quad b_{11} \neq 0, \]

where other parameters are arbitrary real constants.

**III. Breather solutions and solitary wave solutions:**
When \(a_i=0 (i=1,2,3; k=1,2,3,4)\), \(m_3=0\) or \(m_1=0\), \(\eta_i=\eta_3\) in (8), the solution (8) represents the breather solutions \(f=a_0+m_1 e^{\eta}+m_2 e^{\eta}+m_3 \cos \eta_3\) and the solitary wave solutions \(f=a_0+m_1 e^{\eta}+m_2 e^{\eta}+m_3 \cos \eta_3\).

**Case 3.1:** \(b_{11}=b_{14}=b_{21}=b_{24}=b_{32}=b_{33}=0, b_{12}=-\frac{b_{13}b_{31}}{b_{34}}, b_{22}=\frac{b_{23}b_{31}}{b_{34}}, b_{34} \neq 0\).

**Case 3.2:** \(b_{11}=b_{14}=b_{22}=b_{23}=b_{32}=b_{33}=0, b_{13}=-\frac{b_{12}b_{24}}{b_{21}}, b_{34}=\frac{b_{24}b_{31}}{b_{21}}, b_{21} \neq 0\).

**Case 3.3:** \(b_{11}=b_{14}=b_{22}=b_{23}=b_{31}=b_{34}=0, b_{13}=-\frac{b_{12}b_{24}}{b_{21}}, b_{33}=\frac{b_{24}b_{31}}{b_{21}}, b_{21} \neq 0\).

**Case 3.4:** \(b_{12}=b_{13}=b_{21}=b_{24}=b_{31}=b_{34}=0, b_{22}=\frac{b_{13}b_{31}}{b_{34}}, b_{32}=\frac{b_{13}b_{31}}{b_{34}}, b_{14} \neq 0\).

**Case 3.5:** \(a_0=b_{12}=b_{13}=b_{21}=b_{24}=b_{32}=b_{33}=0, b_{11}=\frac{b_{13}b_{31}}{b_{34}}, b_{22}=\frac{b_{23}b_{31}}{b_{34}}, b_{34} \neq 0\).

**Case 3.6:** \(a_0=b_{12}=b_{13}=b_{22}=b_{23}=b_{31}=b_{34}=0, b_{14}=\frac{b_{13}b_{31}}{b_{34}}, b_{24}=\frac{b_{23}b_{31}}{b_{34}}, b_{32} \neq 0\),

where other parameters are arbitrary real constants.

**Remark 2.1:** In addition to Case 3.1-3.6, we can get the special results of Case 2.2.2.4 when \(b_{43}=0\).

### 2.2 Collisions between lump and soliton solutions:

**I. Between lumps and a pair of line soliton solution:**

When \(m_1=0, m_2=0\) in (8), the solution (8) represents the interaction solutions between the high-order lump-type solution and a pair of line soliton solution \(f=a_0+\eta_1^2+\eta_2^2+\eta_3^2+m_1 e^\eta+m_2 e^{\eta}\).

(1) When \(m_2=m_1\), the interaction solution is \(f=a_0+\eta_1^2+\eta_2^2+\eta_3^2+m_1 e^\eta+m_2 e^{\eta}\).

**Case 4.1:** \(a_{11}=0, a_{13}=-\frac{a_{12}a_{24}}{a_{21}}, a_{14}=a_{22}=a_{23}=a_{31}=0, a_{33}=-\frac{a_{31}a_{32}}{a_{34}}, a_{34}=b_{12}=b_{13}=0\),

\[ b_{14}=\frac{a_{24}b_{11}}{a_{21}}, b_{22}=b_{23}=0, b_{24}=\frac{a_{13}b_{21}}{a_{12}}(a_{21} \neq 0) \]

**Case 4.2:** \(a_{11}=a_{14}=0, a_{21}=\frac{a_{13}a_{34}}{a_{13}}, a_{22}=a_{23}=0, a_{31}=-\frac{a_{31}a_{33}}{a_{34}}, a_{32}=0, a_{33}=b_{12}=b_{13}=0\),

\[ b_{14}=\frac{a_{13}b_{11}}{a_{12}}, b_{22}=b_{23}=0, b_{24}=\frac{a_{13}b_{21}}{a_{12}}(a_{12}a_{13} \neq 0) \]

**Case 4.3:** \(a_{10}=a_{11}=0, a_{12}=-\frac{a_{13}a_{31}}{a_{34}}, a_{14}=a_{21}=a_{22}=a_{23}=a_{31}=a_{32}=a_{33}=b_{12}=b_{13}=0\),

\[ b_{14}=\frac{a_{34}b_{11}}{a_{31}}, b_{22}=b_{23}=0, b_{24}=\frac{a_{34}b_{21}}{a_{31}}(a_{31}a_{34} \neq 0) \]

**Case 4.4:** \(a_{11}=a_{14}=a_{21}=a_{22}=a_{23}=a_{24}=a_{31}=0, a_{33}=-\frac{a_{13}a_{12}}{a_{12}}, a_{34}=b_{12}=b_{13}=0\),

\[ b_{14}=\frac{a_{13}b_{11}}{a_{12}}, b_{22}=b_{23}=0, b_{24}=\frac{a_{13}b_{21}}{a_{12}}(a_{12} \neq 0) \]

where other parameters are arbitrary real constants.
(2) When \( m_2=m_1, \eta_2=-\eta_1 \), the interaction solution is \( f=a_0+\zeta_1^4+\zeta_2^2+\zeta_3^2+m_1 e^{\eta}+m_1 e^{-\eta} \).

**Case 4.5:** \( a_{11}=0, a_{13}=-\frac{a_{22}a_{34}}{a_{31}}, a_{14}=0, a_{21}=-\frac{a_{13}a_{32}}{a_{22}}, a_{23}=-\frac{a_{32}a_{34}}{a_{31}}, a_{24}=-\frac{a_{22}a_{34}}{a_{21}}, a_{33}=-\frac{a_{32}a_{34}}{a_{31}}, b_{12}=b_{13}=0, b_{14}=-\frac{a_{23}b_{11}}{a_{34}}(a_{12}a_{31} \neq 0) \)

**Case 4.6:** \( a_{11}=a_{14}=a_{21}=0, a_{23}=\frac{a_{13}a_{22}}{a_{12}}, a_{24}=a_{31}=0, a_{32}=\frac{a_{13}a_{31}}{a_{12}}, a_{34}=b_{12}=b_{13}=0, b_{14}=-\frac{a_{13}b_{11}}{a_{12}}(a_{12}a_{31} \neq 0) \)

**Case 4.7:** \( a_{11}=0, a_{13}=-\frac{a_{21}a_{34}}{a_{31}}, a_{14}=a_{21}=0, a_{22}=-\frac{a_{21}a_{31}}{a_{34}}, a_{24}=a_{32}=a_{33}=b_{11}=0, a_{34}=-\frac{a_{21}b_{12}}{a_{31}}, b_{14}=0, (a_{31}a_{34} \neq 0) \)

**Case 4.8:** \( a_{11}=0, a_{12}=-\frac{a_{13}a_{11}}{a_{34}}, a_{14}=a_{21}=a_{23}=a_{24}=a_{32}=a_{33}=b_{12}=b_{13}=0, b_{14}=-\frac{a_{32}b_{11}}{a_{31}}(a_{31}a_{34} \neq 0) \)

where other parameters are arbitrary real constants.

**Remark 2.2:** In addition to Case 4.5-4.8, we can get the special results of Case 4.1,4.2,4.4 when \( b_{21}=b_{11} \).

II. Between lumps and one line-soliton solution:

When \( m_3=0, m_4=0 \) and \( m_2=0 \) or \( m_1=0, \eta_2=-\eta_1 \) in (8), the solution (8) represents the interaction solutions between the high-order lump-type soliton and one line-soliton solution \( f=a_0+\zeta_1^4+\zeta_2^2+\zeta_3^2+m_1 e^{\eta} \) or \( f=a_0+\zeta_1^4+\zeta_2^2+\zeta_3^2+m_2 e^{-\eta} \).

**Case 5.1:** \( a_{11}=a_{14}=0, a_{22}=-\frac{a_{33}a_{32}}{a_{21}}, a_{23}=-\frac{a_{33}a_{31}}{a_{21}}, a_{24}=-\frac{a_{33}a_{21}}{a_{22}}, a_{33}=\frac{a_{33}a_{21}}{a_{22}}, b_{11}=0, b_{13}=\frac{a_{12}b_{12}}{a_{12}}, b_{14}=0, (a_{12}a_{21} \neq 0) \)

where other parameters are arbitrary real constants.

**Remark 2.3:** In addition to Case 5.1, we can get the same results as Case 4.7,4.8 and the special results of Case 4.1,4.2,4.4 when \( b_{21}=b_{11} \).

III. Between lumps and three-wave solution:

When \( m_3=m_4(i=2,3,4), m_3 \neq 0, m_4 \neq -\eta_1 \) in (8), the solution (8) represents the interaction solutions between the high-order lump-type soliton and three-wave solution \( f=a_0+\zeta_1^4+\zeta_2^2+\zeta_3^2+m_1 e^{\eta}+m_1 e^{-\eta}+m_2 \cos \eta \eta_1+m_2 \cos \eta \eta_4 \).

**Case 6.1:** \( a_{11}=0, a_{13}=-\frac{a_{24}a_{21}}{a_{21}}, a_{14}=a_{23}=a_{31}=0, a_{33}=-\frac{a_{32}a_{22}}{a_{21}}, a_{34}=b_{12}=b_{13}=0, b_{14}=-\frac{a_{23}b_{11}}{a_{21}}(a_{21} \neq 0) \).
Case 6.2: \( a_{10} = a_{11} = a_{14} = 0, a_{21} = -\frac{a_{12}a_{24}}{a_{13}}, a_{22} = a_{23} = 0, a_{31} = -\frac{a_{13}a_{34}}{a_{13}}, a_{32} = a_{33} = b_{12} = 0, b_{13} = 0, b_{14} = -\frac{a_{14}b_{11}}{a_{12}}, b_{32} = b_{33} = 0, b_{34} = -\frac{a_{14}b_{31}}{a_{12}}, b_{42} = b_{43} = 0, b_{44} = -\frac{a_{14}b_{41}}{a_{12}} \) \((a_{12}a_{13} \neq 0)\)

Case 6.3: \( a_{11} = a_{14} = a_{21} = a_{22} = a_{23} = a_{24} = 0, a_{31} = a_{12}a_{12}, a_{34} = b_{12} = b_{13} = 0, b_{14} = -\frac{a_{14}b_{11}}{a_{12}}, b_{32} = b_{33} = 0, b_{34} = -\frac{a_{14}b_{31}}{a_{12}}, b_{42} = b_{43} = 0, b_{44} = -\frac{a_{14}b_{41}}{a_{12}} \) \((a_{12} \neq 0)\)

Case 6.4: \( a_{11} = 0, a_{12} = -\frac{a_{12}a_{31}}{a_{34}}, a_{14} = a_{21} = a_{22} = a_{23} = a_{24} = a_{32} = a_{33} = b_{12} = b_{13} = 0, b_{14} = \frac{a_{34}b_{11}}{a_{31}}, b_{31} = b_{32} = b_{33} = 0, b_{34} = \frac{a_{34}b_{31}}{a_{31}}, b_{42} = b_{43} = 0, b_{44} = \frac{a_{34}b_{41}}{a_{31}} \) \((a_{12}a_{34} \neq 0)\)

Case 6.5: \( a_{11} = 0, a_{13} = -\frac{a_{12}a_{34}}{a_{31}}, a_{14} = a_{21} = 0, a_{22} = -\frac{a_{23}a_{31}}{a_{34}}, a_{24} = a_{32} = a_{33} = b_{11} = 0, b_{13} = -\frac{a_{34}b_{31}}{a_{31}}, b_{14} = b_{31} = b_{32} = b_{33} = 0, b_{34} = -\frac{a_{34}b_{34}}{a_{31}}, b_{44} = 0 \) \((a_{12}a_{34} \neq 0)\)

Case 6.6: \( a_{11} = a_{14} = 0, a_{21} = -\frac{a_{12}a_{23}a_{34}}{a_{23}a_{32}}, a_{23} = a_{13}a_{23}, a_{24} = -\frac{a_{12}a_{34}}{a_{22}}, a_{31} = a_{12}a_{34}, a_{32} = a_{33}, b_{12} = b_{13} = 0, b_{14} = -\frac{a_{13}b_{11}}{a_{12}}, b_{31} = b_{32} = b_{33} = 0, b_{34} = -\frac{a_{13}b_{31}}{a_{12}}, b_{42} = 0, b_{44} = -\frac{a_{13}b_{41}}{a_{12}} \) \((a_{12}a_{13}a_{22} \neq 0)\)

Case 6.7: \( a_{11} = 0, a_{13} = -\frac{a_{12}a_{34}}{a_{31}}, a_{14} = 0, a_{21} = -\frac{a_{13}a_{32}}{a_{31}}, a_{23} = -\frac{a_{12}a_{34}}{a_{31}}, a_{24} = -\frac{a_{12}a_{34}}{a_{22}}, a_{30} = \frac{a_{20}a_{32}}{a_{22}}, a_{33} = -\frac{a_{12}a_{34}}{a_{31}}, b_{2} = b_{3} = 0, b_{4} = \frac{a_{34}b_{11}}{a_{31}}, b_{31} = b_{32} = b_{33} = 0, b_{34} = -\frac{a_{13}b_{31}}{a_{12}}, b_{42} = b_{43} = 0 \) \((a_{22}a_{31} \neq 0)\)

Case 6.8: \( a_{11} = a_{14} = a_{21} = 0, a_{23} = -\frac{a_{12}a_{23}}{a_{12}}, a_{24} = 0, a_{30} = -\frac{a_{12}a_{20}a_{33}}{a_{13}a_{22}}, a_{31} = 0, a_{32} = a_{12}a_{33}, a_{33} = a_{13}a_{33}, a_{34} = b_{12} = b_{13} = 0, b_{14} = -\frac{a_{13}b_{11}}{a_{12}}, b_{31} = b_{32} = b_{33} = 0, b_{34} = -\frac{a_{13}b_{31}}{a_{12}}, b_{42} = b_{43} = 0, b_{44} = -\frac{a_{13}b_{41}}{a_{12}} \) \((a_{12}a_{13}a_{22} \neq 0)\)

where other parameters are arbitrary real constants.

IV. Between lumps and breather solution (or solitary wave solution):

When \( \eta_2 = -\eta_1, m = m_1 (i=2,3), m_4 = 0 \) (or \( m = m_1 (i=2,4), m_3 = 0, \eta_1 = \eta_3 \)) in (8), the solution (8) represents the interaction solutions between the high-order lump-type solution and breather solution \( f = a_0 + \xi_1^4 + \xi_2^2 + \xi_3^2 + m_1 e^{\eta_1} + m_1 e^{-\eta_1} + m_1 \cos \eta_1 \) or the solitary wave solution \( f = a_0 + \xi_1^4 + \xi_2^2 + \xi_3^2 + m_1 e^{\eta_1} + m_1 e^{-\eta_1} + m_1 \cos \eta_1 \).
Case 7.1: \( a_{11} = a_{41} = 0, a_{21} = -\frac{a_{12}a_{34}}{a_{13}}, a_{22} = a_{23} = 0, a_{31} = -\frac{a_{12}a_{34}}{a_{13}}, a_{32} = a_{33} = b_{12} = b_{13} = 0, \)
\[ b_{14} = -\frac{a_{14}b_{11}}{a_{12}}, b_{32} = b_{33} = 0, b_{34} = -\frac{a_{14}b_{11}}{a_{12}}, (a_{12}a_{13} \neq 0) \]

Case 7.2: \( a_{11} = 0, a_{12} = -\frac{a_{14}a_{31}}{a_{34}}, a_{44} = a_{21} = a_{22} = a_{24} = a_{32} = a_{33} = b_{12} = b_{13} = 0, \)
\[ b_{34} = \frac{a_{34}b_{11}}{a_{31}}, b_{32} = b_{33} = 0, b_{34} = -\frac{a_{34}b_{11}}{a_{31}}, (a_{34}a_{31} \neq 0) \]

where other parameters are arbitrary real constants.

**Remark 2.4:** In addition to Case 7.1,7.2, we can get the special results of Case 6.1,6.3,6.5-6.8 as long as \( b_{41}=0. \)

V. Between lumps and cosh periodic wave solution:

When \( m=0(i=1,2), m \neq 0 \) in (8), the solution (8) represents the interaction solutions between the high-order lump-type solution and solitary wave solution \( f=a_0+\zeta_1^4+\zeta_2^4+\zeta_3^4+m_4\cos\eta_4+m_4\cosh\eta_4. \)

Case 8.1: \( a_{10} = a_{11} = 0, a_{13} = \frac{a_{12}a_{33}}{a_{23}}, a_{14} = 0, a_{21} = -\frac{a_{24}a_{52}}{a_{33}}, a_{22} = a_{23} = a_{31} = a_{34} = b_{12} = b_{33} = 0, \)
\[ b_{34} = -\frac{a_{34}b_{11}}{a_{32}}, b_{42} = b_{43} = 0, b_{44} = -\frac{a_{34}b_{11}}{a_{32}}, (a_{34}a_{32} \neq 0) \]

Case 8.2: \( a_{10} = a_{11} = a_{14} = 0, a_{21} = \frac{a_{12}a_{35}a_{44}}{a_{13}a_{22}}, a_{23} = a_{24} = -\frac{a_{22}a_{34}}{a_{12}}, a_{31} = -\frac{a_{12}a_{34}}{a_{13}}, \)
\[ a_{33} = \frac{a_{33}a_{32}}{a_{12}}, b_{32} = b_{33} = 0, b_{34} = -\frac{a_{33}b_{11}}{a_{12}}, b_{42} = b_{43} = 0, b_{44} = -\frac{a_{33}b_{11}}{a_{12}}, (a_{12}a_{13}a_{14} \neq 0) \]

Case 8.3: \( a_{10} = a_{11} = 0, a_{12} = -\frac{a_{14}a_{31}}{a_{34}}, a_{14} = 0, a_{21} = -\frac{a_{24}a_{32}}{a_{34}}, a_{22} = a_{23} = -\frac{a_{22}a_{34}}{a_{14}}, a_{24} = -\frac{a_{24}a_{32}}{a_{34}}, \)
\[ a_{30} = \frac{a_{30}a_{32}}{a_{12}}, a_{33} = -\frac{a_{32}a_{34}}{a_{31}}, b_{32} = b_{33} = 0, b_{34} = -\frac{a_{33}b_{11}}{a_{31}}, b_{42} = b_{43} = 0, b_{44} = -\frac{a_{33}b_{11}}{a_{31}}, (a_{24}a_{31}a_{34} \neq 0) \]

where other parameters are arbitrary real constants.

**Remark 2.5:** In addition to Case 8.1-8.3, we can get the special results of Case 4.3 (\( \eta_3=\eta_1, \eta_4=\eta_2 \), Case 5.1 (\( \eta_3=\eta_1=\eta_2, a_{10}=0 \), Case 6.2 (\( b_{11}=0 \)), Case 6.5 (\( b_{12}=0 \)) and Case 9.1-9.2 (\( \eta_4=\eta_3 \)).

VI. Between lumps and cos periodic wave solution (or hyperbolic function solution):

When \( m=0(i=1,2,4), m \neq 0 \) (or \( m=0(i=1,2,3), m \neq 0, \eta_4=\eta_3 \)) in (8), the solution (8) represents the interaction solutions between the high-order lump-type solution and cos periodic wave solution \( f=a_0+\zeta_1^4+\zeta_2^2+\zeta_3^4+m_3\cos\eta_3+m_3\cosh\eta_3 \) or solitary wave solution \( f=a_0+\zeta_1^4+\zeta_2^2+\zeta_3^4+m_3\cosh\eta_3 \).

Case 9.1: \( a_{10} = a_{11} = 0, a_{13} = \frac{a_{12}a_{33}}{a_{23}}, a_{14} = 0, a_{21} = -\frac{a_{24}a_{32}}{a_{21}}, a_{22} = a_{23} = -\frac{a_{31}a_{33}}{a_{21}}, a_{24} = -\frac{a_{24}a_{32}}{a_{32}}, \)
\[ a_{30} = \frac{a_{30}a_{31}}{a_{21}}, a_{33} = -\frac{a_{32}a_{34}}{a_{31}}, b_{32} = b_{33} = 0, b_{34} = -\frac{a_{33}b_{11}}{a_{31}}, b_{42} = b_{43} = 0, b_{44} = -\frac{a_{33}b_{11}}{a_{31}}, (a_{24}a_{31}a_{34} \neq 0) \]

Case 9.2: \( a_{10} = a_{11} = a_{14} = a_{22} = a_{23} = 0, a_{24} = -\frac{a_{14}a_{21}}{a_{12}}, a_{30} = -\frac{a_{24}a_{30}a_{34}}{a_{13}a_{21}}, a_{33} = -\frac{a_{14}a_{34}}{a_{13}} \),
\[ a_{32} = a_{33} = b_{31} = 0, b_{33} = \frac{a_{13} b_{32}}{a_{12}}, b_{34} = 0, \quad (a_{12} a_{33} a_{21} \neq 0) \]

where other parameters are arbitrary real constants.

**Remark 2.6:** In addition to Case 9.1-9.2, we can get the special results of Case 4.3 \((\eta_3 = \eta_1, b_{21}=0),\) Case 5.1 \((\eta_3 = \eta_1, a_{10}=0),\) Case 6.2 \((b_{11}=b_{41}=0),\) Case 6.5 \((a_{10}=b_{11}=b_{41}=0)\) and Case 8.1-8.3 \((b_{41}=0).\)

**Step 4.** By substituting the parameters \(a_{10}, a_{11}, b_{8}, m_j\) in Case 1.1-Case 9.2 into the expression (8) and using the transformation (2), we can obtain abundant exact analytical solutions and novel interaction phenomena of the generalized (3+1)-dimensional Shallow Water equation (1).

As the example, substituting the results of Case 6.1 to the expression (8), we can get the exact solution \(f\) of the generalize bilinear Shallow Water equation (4) as follow

\[
f_x = a_0 + \left( a_{12} y - \frac{a_{12} a_{24}}{a_{21}} z + a_{10} \right) x + \left( a_{21} x + a_{22} t + a_{20} \right) y + \left( a_{32} y - \frac{a_{32} a_{24}}{a_{21}} z + a_{30} \right) z + m_1 \exp(b_{11} x + \frac{a_{23} b_{11}}{a_{21}} t + b_{10}) + m_2 \exp(-b_{11} x - \frac{a_{23} b_{11}}{a_{21}} t - b_{10}) + m_3 \cos(b_{31} x + \frac{a_{23} b_{31}}{a_{21}} t + b_{30}) + m_4 \cosh(b_{41} x + \frac{a_{23} b_{41}}{a_{21}} t + b_{40}).
\]

By using the transformation (2), we get the following interaction solution between the high-order lump-type solution and three-wave solution of the generalize Shallow Water equation (1)

\[
u(x, y, z, t) = \frac{2 f_x(x, y, z, t)}{f(x, y, z, t)},
\]

where \(f(x, y, z, t)\) is given in (9).

In order to exhibit the dynamical characteristics of these waves, we can plot various three-dimensional, contour and density plots. These Figures can show the physical properties, structures and the energy distribution for the exact solutions (10). The phenomenon of the interaction solution is very strange and analogous to rogue wave. The process of interaction changes the amplitudes, shapes and velocities of both waves. This type of interaction solutions provide a method to forecast the appearance of rogue waves, such as financial rogue wave, optical rogue wave and plasma rogue wave, through analyzing the relations between lump wave part and soliton wave part.

### 3. Conclusion

In this paper, we gave a novel form of exact analytical solution to the generalized (3+1)-dimensional Shallow Water equation. To search for various kinds of exact analytical solutions, we are free to choose the values of \(N, M\) and the basis function \(g(\eta)\) in the expression (5), such as, lump solution \((N=2, n_1=n_2=1, m=0);\) lump-type solution \((N=3, n_1=n_2=n_3=1, m=0);\) high-order lump solution \((N=2, n_1=n_2=2, m=0);\) high-order lump-type solution \((N=3, n_1=2, n_2=n_3=1, m=0);\) lump-kink solution \((N=2, n_1=n_2=1; M=1, g(\eta)=e^\eta);\) lump-soliton solution \((N=2, n_1=n_2=1; M=1, g(\eta)=\cosh\eta,\) etc.

As the example, by choosing the basis functions \(g_1(\eta_1)=e^{\eta_1}, g_2(\eta_2)=e^{\eta_2}, g_3(\eta_3)=\cos\eta_3, g_4(\eta_4)=\cosh\eta_4,\) we successfully constructed abundant exact analytical solutions of the generalized (3+1)-dimensional Shallow Water equation based on the generalized bilinear method, and these solutions contained the high-order lump-type solutions, the three-wave solutions, the breather solutions, the interaction solution between high-order lump-type solutions and soliton solutions. These solutions will
greatly expand the exact solutions of the generalized (3+1)-dimensional Shallow Water equation on the existing literature [14]. These new solutions are significant to understand the propagation processes for nonlinear waves in fluid mechanics and the explanation of some special physical problems.

By using the expression (5), we can construct other rational solution and their interaction solution to the generalized (3+1)-dimensional Shallow Water equation. Such as, when \( g_1(\eta_1)=e^{\eta_1}, \) \( g_2(\eta_1)=e^{\eta_1}, \) \( g_3(\eta_1)=\sin\eta_1, \) \( g_4(\eta_1)=\sinh\eta_1 \) or \( g_3(\eta_1)=\tan\eta_1, \) \( g_4(\eta_1)=\tanh\eta_1, \) we can obtain the interaction solution between rational solution and soliton solutions, periodic wave solution. But due to the lack of space, we will discuss these solutions in another paper. The method can be used for many other NLEEs in mathematical physics.

4. Acknowledgments

This work is supported by the National Natural Science Foundation of China (11661060) and the Natural Science Foundation of Inner Mongolia Autonomous Region of China (2018LH01013).

5. References


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**Paper submitted: November 23, 2019**

**Revised: July 3, 2020**

**Accepted: July 5, 2020**