ENTANGLEMENT AND GEOMETRIC PHASE OF THE COHERENT FIELD INTERACTING WITH A THREE TWO-LEVEL ATOMS IN THE PRESENCE OF NON-LINEAR TERMS

by

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We study the interaction of a three two-level atoms with a one-mode optical coherent field in coherent state in the presence of non-linear Kerr medim. The three atoms are initially prepared in upper and entangled states while the field mode is in a coherent state. The constants of motion, three two-level atoms and field density matrix are obtained. The analytic results are employed to perform some investigations of the temporal evolution of the von Neumann entropy as measure of the degree of entanglement between the three two-level atoms and optical coherent field. The effect of the detuning and the initial atomic states on the evolution of geometric phase and entanglement is analyzed. Also, we demonstrate the link between the geometric phase and non-classical properties during the evolution time. Additionally the effect of detuning and initial conditions on the Mandel parameter is studied. The obtained results are emphasize the impact of the detuning and the initial atomic states of the feature of the entanglement, geometric phase and photon statistics of the optical coherent field.

Key words: *non-linear Kerr medium, three two-level atoms, entanglement, von Neumann entropy, Mandel parameter*

Introduction

The quantum correlations (QC) in the composite quantum system especially the entanglement are related to the superposition technique [1]. The detecting and controlling of different correlations among quantum subsystems in context of classical and quantum techniques have been introduced as one of the major objectives of the technology of quantum information [2-4]. Entanglement is the main type QC that can be experimentally generated via quantum devices such as beam splitter [5-7], cavity quantum electrodynamics [8, 9], quantum nanoresonators [10, 11], and nuclear magnetic resonance spin systems [12].

Entanglement is connected with the state of the target separable or entangled system. Both of separable and entangled states are applied in various trends of the quantum information technology (QIT) [13, 14]. Many investigations have quantified the entanglement in composed

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and simple quantum system by the evolution of the von Neumann entropy (VNE) [15, 16]. Most investigations have been processed on the multi-level atom and coherent state field (CSF). In this regard, as some kind of non-local correlation between a three-level atom and OCF in [17], and four-level atom and optical coherent field (OCF) in [18]. Under these models, different relationships between the entanglement and non-classical properties have been explored. For example, the relationship between the two-level atom and photon added to the CS of Morse potential has been studied in [19]. Also, the effect of different types of entanglement has been discussed [20-23].

As known that the geometric phase can be considered as an intrinsic property in the theory of the quantum mechanics that has been treated through two generations of physicists [24-26]. With regard appearing the phase in the wave function, the later keeps the memory of its evolution, which can be only related to the path geometry crossed by the quantum system, and it is introduced as the geometric phase factor [25]. This factor is resistive to the control imperfection and the effect of environmental perturbation.

In recent years, some papers have been shed the light on the properties of the (TCM) [27]. Considering the time dependent effects on the TCM has been pointed out in [28] by Deng and Fang. An interesting entanglement sudden death (ESD) and entanglement sudden birth (ESB) phenomena have been explored by Vaglica and Vetric in [29]. This work has been modified considering the effect of initial state preparation of the two-qubit system on the qubit-qubit entanglement measured by the concurrence [30, 31]. An important model quantum system of the interaction between an OCF and two three-level atoms (3LA) has been introduced when their coupling depends on the time [32]. Our target in this work is to develop this work by considering the OCF interacting with 3-2LA. Also, we discuss the entanglement between the first, second and the third 2LA, in addition the non-local correlation between the atomic system (3-2LA) and the OCF.

Model Hamiltonian of 3-2LA in OCF

In this section, we investigate our problem which includes that three two-level atom (TTLA) interacting with a single mode of a field:

$$\frac{\hat{H}}{\hbar}=\omega\hat{\psi}^{+}\hat{\psi}+\chi\hat{\psi}^{+}\hat{\psi}\left(\hat{\psi}^{+}\hat{\psi}-\hat{I}\right)+\frac{\Omega}{2}\sum_{j=1}^{3}\sigma_{z}^{(j)}+\lambda\sum_{j=1}^{3}\left[\sigma_{+}^{(j)}\hat{\psi}+\sigma_{-}^{(j)}\hat{\psi}^{+}\right] \tag{1}$$

where j = 1, 2, 3 indicat the atoms A, B, C, respectivly, $\hat{\psi}^{+}$ and $\hat{\psi}$ symbolize the boson operators for the field which obeys the relation $[\hat{\psi}, \hat{\psi}^{+}] = I$, ω , and Ω are the field and the atomic frequencies, respectively, while λ is the coupling between the atoms and the cavity field, and χ – the coupling of Kerr-like medium. Where $\hat{\sigma}_{+}$, $\hat{\sigma}_{-}$, and $\hat{\sigma}_{z}$ are the Pauli operators, su(2) Lie group generator, which obey the relations:

$$\left[\hat{\sigma}_{\pm},\hat{\sigma}_{z}\right]=\mp 2\hat{\sigma}_{\pm},\ \left[\hat{\sigma}_{-},\hat{\sigma}_{+}\right]=-\hat{\sigma}_{z} \tag{2}$$

The main purpose of this project is to discuss with details the statistical properties of the system Hamiltonian (1) as well as to see the effect of the motion of the su(1, 1) on the TTLA.

For calculating the general form for the analytical solution of the Schrodinger equations, the system of differential equations for the operators must be solved by using Heisenberg formula $\hat{a}^{+}\hat{a}$, $\hat{\sigma}_{z}^{(1)}$, $\hat{\sigma}_{z}^{(2)}$, and $\hat{\sigma}_{z}^{(3)}$ thus:

$$\frac{\mathrm{d}\hat{\psi}^{+}\hat{\psi}}{\mathrm{d}t}=-i\lambda\sum_{j=1}^{3}\left[\sigma_{+}^{(j)}\hat{\psi}-\sigma_{-}^{(j)}\hat{\psi}^{+}\right] \tag{3}$$

$$\frac{\mathrm{d}\sigma_z^{(j)}}{\mathrm{d}t} = 2i\lambda\left[\sigma_+^{(j)}\hat{\psi} - \sigma_-^{(j)}\hat{\psi}^+\right], j = 1,2,3 \tag{4}$$

From which we can show:

$$\hat{N} = \hat{n} + \frac{\hat{\sigma}_z}{2}, \quad \hat{\sigma}_z = \hat{\sigma}_z^{(2)} + \hat{\sigma}_z^{(1)} + \hat{\sigma}_z^{(3)} \tag{5}$$

where $\hat{N}$ is a constant of motion operator, so the eq. (1) can be re-written

$$\hat{H} = \omega\hat{N} + \hat{C} \tag{6}$$

the operator $\hat{C}$ takes the form

$$\hat{C} = \frac{\Delta}{2}\hat{\sigma}_z + \lambda\sum_{j=1}^{3}\left[\sigma_+^{(j)}\hat{\psi} + \sigma_-^{(j)}\hat{\psi}^+\right] \tag{7}$$

with

$$\Delta = \Omega - \omega - \chi\left(n - \frac{1}{2}\right)$$

and we consider the identical case of the three atoms. From the aforementioned equation, it is clear that the operators $\hat{C}$ and $\hat{N}$ are constants of motion.

The wavefunction corresponding to eq. (7) can be assumed:

$$\begin{gathered}\left|\Theta(t)\right\rangle = \sum_{m=0}^{\infty}\Big(X_1(m,t)\left|eee\right\rangle\left|m\right\rangle + X_8(m,t)\left|ggg\right\rangle\left|m+3\right\rangle + \\ +\left\{X_2(m,t)\left|eeg\right\rangle + X_3(m,t)\left|ege\right\rangle + X_5(m,t)\left|gee\right\rangle + X_7(m,t)\left|geg\right\rangle\right\}\left|m+1\right\rangle + \\ +\left\{X_4(m,t)\left|egg\right\rangle + X_6(m,t)\left|gge\right\rangle\right\}\left|m+2\right\rangle\Big)\end{gathered} \tag{8}$$

where the coefficients $Xj(n, t)$ are the solution of the differential equations:

$$i\hbar\frac{\partial\left|\Theta(t)\right\rangle}{\partial t} = \hat{H}\left|\Theta(t)\right\rangle \tag{9}$$

Equation (9) can be formulated in matrix form:

$$\frac{\mathrm{d}X_j}{\mathrm{d}t} = \lambda\Lambda X \tag{10}$$

where

$$\Lambda = \begin{bmatrix} 3\Delta_1 & \alpha & \alpha & 0 & \alpha & 0 & 0 & 0 \\ \alpha & \Delta_1 & 0 & \beta & 0 & 0 & \beta & 0 \\ \alpha & 0 & -\Delta_1 & \beta & 0 & \beta & 0 & 0 \\ 0 & \beta & \beta & -\Delta_1 & 0 & 0 & 0 & \gamma \\ \alpha & 0 & 0 & 0 & \Delta_1 & \beta & \beta & 0 \\ 0 & 0 & \beta & 0 & \beta & \Delta_1 & 0 & \gamma \\ 0 & \beta & 0 & 0 & \beta & 0 & -\Delta_1 & \gamma \\ 0 & 0 & 0 & \gamma & 0 & \gamma & \gamma & -3\Delta_1 \end{bmatrix} \tag{11}$$

and $\Delta_1 = \Delta/2\lambda$, $\alpha = (n + 1)^{1/2}$, $\beta = (n + 2)^{1/2}$, and $\gamma = (n + 3)^{1/2}$.

Now assuming that the field initially is in coherent state:

$$\left|\Theta_f(0)\right\rangle = \sum_{m=0}^{\infty} Q_m \,|\, m\rangle,\;\; Q_m = \exp\left(-\frac{|B|^2}{2}\right)\frac{B^m}{\sqrt{m!}},\;\; \bar{n} = |B|^2$$

but the atoms starts from an entangled states:

$$\left[\cos(\theta)\,|\,\text{eee}\rangle + \sin(\theta)\left|\text{ggg}\right\rangle\right]$$

The solution at $t = 0$ takes the form:

$$\left|\Theta(0)\right\rangle = \left[\cos(\theta)\,|\,\text{eee}\rangle + \sin(\theta)\left|\text{ggg}\right\rangle\right] \otimes \sum_{m=0}^{\infty} Q_m \,|\, m\rangle \tag{12}$$

Some statistical aspects will be performed in the next sections.

Atomic population

It is known that the population inversion (PI) in quantum optics gives more information about the properties of the interaction between the sphere and the atom:

$$W(t) = \rho_{11}(t) - \rho_{88}(t) \tag{13}$$

Here, in absence of the detuning parameter ($\Delta = 0$) and the atoms beginning from, $|\text{eee}\rangle$ the population function shows little collapse in the beginning of interaction followed by short region of revival with small amplitude followed by revival region with hug amplitude oscillations that belong to the period (–0.5, 0.5) and followed by another revival region as observed in fig. 1(a). After taking the atoms prepared in entangled state $\theta = \pi/4$ (linear combination between $|\text{eee}\rangle$ and $|\text{ggg}\rangle$), the population inversion shows more regular oscillations with reduction of the amplitude and the collapse region in the earlier case vanishes after taking the entangled state as initial state see fig. 1(b). Note that there is an increase in the speed of the atom moving from the lowest energy level to the higher energy level and *vice versa*, therefore, the interaction between the atoms and the electromagnetic field is increasing after taking the entangled state into account. When we take the detuning into consideration and the atoms prepared in pure excited states $|\text{eee}\rangle$, the periods of revels increases with reduction of amplitude, the collapses phenomena never occurred in the time interaction consideration see fig. 1(c). When we take the entangled states into account, the amplitude of oscillations decreases and more fasting fluctuations occur between the upper and the lower states as shown in fig. 1(d).

Mandel parameter

The Mandel parameter, P_M, which describes the photon statistics of the field during the interaction is defined [33]:

$$P_M = \frac{\left\langle \hat{n}^2 \right\rangle - \left\langle \hat{n} \right\rangle^2 - \left\langle \hat{n} \right\rangle}{\left\langle \hat{n} \right\rangle} \tag{14}$$

Here $\langle \hat{n} \rangle$ defines the mean photon number. Through the Mandel's parameter, we can determine whether the distribution of the photons is sub-Poissonian (classical case), $P_M < 0$, Poissonian (semiclassical states), $P_M = 0$, or super-Poissonian, $P_M > 0$.

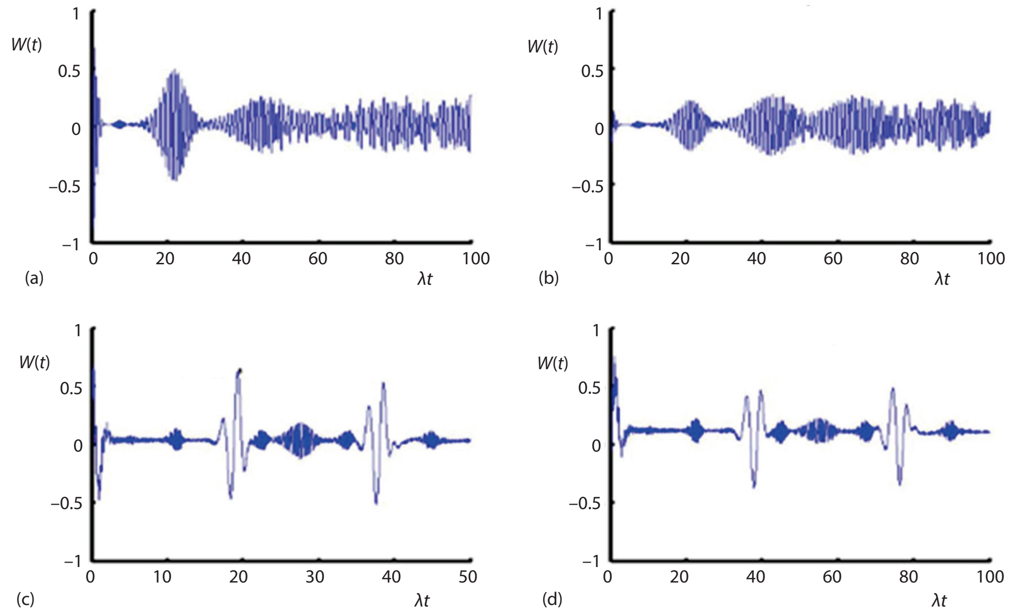

Figure 1. Population inversion $W(t)$ for a three two-level atom interaction with electro-magnetic field prepared in coherent state with initial mean value $\bar{n} = 10$; the three two-level atom are beginning in the: excited states $\theta = 0$ (a) and (c) , and entangled states $\theta = \pi/4$ (b) and (d); the effect of detuning has neglected $\Delta = 0$ (a) and (b), and considered as $\Delta = 10\lambda$ in (c) and (d)

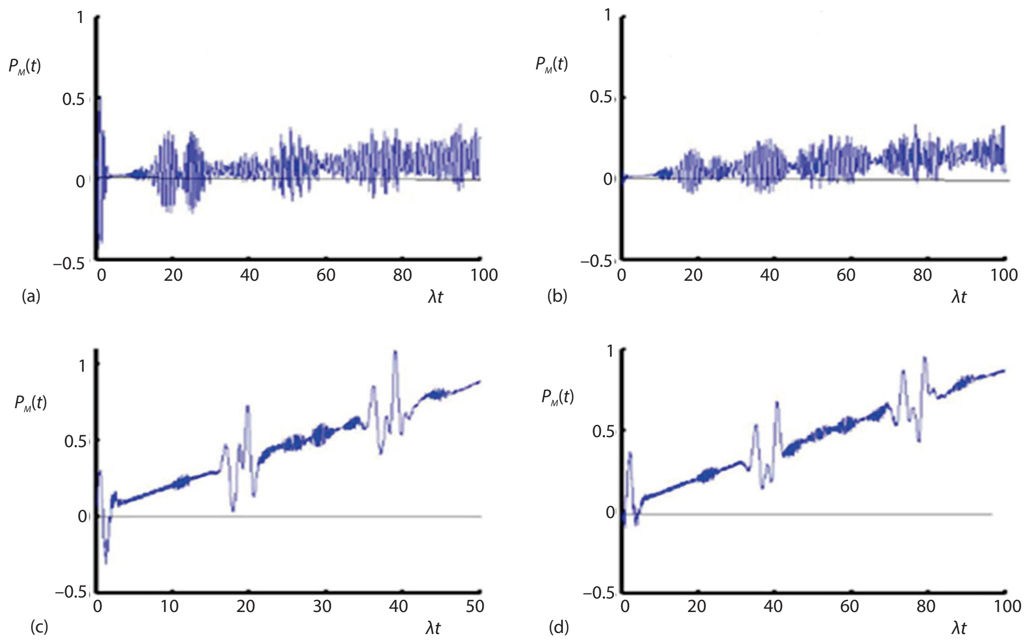

Figure 2. Mandel parameter P_M for the same conditions and parameters as in fig. 1

In absence of the detuning parameter, the Mandel parameter P_M occur between negative and positive values (classical and non-classical) and the distribution approach gradually to non-classical as the time of interaction goes on, see fig. 2(a). We noted that the system takes the poissonian distribution in the collapse interval but the oscillations between the negative and positive hold at the revivals interval as compared between figs. 1(a) and 2(a). When the entangled states are taken into account the amplitude of oscillations reduces and the intervals of non-classical are decreased and the system reach to classical distribution after short time, see in fig. 2(b). The negative values are washed after adding the detuning parameter and the system almost represents classical distribution as the time of interaction increasing. In addition, oscillations in the earlier case are reduced and the amplitude of oscillations increases after taking the detuning parameter, see fig. 2(c). The distribution is more classical behavior after taking the entangled states into consideration and the non-classical interval never occur except little interval as observed in fig. 2(d). Therefore, the results show that the Mandel parameter is sensitive to the initial states and the dunning parameter.

Geometric phase

The geometric phase is defined as the argument of the projection between the initial state and the wave function [24]:

$$\phi_G = \arg\langle\Theta(0)\,|\,\Theta(t)\rangle \tag{15}$$

For the same conditions as population inversion, in resonance case $\Delta = 0$, the geometric phase has regular oscillations between 0 and π as shown in fig. 3(a). After taking the entangled states into account the negative values occur and the fluctuations between the maximum and minimum values are increased, therefore, the amplitude of oscillation increases, see fig. 3(b). When we take the off-resonance case $\Delta = 10\lambda$, the regular oscillation becomes irregular after taking the detuning into account, little faster oscillations are compared with the previous case as shown in fig. 3(c). For the entangled states and off-resonance case, we see that more increasing of the oscillations happen and the amplitude of geometric phase decreases, more fluctuations, raped oscillations and the minimum values are reduced as observed in fig. 3(d).

Entanglement transfer and non-classical properties

Here, we employ the von-Neumann entropy to quantify the ABC-OF entanglement defined by [34, 35]:

$$S(\rho_{ABC}) = -tr\{\rho_{ABC}\ln\rho_{ABC}\} \quad \text{where} \quad \rho_{ABC} = tr\{|\Theta(t)\rangle\langle\Theta(t)|\} \tag{16}$$

where the atomic density matrix is denoted by ρ_{ABC} and is obtained by taking the trace over the field basis, *i. e.* $\rho_{ABC} = tr_F\{\rho(t)\}$. Equation (16) can be written in terms of the eigenvalues of the TTLA density matrix υ_j:

$$S_A = S(\rho_{ABC}) = -\sum_{j=1}^{8}\upsilon_j\ln\upsilon_j \tag{17}$$

Quantum entropy takes ln2 in the case of a single two-level atom for maximally entangled state and zero value for the pure state.

In fig. (3), we study the degree of entanglement between the atoms and the field for the present system with same conditions as earlier. In absence of the detuning parameter ($\Delta = 0$) and the atoms starts from their excited states $|eee\rangle$. In general the degree of entanglement

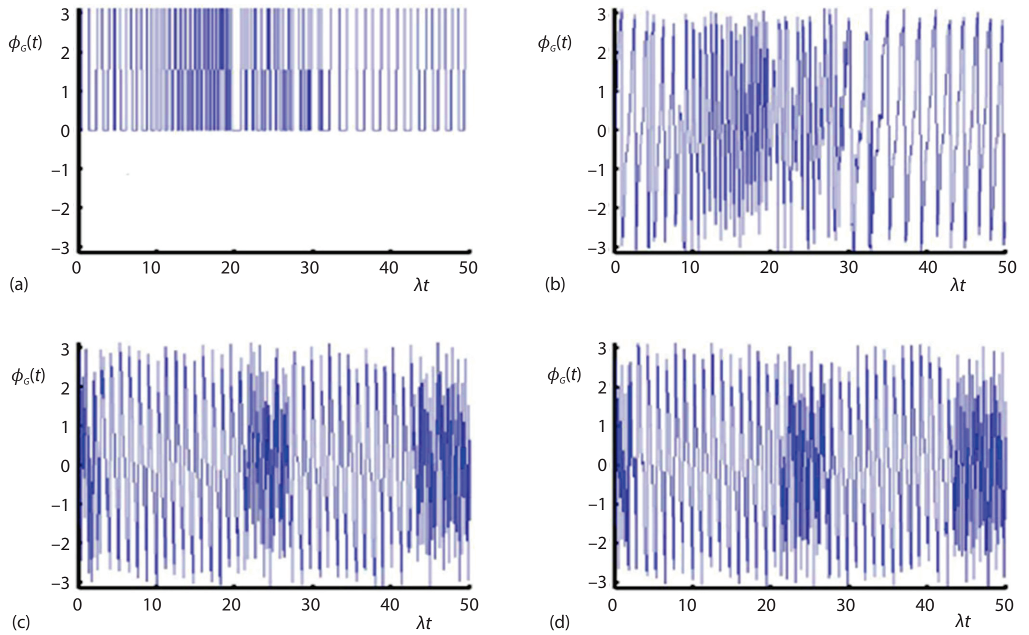

Figure 3. Geometric phase ϕ_G for same conditions and parameters as in fig. 1

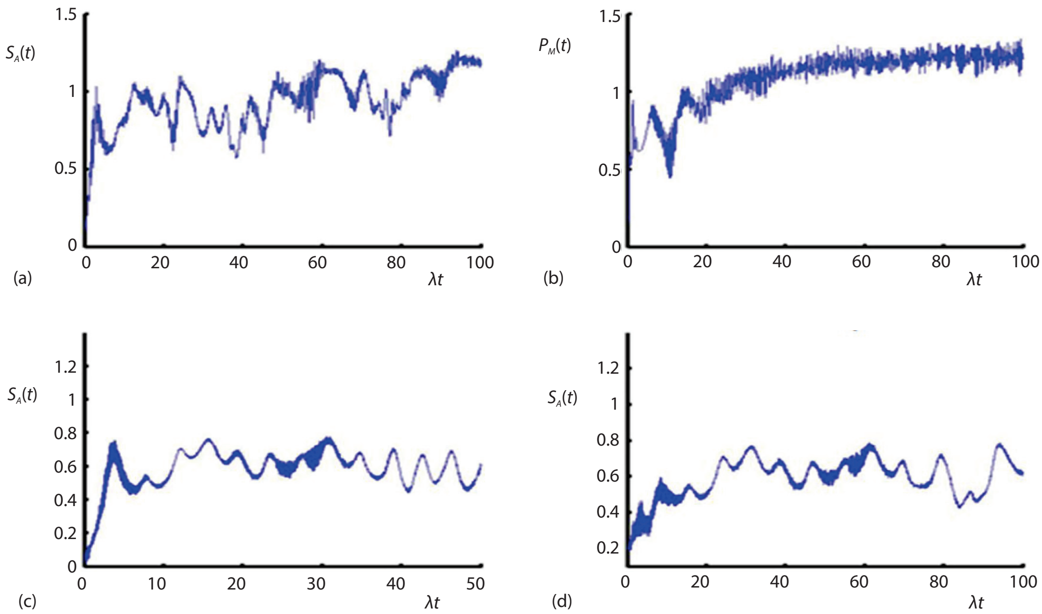

Figure 4. The ABC-OF entanglement measured by the atomic entropy S_A for the same conditions of fig. 1

which described by von-Neumann entropy function $S(\rho_{ABC})$ satisfies $0 < S_A(t) < \ln(6)$. By comparison between the population and the von-Neumann entropy, we see that the system reaches the minimum values at the central of the collapses intervals and the maximum values at the center of the revivals intervals figs. 1(a) and 4(a). When the atoms start from the entangled states, the asymmetrical oscillations in earlier case convert to regular, while the minimum values of the $S_A(t)$ are reduced, while the maximum values increase after taking the entangled states into consideration, see fig. 4(b). The extreme values of the $S_A(t)$ function are reduced and more oscillations appear between 0, 0.8 after adding the detuning and the atoms prepared in pure excited states into account as shown in fig. 4(c). In final case, degree of entanglement increases gradually and reaches the maximum values after short time and the amplitude of oscillations decreases, see fig. 4(d). Therefore, the von-Neumann entropy is strongly changed by varying of both the entangled states and the detuning parameter. Finally, it is seen the entanglement decresses when the OCF is more classical see fig. 2(d).

Concussion

The wave function for the interaction between the TTLA and the field has been obtained numerically. The equations of motion have been calculated through the Heisenberg formula. The atomic population inversion has been studied and the collapses and revivals regions have been estimated in both of the atoms prepared in excited state and entangled states. The results show that the amplitude of oscillations is reduced after taking the entangled states into consideration. The Mandel parameter indicates that the distribution oscillates between the classical and non-classical behavior when absence of the detuning parameter and the atoms beginning in |eee> state. The distribution gradually becomes classical and never reaches to non-classical distribution after adding the detuning into account. The degree of entanglements between the atoms and the electromagnetic field has been measured by using the von-Neumann formula, the results show that the minimum values of the degree of entanglement occur at the central point of the collapses intervals while the maximum values occur at the center of revivals intervals. On the other hand, the degree of entanglement reduces in the resonance case, while it increases after taking the off-resonance case and the entangled states into account.

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