# NEW ANALYTICAL SOLUTIONS OF HEAT TRANSFER FLOW OF CLAY-WATER BASE NANOPARTICLES WITH THE APPLICATION OF NOVEL HYBRID FRACTIONAL DERIVATIVE

by

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*Clay nanoparticles are hanging in three different based fluids (water, kerosene, and engine oil). The exact terminologies of Maxwell-Garnett and Brinkman for the current thermophysical properties of clay nanofluids are used, while the flow occurrence is directed by a set linear PDE with physical initial and boundary conditions. The classical governing equations are extended to non-integer order hybrid fractional derivative which is introduced in [33]. Analytical solutions for temperature and velocity fields are attained via Laplace transform technique. Some limiting solutions are also obtained from the existing literature and compared for different values of fractional parameter. To vision the impact of several flow parameters on the temperature and velocity some graphs are drawn using Mathcad software and designed in different figures. As a result, we found that hybrid fractional model is better in describing the decay behavior of temperature and velocity in comparison of classical derivatives. In comparison of nanofluid with different base fluids, it is concluded that water-based nanofluid has higher velocity than others.*

Key words: *hybrid fractional derivative, power law kernel, Clay-nanoparticles, analytical solutions*

## Introduction

First of all, Choi [1] offered the ideas of nanofluid. Nanofluids are the interruptions of nanometer sized elements of metals, oxides, silica, carbon nanotubes, *etc*. Recently, the learning of nanofluids developed a burning zone of investigation for the technologists and manufacturer to stunned the heat transfer difficulties in manufacturing areas. Heat transfer procedures play exciting role in several manufacturing areas. Together, the manufacturers are complex to discover the thermophysical properties of different nanoparticles and base fluids by various approaches as nanofluid is measured as a succeeding bearing of fluid as a standard for heat transfer which may suggestion additional thermal act in various manufacturing areas which

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comprise control genesis, hyperthermia, carriage, microfabrication, drying, and air-cooling system. For detail see the references [2-28].

Recently, Khan *et al*. [29] studied the suspension of clay nanoparticles for cleaning water and heat transfer of drilling nanofluids. He obtained the exact solution of classical model of governing equation with Laplace transform method and made comparison between different base fluid. Boring fluids are weighty in the boring approaches of gases and oils from stuns and acreage. To advance the act of fractional calculus represents the more general differential calculus which employs the derivatives/integrals of non-integer order. Not long ago the differential calculus of non-integer order was found to be more appropriate to modelling of many practical processes [30]. In many areas, such as bioengineering, viscoelasticity, polymer chemistry, the fractional calculus provides much adequate models to physical systems than the models described by the classical ordinary differential equations. It is known that the memory and hereditary properties of the materials are described by the time-fractional and space-fractional derivatives [31]. Mathematical models illustrated by the fractional differential equations are used in electrical circuits, fractal theory, electromagnetic theory, *etc*. [32].

In 2020 a new kind of fractional operator with power law recently suggested by Baleanu *et al*. [33] and called it a hybrid fractional derivatives because it is linear combination of two fractional operators known as constant proportional and Caputo type fractional derivative.

For the moment, there is no single result in the literature discussed with this new operator. Therefore, our interest here to apply the most recent fractional operator namely hybrid fractional operator to the heat transfer of caly-water base nanofluids over an infinite vertical surface moving with constant velocity. In this paper, we show that which operator is better in exhibiting decay of the velocity of the fluid. We have plotted some graphs for fractional parameters for small and large time. Also, drawn some comparison between present and classical fractional derivatives and presented in graphically.

## Mathematical formulation

Considering the movement of unsteady motion of viscous fluid with clay nanoparticles over an infinite vertical plate situated in the region $y > 0$ in the plane OXY with $y$-axis normal to it and $x$-axis along the plate. We have taken water, kerosene and engine oils as base fluids. The plate is static along $x$-axis and $y$-axis is taken normal to it. At the beginning, the nanofluid at the geometry has fixed temperature $T_\infty$. After passing some time, the fluid temperature changes from $T_\infty$ to $T_w$ and surface starts motion constantly. At this point, the fluid begins motion in its plane due to the motion of the surface and difference in temperature. In this way the heat transfer process occurs and partial differential equations that govern the flow are given [29]:

Thermo-physical properties are defined in [29]:

$$\rho_{\rm nf}\partial_t V(y,t) - \mu_{\rm nf}\partial_{yy}V(y,t) - {\rm g}(\rho\beta_T)_{\rm nf}[T(y,t) - T_\infty] = 0 \tag{1}$$

$$\frac{(\rho C_p)_{\rm nf}}{K_{\rm nf}}\partial_t T(y,t) - \partial_{yy}T(y,t) = 0 \tag{2}$$

with the set of initial and boundary conditions are:

$$V(y,0) = 0,\ \ T(y,0) = T_\infty,\ \ \text{for all}\ \ y \ge 0 \tag{3}$$

$$V(0,t) = V_0 H(t),\ \ T(0,t) = T_w,\ \ t > 0 \tag{4}$$

$$V(\infty,t) \to 0,\ \ T(\infty,t) \to T_\infty,\ \ t > 0 \tag{5}$$

Thermophisical properties are defined in [29] as:

$$(\rho C_p)_{\rm nf} - (1-\phi)(\rho C_p)_f + \phi(\rho C_p)_s = 0,\ \rho_{\rm nf} - (1-\phi)\rho_f + \phi\rho_s = 0,\ \frac{\mu_{\rm nf}}{\mu_f} - \frac{1}{(1-\phi)^{2.5}} = 0$$

$$K_{\rm nf}\left[K_s + 2K_f + 2\phi(K_f - K_s)\right] - K_f\left[K_s + 2K_f - 2\phi(K_f - K_s)\right] = 0$$

**Table A. Thermophysical properties of nanofluids [29]**

| Material | Symbol | $\rho$ [kgm$^{-3}$] | $C_p$ [Jkgm$^{-1}$K$^{-1}$] | $K$ [Wm$^{-1}$K$^{-1}$] | $\beta$ [10$^{-1}$K$^{-1}$] | Pr |
|---|---|---|---|---|---|---|
| Clay | Nanoparticles | 6320 | 531.8 | 76.5 | 1.80 | – |
| Water | $H_2O$ | 997 | 4179 | 0.613 | 21 | 6.2 |
| Kerosene oil | KO | 783 | 2090 | 0.145 | 99 | 21 |
| Engine oil | EO | 884 | 1910 | 0.114 | 70 | 500 |

We took the dimensionless variables:

$$y^* = \frac{V_o}{\nu_f}y,\ t^* = \frac{V^2}{\nu_f}t,\ v^* = \frac{V}{V_o},\ \psi^* = \frac{T - T_\infty}{T_w - T_\infty} \tag{6}$$

We introduced some variables in non-dimensional form into eqs. (1)-(5) and get the expressions for temperature and velocity free from flow regime:

$$\partial_t v(y,t) - a_6 \partial_{yy} v(y,t) - a_7 \text{Gr}\,\psi(y,t) = 0 \tag{7}$$

$$\partial_t \psi(y,t) - a_5 \partial_{yy}\psi(y,t) = 0 \tag{8}$$

with conditions

$$v(y,0) = 0,\ \psi(y,0) = 0,\ \forall y \ge 0 \tag{9}$$

$$v(0,t) = H(t),\ \psi(0,t) = 1,\ t > 0 \tag{10}$$

$$v(y,t) \to 0,\ \psi(y,t) \to T_\infty,\ \text{as}\, y \to \infty,\ t > 0 \tag{11}$$

where

$$a_0 = \left[(1-\phi) + \phi\frac{\rho_s}{\rho_f}\right],\ a_1 = \left[\frac{1}{(1-\phi)^{2.5}}\right],\ a_2 = \left[(1-\phi) + \phi\frac{(\rho\beta_T)_s}{(\rho\beta_T)_f}\right]$$

$$a_6 = \frac{a_1}{a_0},\ a_7 = \frac{a_2}{a_0}, a_3 = \left[(1-\phi) + \phi\frac{(\rho C_p)_s}{(\rho C_p)_f}\right],\ a_4 = \frac{K_{\rm nf}}{K_f},\ a_5 = \frac{a_4}{a_3\,\text{Pr}} \tag{12}$$

$$\text{Pr} = \frac{(\mu C_p)_f}{K_f},\ \text{Gr} = \frac{g(\beta)_f (T_w - T_\infty)}{U_o^3}$$

## Fractional model and its solution

In this section we will develop fractional model of the physical problem given in eqs. (7) and (8):

$$^{\rm CPC}D_t^\alpha v(y,t) = a_6\frac{\partial^2 v(y,t)}{\partial y^2} + a_7 \text{Gr}\,\psi(y,t) \tag{13}$$

$$ {}^{\mathrm{CPC}}D_t^{\alpha}\psi(y,t) = a_5 \frac{\partial^2 \psi(y,t)}{\partial y^2} \tag{14} $$

Where ${}^{\mathrm{CPC}}D_t^{\alpha}v(y, t)$ is the constant proportional-Caputo hybrid derivative and is defined as [33]. Taking Laplace transform of eqs. (13) and (14) and eqs. (9)-(11), we have:

$$ \left[\frac{K_1(\alpha)}{s} + K_0(\alpha)\right] s^{\alpha}\bar{v}(y,s) - a_6\partial_{yy}\bar{v}(y,s) - a_7\mathrm{Gr}\bar{\psi}(y,s) = 0 \tag{15} $$

$$ \left[\frac{K_1(\alpha)}{s} + K_0(\alpha)\right] s^{\alpha}\bar{\psi}(y,s) - a_5\partial_{yy}\bar{\psi}(y,s) = 0 \tag{16} $$

subject to the following conditions:

$$ \bar{v}(0,s) = \frac{1}{s},\ \bar{\psi}(0,s) = \frac{1}{s} \tag{17a} $$

$$ \bar{v}(\infty,s) = 0,\ \bar{\psi}(\infty,s) = 0 \tag{17b} $$

The solution of eq. (16) using conditions from eqs. $(17a)_2$-$(17b)_2$:

$$ \bar{\psi}(y,s) = \frac{1}{s}\mathrm{e}^{-y\sqrt{\frac{1}{a_5}\left[\frac{K_1(\alpha)}{s}+K_0(\alpha)\right]s^{\alpha}}} \tag{18} $$

The expression appear in eq. (18) in exponential form is complicated and difficult to obtain analytically, so we express the this form in its equivalent form:

$$ \bar{\psi}(y,s) = \frac{1}{s} + \sum_{p=1}^{\infty}\sum_{q=0}^{\infty} \frac{(-y)^p\left[K_1(\alpha)\right]^q}{p!q!\left(\sqrt{a_5}\right)^p\left[K_0(\alpha)\right]^{q-\frac{p}{2}} s^{1+q-\frac{\alpha p}{2}}} \frac{\Gamma\left(\frac{p}{2}+1\right)}{\Gamma\left(\frac{p}{2}+1+q\right)} \tag{19} $$

By applying inverse Laplace transform of eq. (19) we have:

$$ \psi(y,t) = 1 + \sum_{p=1}^{\infty}\sum_{q=0}^{\infty} \frac{(-y)^p\left[K_1(\alpha)\right]^q}{p!q!\left(\sqrt{a_5}\right)^p\left[K_0(\alpha)\right]^{q-\frac{p}{2}}} \frac{t^{q-\frac{\alpha p}{2}}\Gamma\left(\frac{p}{2}+1\right)}{\Gamma\left(1+q-\frac{\alpha p}{2}\right)\Gamma\left(\frac{p}{2}+1-q\right)} \tag{20} $$

The solution of eq. (15) using conditions from eqs. $(17a)_1$-$(17b)_1$:

$$ \begin{aligned} \bar{v}(y,s) = {} & \frac{1}{s}\mathrm{e}^{-y\sqrt{\frac{1}{a_6}\left[\frac{K_1(\alpha)}{s}+K_0(\alpha)\right]s^{\alpha}}} + \left(\frac{a_7a_5\mathrm{Gr}}{s\left\{(a_6-a_5)\left[\frac{K_1(\alpha)}{s}+K_0(\alpha)\right]s^{\alpha}\right\}}\right)\cdot \\ & \cdot\mathrm{e}^{-y\sqrt{\frac{1}{a_6}\left[\frac{K_1(\alpha)}{s}+K_0(\alpha)\right]s^{\alpha}}} - \left(\frac{a_7a_5\mathrm{Gr}}{s\left\{(a_6-a_5)\left[\frac{K_1(\alpha)}{s}+K_0(\alpha)\right]s^{\alpha}\right\}}\right)\mathrm{e}^{-y\sqrt{\frac{1}{a_5}\left[\frac{K_1(\alpha)}{s}+K_0(\alpha)\right]s^{\alpha}}} \end{aligned} \tag{21} $$

By writing eq. (21) in suitable form:

$$\bar{v}(y,s)=\frac{1}{s}+\sum_{l=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-y)^l\left[K_1(\alpha)\right]^m}{l!m!\left(\sqrt{a_5}\right)^p\left[K_0(\alpha)\right]^{m-\frac{1}{2}}s^{1+m-\frac{\alpha l}{2}}}\frac{\Gamma\left(\frac{l}{2}+1\right)}{\Gamma\left(\frac{l}{2}+1-m\right)}+\frac{a_7a_2\mathrm{Gr}}{a_6-a_5}\cdot$$

$$\cdot\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-y)^l(-1)^n\left[K_1(\alpha)\right]^{m+n}}{l!m!\left[K_0(\alpha)\right]^{m-\frac{l}{2}+n+1}s^{1+m-\frac{\alpha l}{2}+n}}\frac{\Gamma\left(\frac{l}{2}+1\right)}{\Gamma\left(\frac{l}{2}+1-m\right)}+\frac{a_7a_2\mathrm{Gr}}{a_6-a_5}\cdot$$

$$\cdot\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-y)^p(-1)^n\left[K_1(\alpha)\right]^{q+n}}{p!q!\left(\sqrt{a_2}\right)^p\left[K_0(\alpha)\right]^{q-\frac{p}{2}+n+1}s^{1+q-\frac{\alpha p}{2}+n+\alpha}}\frac{\Gamma\left(\frac{p}{2}+1\right)}{\Gamma\left(\frac{p}{2}+1-q\right)} \tag{22}$$

Finally, by taking Laplace inverse of eq. (22), we have solution for velocity field in series form:

$$v(y,t)=1+\sum_{l=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-y)^l\left[K_1(\alpha)\right]^m}{l!m!\left[K_0(\alpha)\right]^{m-\frac{1}{2}}}\frac{t^{m-\frac{\alpha l}{2}}\Gamma\left(\frac{l}{2}+1\right)}{\Gamma\left(1+m-\frac{\alpha l}{2}\right)\Gamma\left(\frac{l}{2}+1-m\right)}+\frac{a_7a_5\mathrm{Gr}}{a_6-a_5}\cdot$$

$$\cdot\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-y)^l(-1)^n\left[K_1(\alpha)\right]^{m+n}}{l!m!\left[K_0(\alpha)\right]^{m-\frac{l}{2}+n+1}}\frac{t^{m-\frac{\alpha l}{2}+n}\Gamma\left(\frac{l}{2}+1\right)}{\Gamma\left(1+m-\frac{\alpha l}{2}+n\right)\Gamma\left(\frac{l}{2}+1-m\right)}+\frac{a_7a_5\mathrm{Gr}}{a_6-a_5}\cdot$$

$$\cdot\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-y)^p(-1)^n\left[K_1(\alpha)\right]^{q+n}}{p!q!\left(\sqrt{a_5}\right)^p\left[K_0(\alpha)\right]^{q-\frac{p}{2}+n+1}s^{1+q-\frac{\alpha p}{2}+n+\alpha}}\frac{t^{q-\frac{\alpha p}{2}+n+\alpha}\Gamma\left(\frac{p}{2}+1\right)}{\Gamma\left(1+q-\frac{\alpha p}{2}+n+\alpha\right)\Gamma\left(\frac{p}{2}+1-q\right)} \tag{23}$$

*Note*: In the absence of fractional derivative $\alpha \to 1$, we recovered the solutions of temperature and velocity obtained by Khan *et al*. [29].

## Numerical results and discussion

Figure 1 is designed to detect the effect of fractional parameter $\alpha$ for small value of time in temperature field. When we raise the values of fractional parameter $\alpha$, the temperature reduces for small value of time. The $\alpha$ variation on velocity field is presented in fig. 2 for small value of time. The fluid velocity decreases as we raise the values of fractional parameter $\alpha$ for small time. This can be physically sustained as when $\alpha$ is raised, the momentum and thermal boundary-layer declined and became thinnest at $\alpha = 1$ as a result the temperature profiles decreased. In figs. 3 and 4 large times influence is depicted for different values of fractional parameter $\alpha$ and opposite trend can be observed. In fig. 5 it is noticed that the thermal conductivity of water is higher than engine and kerosene oil. It is found that water based clay nanoparticles have higher temperature in comparison with others base fluids. The evaluation of velocity profile for three different base fluids (water, kerosene and engine oil) is revealed in fig. 6. It is observed that the velocity of water-based clay nanofluid is higher than other based clay nanofluids.

The thermal conductivity of water is relatively greater than that of kerosene and engine oil. It is concluded that the velocity of water-based clay nanofluid is higher than the others. Figure 7 is drawn to see the temperature and velocity comparison of hybrid fractional derivative and Khan *et al.* [29] by taking different values of fractional parameter $\alpha$. We found that temperature and velocity are smaller for hybrid fractional model.

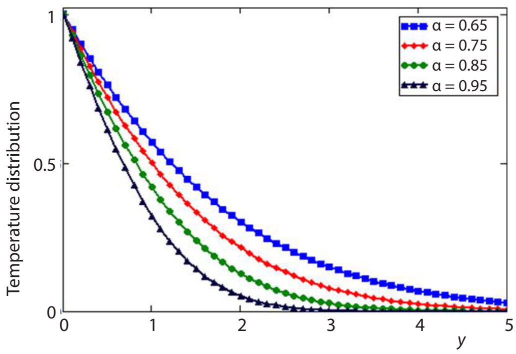


**Figure 1. Small time behavior of temperature profile when: $t = 0.01$, $\phi = 0.1$, Pr = 6.2**

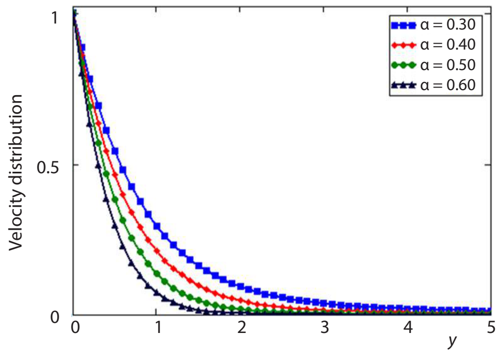


**Figure 2. Small time behavior of velocity profile when: $t = 0.01$, $\phi = 0.01$, Pr = 6.2, Gr = 0.1**

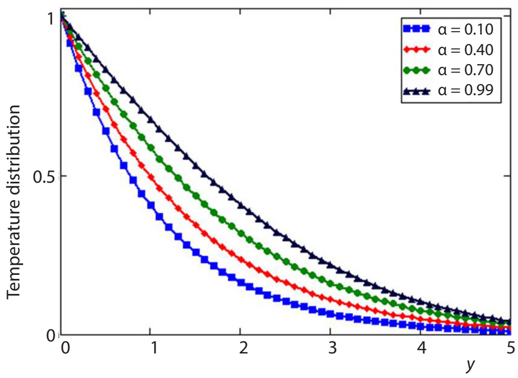


**Figure 3. Large time behavior of temperature profile when: $t = 2.8$, $\phi = 0.5$, Pr = 6.2**

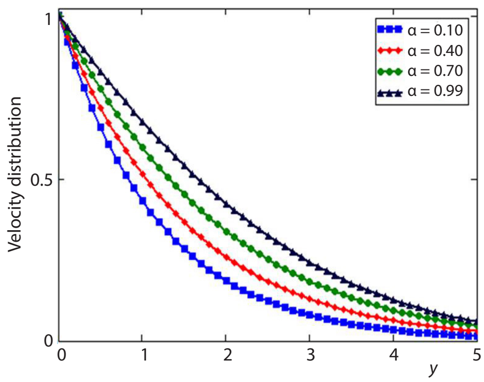


**Figure 4. Large time behavior of velocity profile when $t = 2.5$, $\phi = 0.5$, Pr = 6.2, Gr = 0.001**

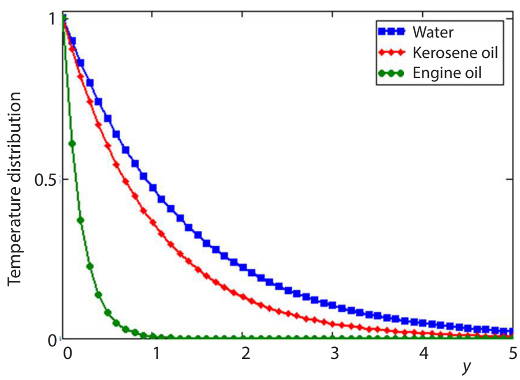


**Figure 5. Temperature distribution with different base fluids, when: $t = 1.5$, $\phi = 0.03$, Pr = 6.2, $\alpha = 0.1$**

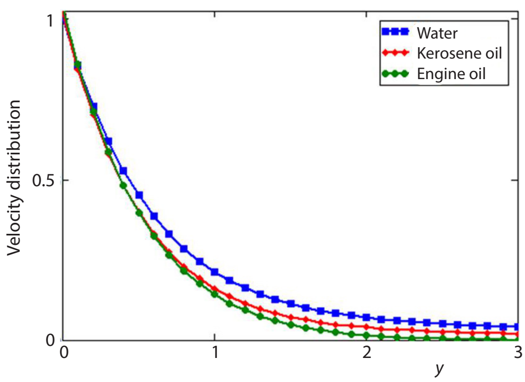


**Figure 6. Velocity distribution with different base fluids, when: $t = 0.001$, $\phi = 0.04$, Pr = 1.2, Gr = 4.5, $\alpha = 0.39$**

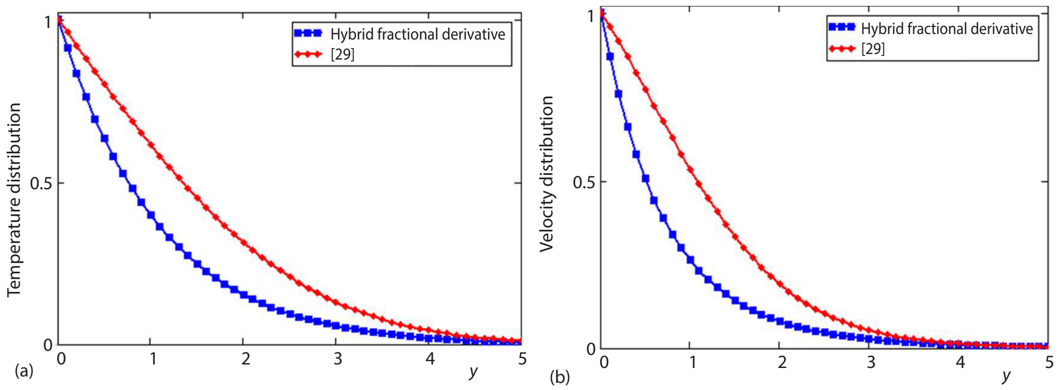


**Figure 7. Temperature and velocity distribution comparisons with [29] for $\alpha = 0.4$**

## Conclusions

In this section, we draw some main outcomes of the present study which deals with applications of innovative fractional derivative in nanotechnology. Heat transfer of clay nanoparticles over an infinite vertical plate with Laplace transform method with different base fluids is considered. Some key findings of the present study are:

- Temperature and velocity showed dual behavior for fractional parameter $\alpha$ for small and large time, respectively.
- Kerosene and engine oil based clay nanoparticles have lower temperature and velocity than water.
- Constant proportional Caputo fractional model is showing more decay nature of temperature and velocity when compared with classical [29].

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## References

[1] Choi, S. U. S., Enhancing Thermal Conductivity of Fluids with Nanoparticles Developments and Applications of Non-Newtonian Flows, *ASME*, *231* (1995), Jan., pp. 99-105
[2] Saqib, M., *et al*., Entropy Generation in Different Types of Fractionalized Nanofluids, *Arabian Journal for Science and Engineering*, *44* (2018), June, pp. 531-540
[3] Ahmed, N., *et al*., Transient MHD Convective Flow of Fractional Nanofluid Between Vertical Plates, *Journal of Applied and Computational Mechanics*, *5* (2019), 4, pp. 592-602
[4] Rahimi-Gorji, M., *et al*., Modelling of the Air Conditions Effects on the Power and Fuel Consumption of the SI Engine Using Neural Networks and Regression, *Journal of the Brazilian Society of Mechanical Sciences and Engineering*, *39* (2017), 2, pp. 375-384
[5] Rahimi-Gorji, M., *et al*., The CFD Simulation of Air-flow Behavior and Particle Transport and Deposition in Different Breathing Conditions through the Realistic Model of Human Airways, *Journal of Molecular Liquids*, *209* (2015), Sept., pp. 121-133
[6] Saqib, M., *et al*., Application of Atangana-Baleanu fractional Derivative to MHD Channel Flow of CMC-Based- CNT's Nanofluid through a Porous Medium, *Chaos, Solitons and Fractals*, *116* (2018), Nov., pp. 79-85
[7] Khanafer, K., *et al*., Buoyancy-Driven Heat Transfer Enhancement in a 2-D Enclosure Utilizing Nanofluids, *International Journal of Heat and Mass Transfer*, *46* (2003), 19, pp. 3639-3653
[8] Buongiorno, J., Convective Transport in Nanofluids, *Journal of Heat Transfer*, *128* (2006), 3, pp. 240-250
[9] Tiwari, R. K., Das, M. K., Heat Transfer Augmentation in a Two-Sided Lid-Driven Differentially Heated Square Cavity Utilizing Nanofluids, *International Journal of Heat and Mass Transfer*, *50* (2007), 9-10, pp. 2002-2018

[10] Ahmad, S., Pop, I., Mixed Convection Boundary-Layer Flow from a Vertical Flat Plate Embedded in a Porous Medium Filled with Nanofluids, *International Communications in Heat and Mass Transfer*, *37* (2010), 8, pp. 987-991
[11] Biglarian, M., *et al*., The $H_2O$ Based Different Nanofluids with Unsteady Condition and an External Magnetic Field on Permeable Channel Heat Transfer, *International Journal of Hydrogen Energy*, *42* (2017), 34, pp. 22005-22014
[12] Mosayebidorcheh, S., *et al.*, Transient Thermal Behavior of Radial Fins of Rectangular, Triangular and Hyperbolic Profiles with Temperature-Dependent Properties Using DTM-FDM, *Journal of Central South University*, *2*4 (2017), 3, pp. 675-682
[13] Pourmehran, O., *et al*., Rheological Behaviour of Various Metal-Based Nanofluids Between Rotating Discs: A New Insight, *Journal of the Taiwan Institute of Chemical Engineers*, *88* (2018), July, pp. 37-48
[14] Rahimi-Gorji, M., *et al*., Unsteady Squeezing Nanofluid Simulation and Investigation of its Effect on Important heat Transfer Parameters in Presence of Magnetic Field, *Journal of the Taiwan Institute of Chemical Engineers*, *67* (2016), Oct., pp. 467-475
[15] Saqib, M., *et al.*, Convection in Ethylene Glycol-Based Molybdenum Disulfide Nanofluid, *Journal of Thermal Analysis and Calorimetry*, *135* (2018), Feb., pp. 523-532
[16] Trisaksri, V., Wongwises, S., Critical Review of Heat Transfer Characteristics of Nanofluids, *Renewable and Sustainable Energy Reviews*, *11* (2007), 3, pp. 512-523
[17] Sheikh, N. A., *et al*., On the Applications of Nanofluids to Enhance the Performance of Solar Collectors: A Comparative Analysis of Atangana-Baleanu and Caputo-Fabrizio fractional Models, *The European Physical Journal Plus*, *132* (2017), 12, 540
[18] Oztop, H. F., *et al.*, A Brief Review of Natural-Convection in Enclosures under Localized Heating with and without Nanofluids, *International Communications in Heat and Mass Transfer*, *60* (2015), Jan., pp. 37-44
[19] Hussanan, A., *et al.*, Convection Heat Transfer in Micropolar Nanofluids with Oxide Nanoparticles in Water, Kerosene and Engine Oil, *Journal of Molecular Liquids*, *229* (2017), Mar., pp. 482-488
[20] Tesfai, W., *et al*., Rheology and Microstructure of Dilute Graphene Oxide Suspension, *Journal of Nanoparticle Research*, *15* (2013), 10, 1989
[21] Wu, J. M., Zhao, J., A Review of Nanofluid Heat Transferand Critical Heat Flux Enhancement – Research Gap to EngineeringApplication, *Progress in Nuclear Energy*, *66* (2013), July, pp. 13-24
[22] Khan, I., Shape Effects of MoS2 Nanoparticles on MHD Slip Flow of Molybdenum Disulphide Nanofluid in a Porous Medium, *Journal of Molecular Liquids*, *233* (2017), May, pp. 442-451
[23] Sheikholeslami, M., Bhatti, M. M., Active Method for Nanofluid Heat Transfer Enhancement by Means of EHD, *International Journal of Heat and Mass Transfer*, *109* (2017), June, pp. 115-122
[24] Abdelsalam, S. I., Bhatti, M. M., The Impact of Impinging $TiO_2$ Nanoparticles in Prandtl Nanofluid Along with Endoscopic and Variable Magnetic Field Effects on Peristaltic Blood Flow, *Multidiscipline Modelling in Materials and Structures*, *14* (2018), 3, pp. 530-548
[25] Abdelsalam, S. I., Bhatti, M. M., The Study of Non-Newtonian Nanofluid with Hall and Ion Slip Effects on PeristalticallyInduced Motion in a Non-Uniform Channel, *RSC Advances*, *8* (2018), 15, pp. 7904-7915
[26] Brinkman, H. C., The Viscosity of Concentrated Suspensions and Solutions, *The Journal of Chemical Physics*, *20* (1952), 4, pp. 571-571
[27] Aminossadati, S. M., Ghasemi, B., Natural-Convection Cooling of a Localised Heat Source at the Bottom of a Nanofluid Filled Enclosure, *European Journal of Mechanics – B/Fluids*, *28* (2009), 5, pp. 630-640
[28] Matin, M. H., Pop, I., Forced Convection Heat and Mass Transfer Flow of a Nanofluid through a Porous Channel with a First Order Chemical Reaction on the Wall, *International Communication in Heat and Mass Transfer*, *46* (2013), Aug., pp. 134-141
[29] Khan, I., *et al*., Convective Heat Transfer in Drilling Nanofluid with Clay Nanoparticles: *Applications in* Water Cleaning Process, *BioNanoScience*, *9* (2019), Mar., pp. 453-460
[30] Baleanu, D., *Fractional Calculus Models and Numerical Method*s, World Scientific, Singapore, 2012
[31] Petras, I., *Fractional-order Non-linear Systems: Modelling Analysis and Simulation*, Springer and Beijing, London, HEP, 2011
[32] Kilbas, A. A., *et al*., *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, The Netherlands, 2006
[33] Baleanu, *et al*., On a Fractional Operator Combining Proportional and Classical Differintegrals, *Mathematics*, *8* (2020), 3, 860

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