INFLUENCE OF CLASSICAL FIELD ON ENTANGLEMENT AND PHOTON STATISTICS OF *N*-LEVEL ATOM INTERACTING WITH A TWO TWO-LEVEL ATOM

by

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In this study, we consider three interacting atoms, one of them represented by N-level atom based on SU(2) Lie algebra and the other represented by a two two-level atom in presence of the external field. The effect of the external field on the dynamics of the proposed system is discussed in detail for certain values for the external field. The dynamical expression of the observable operators is obtained by using the Heisenberg equation of motion. The general solution via solving Schrodinger equation is obtained. The fidelity and concurrence formula as a measure of entanglement between two two-level atom are calculated and discussed in detail. We explore the sudden death and sudden birth phenomena with and without the presence of external field. Finally, we compare the results of the fidelity, concurrence and second-order correlation function for some values of the initial state and the external field parameters.

Key words: *entanglement, external classical field, fidelity, concurrence, second order correlation*

Introduction

Quantum information processing depends on an essential tool which is called quantum entanglement (QE). The QE is related by the proposed quantum system density matrix or operator. The density matrix as a separate product separable state while it cannot be formulated as a separate product state in the case of entangled state [1]. Different investigations of QE from two to five-level atom have been done. Although the study of entanglement in the case of the multi-level atom, especially in the mixed state does not consider widely. Also, the study of QE or non-local correlation between multi-mode fields needs further treatments. Moreover, QE has been treated in both continuous and different variables within the theoretical approaches [2-5]. The measure of QE depends on the type of system density matrix mixed or pure [6]. The comparison between the classical correlation and the QE has seen an explosive growth of activities in technology of quantum information where the realistic systems are generally composite. In order to estimate the quantify of the QE and classical correlation, which are the parts of total

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correlation in quantum system [7, 8], researchers have made tremendous efforts to develop the QE quantifiers with the aim to include different composite systems. The interaction model between a quantized mode of a radiation field and a two-level atom (TLA) are mathematically modelled by the Jaynes-Cummings Model (JCM) [9]. So the JCM has been extensively used in quantum information processing such as cryptography, quantum teleportation, and quantum algorithms [10, 11]. Many different generalizations of JCM based on multi-mode field [12], multi-level atoms [13], time-dependent coupling [14], multi-photon transitions [15, 16], energy dissipation and so on have been proposed by some researchers over recent decades. In fact, all of these studies have been taken into account in an ideal form, where the classical field effect is neglected [17]. Particularly, TLA have been considered where the QE plays a significant role in the new field of quantum information, there have been a lot of studies of the QE properties [18-21].

The von Neumann entropy [22] has been formally used to measure the QE between two partite quantum systems. Whereas, to measure the QE in the case of the complex system we should find another measure. For this reason different QE measures of quantifiers have been proposed such as (atomic) Wehrl entropy [23] and quantum (atomic) Fisher information [24]. The non-classical properties of a TLA interacting with a resonator in the coherent state within the evolution of Wehrl entropy have been studied [25]. Concerning the two SC-qubits interaction case, the effect of the magnetic field of the quantum discord as a measure of quantum correlations between two two-level atoms (TTLA) has been explored [26]. For atomic simulation, Alotaibi *et al*. [27] establish a computational scheme which is the molecular dynamics of a monatomic gas in 3-D space. To predict the desired macroscale spatial dynamics, the microscale, detailed, simulator computes the motions of *N* atoms in a microscale patch as they react to forces originating from interactions with the other atoms in the patch.

The investigation of QE between an optical field and TTLA leads to exploring some unusual features such as entanglement sudden birth (ESB) and the phenomenon of entanglement sudden death (ESD) [28]. For example, in some classes of initial states of the TTL, Obada *et al*. [29] reviewed the ESD and ESB phenomena of a ttls system in the presence of local squeezed reservoirs. The interaction between a two moving three-level atom with the squeezed field has been introduced by [30]. For a non-linear two qubits, the ESD of SU(1,1) group has been studied by [31].

Description model and exact solution

Let us consider the Hamiltonian system which contains a *N*-level atom represented by SU(2) Lie group interacting with a two two-level atom in presence of the external field:

$$\begin{aligned}\frac{\hat{H}}{\hbar} &= \omega\hat{J}_z + \sum_{j=1}^{2}\left(\Omega_1^j\hat{S}_{11}^j + \Omega_2^j\hat{S}_{22}^j\right) + \lambda\left(\hat{S}_{12}^1 - \hat{S}_{21}^1\right)\left(\hat{J}_- - \hat{J}_+\right) + \\ &+\lambda\left(\hat{S}_{12}^2 - \hat{S}_{21}^2\right)\left(\hat{J}_- - \hat{J}_+\right) + g\left(\hat{S}_{12}^1 - \hat{S}_{21}^1\right) + g\left(\hat{S}_{12}^2 - \hat{S}_{21}^2\right)\end{aligned} \tag{1}$$

where $\hat{J}_j^-$, $\hat{J}_j^+$ are the lower and raise operators of SU(2) and satisfy the following commutation relation:

$$\left[\hat{J}_-, \hat{J}_+\right] = -2\hat{J}_z, \quad \left[\hat{J}_z, \hat{J}_\pm\right] = \pm\hat{J}_\pm$$

The parameters ω and Ω_1^j, Ω_2^j are, respectively, the quantum system frequency and atomic frequencies, while λ is the coupling parameter. The $\hat{S}_{kl}^j$ is the generators of operators of SU(2) which obey the commutation relation:

$$\left[\hat{S}_{kl}^{j}, \hat{S}_{mn}^{i}\right] = \delta_{lm}\hat{S}_{kn}^{j}$$

The last two terms in eq. (1) represent the external fields with coupling constant g. Our ultimate aim for this work is to solve the present system eq. (1) to find the wave function in order to calculate some quantities to describe the total Hamiltonian eq. (1). Here we use the conical transformation:

$$\begin{aligned}\hat{S}_{12}^{j} &= \hat{\sigma}_{+}^{j}\cos\vartheta_j - \hat{\sigma}_{+}^{j}\sin\vartheta_j + \frac{1}{2}\hat{\sigma}_{z}^{j}\sin 2\vartheta_j \\ \hat{S}_{21}^{j} &= \hat{\sigma}_{-}^{j}\cos\vartheta_j - \hat{\sigma}_{+}^{j}\sin\vartheta_j + \frac{1}{2}\hat{\sigma}_{z}^{j}\sin 2\vartheta_j \\ \hat{S}_{z}^{j} &= \hat{\sigma}_{z}^{j}\cos 2\vartheta_j - \left(\hat{\sigma}_{+}^{j} + \hat{\sigma}_{-}^{j}\right)\sin 2\vartheta_j\end{aligned} \tag{2}$$

where

$$\vartheta_j = \frac{1}{2}\tan^{-1}\left(\frac{2g}{\Omega_1^j - \Omega_2^j}\right)$$

Substitute the conical transformation eq. (2) into Hamiltonian eq. (1) and then apply the rotating wave approximation (RWA) technic to obtain:

$$\frac{\hat{H}}{\hbar} = \omega\hat{J}_z + \sum_{j=1}^{2}\left[\Omega_0^j\hat{\sigma}_z^j + \lambda\left(\hat{\sigma}_{+}^{j}\hat{J}_{-} + \hat{\sigma}_{-}^{j}\hat{J}_{+}\right)\right] \quad \text{with} \quad \Omega_0^j = \sqrt{(\Omega_1^j - \Omega_2^j)^2 + 4g^2} \tag{3}$$

For describes and explains the physical phenomena for the model eq. (3), we write the differential equations by applying the Heisenberg equations of motion (HEM). Therefore, the mathematical expressions of the hem take the form:

$$\begin{aligned}\frac{\mathrm{d}\hat{J}_z}{\mathrm{d}t} &= -i\lambda\left(\hat{\sigma}_{+}^{1}\hat{J}_{-} - \hat{\sigma}_{-}^{1}\hat{J}_{+}\right) - i\lambda\left(\hat{\sigma}_{+}^{2}\hat{J}_{-} - \hat{\sigma}_{-}^{2}\hat{J}_{+}\right) \\ \frac{\mathrm{d}\hat{\sigma}_z^1}{\mathrm{d}t} &= 2i\lambda\left(\hat{\sigma}_{+}^{1}\hat{J}_{-} - \hat{\sigma}_{-}^{1}\hat{J}_{+}\right) \\ \frac{\mathrm{d}\hat{\sigma}_z^2}{\mathrm{d}t} &= 2i\lambda\left(\hat{\sigma}_{+}^{2}\hat{J}_{-} - \hat{\sigma}_{-}^{2}\hat{J}_{+}\right)\end{aligned} \tag{4}$$

from which we can show that:

$$\hat{N} = \frac{\hat{S}_z^1 + \hat{S}_z^2}{2} + \hat{J}_z \tag{5}$$

where $\hat{N}$ is a constant of motion. Therefore, the Hamiltonian system eq. (3) becomes:

$$\frac{\hat{H}}{\hbar} = \omega\hat{N} + \hat{C} \tag{6}$$

where the operator $\hat{C}$ takes the form:

$$\hat{C} = \sum_{j=1}^{2}\left[\frac{\delta^j\hat{\sigma}_z^j}{2} + \lambda\left(\hat{\sigma}_{+}^{j}\hat{J}_{-} + \hat{\sigma}_{-}^{j}\hat{J}_{+}\right)\right] \tag{7}$$

the quantities δ^j is the detuning function which can be expressed:

$$\delta^j = 2\Omega_0^j - \omega, \quad \text{where} \quad j = 1, 2 \tag{8}$$

Now we assume that the initial conditions for the present system eq. (3), for the atoms start from the general form of the ground and excite states, while the two SU(1, 1) group starts from the atomic coherent state:

$$|\psi(0)\rangle = \left|++\right\rangle \otimes \left|\beta\right\rangle \tag{9}$$

where $|\beta\rangle$ is the initial state of the SU(2) group and can be expressed:

$$\left|\beta\rangle = \sum_{n=-j}^{j} Q_n^j\left(\Theta,\Phi\right)\right|, \ n, j\rangle \tag{10}$$

the coupling coefficients Q_n^j (Θ, Φ) obey the relation:

$$Q_n^j = \sqrt{C_{j+n}^{2j}}\mathrm{e}^{i(j-n)}\Phi\left(\cos\frac{\Theta}{2}\right)^{j+n}\left(\sin\frac{\Theta}{2}\right)^{j-n} \tag{11}$$

where C_{j+2}^{2j} is the binomial coefficient. The exact solution $|\Psi(t)\rangle$ for $t > 0$:

$$\begin{aligned}|\Psi\left(t\right)\rangle = \sum_{n=0}^{\infty}\Big[& X_1\left(n,t\right)\left|ee\right\rangle\left|n, j\right\rangle + X_2\left(n,t\right)\left|eg\right\rangle\left|n+1, j\right\rangle + \\ & + X_3\left(n,t\right)\left|ge\right\rangle\left|n+1, j\right\rangle + X_4\left(n,t\right)\left|gg\right\rangle\left|n+2, j\right\rangle\Big]\end{aligned} \tag{12}$$

The functions X_1, X_2, X_3, and X_4 nd represent the solutions of the system of differential equations, which are given from the Schrodinger equation. Straightforwardly we write the system of differential equations which describe the model eq. (3) under the full resonance case condition ($\delta_1 + \delta_2 = 0$) the differential equations:

$$\begin{aligned}\frac{\mathrm{d}X_1}{\mathrm{d}t} &= -iv_1\left(n\right)\left(X_2 + X_3\right) \\ \frac{\mathrm{d}X_2}{\mathrm{d}t} &= -i\left[\frac{\delta}{2}X_2 + v_1\left(n\right)X_1 + v_2\left(n\right)X_4\right] \\ \frac{\mathrm{d}X_3}{\mathrm{d}t} &= -i\left[\frac{\delta}{2}X_3 + v_1\left(n\right)X_1 - v_2\left(n\right)X_4\right] \\ \frac{\mathrm{d}X_4}{\mathrm{d}t} &= -iv_2\left(n\right)\left(X_2 + X_3\right)\end{aligned} \tag{13}$$

where v_1, v_2 take the form:

$$v_1 = \lambda\sqrt{\left(j-n\right)\left(n+j+1\right)}, \quad v_2 = \lambda\sqrt{\left(j-n-2\right)\left(n+j+3\right)} \tag{14}$$

Therefore, the solutions X_1, X_2, X_3, and X_4 of the aforementioned differential eq. (13). After obtained on the wave function (13) can be used to describe the evolution of some measures of quantum effects as: atomic inversion, entanglement and non-classical correlation, atom-atom entanglement and second-order correlation function.

Atomic inversion

We begin with atomic inversion for the present system eq. (3), which is defined in eq. (2):

$$W\left(t\right) = \langle\hat{\sigma}_z^j\rangle\cos 2\vartheta_j - \left(\langle\hat{\sigma}_+^j\rangle + \langle\hat{\sigma}_-^j\rangle\right)\sin 2\vartheta_j \tag{15}$$

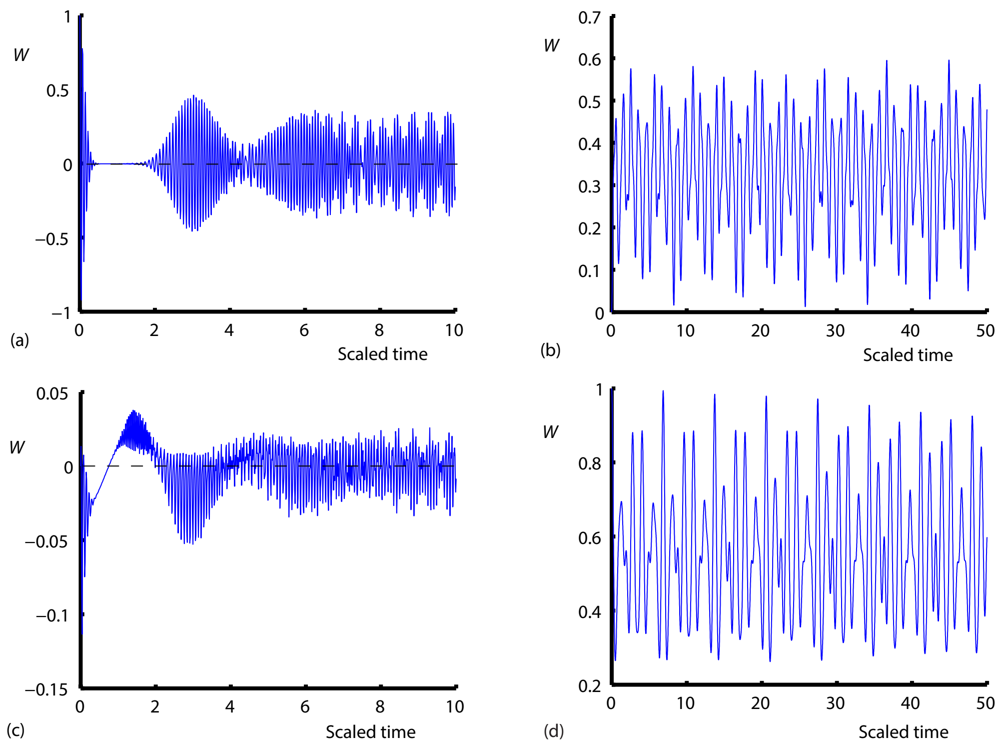

Figure 1. Atomic population $W(t)$ against the scaled time for $\Theta = \pi/4$ and $\Delta = 5$ where the classical field is ignored (a) and (b) and considered (c) and (d); also, the number of level is $J = 60$ (a) and (c), and $J = 3$ (b) and (d)

Figure 1 plots the atomic inversion $W(t)$ against the scaled time λt for fixed detuning parameter $\Delta = 5$ and modification of other parameters. For $\vartheta_1 = \vartheta_2 = 0$ and $j = 60$, the collapse occurs after a very small period of interaction began, followed by more period of revivals. It is noted that the amplitude of oscillation changes between –0.5 to 0.5 and reduces as the time goes on, fig. 1(a). When the number of the level takes small values adjust ($j = 3$), the results are showing that the regular oscillations in the earlier case change completely to a chaotic form, which indicates the effect of the population by the number of levels of the atom as shown in Figure 1(b). After adding the external field $\vartheta_1 = \vartheta_2 = \pi/4$, we notice that the collapse period is asymmetric around the horizontal axis, but it fluctuates smoothly from the lower state to the ex-cited state. We also note that the amplitude of vibrations expands over time and then in revivals periods increases with the increase of the external field coefficient, see fig. 1(c). Note that after reducing the number of atom levels, the atomic inversion becomes somewhat chaotic again. This confirms that the phenomena of revivals and collapses do not occur in the case of the number of small levels as seen in figs. 1(b) and 1(d).

Entanglement and non-classical correlation

In general, the von Neumann entropy measures the QE between the TTLA and a pair of SU(2). The divisible states have several applications in quantum optics [31-33]. The von Neumann entropy function is defined as the atomic entropy and can be written in terms of eigenvalues for the reduced atomic density operator [34, 35]:

$$\rho_{AB}\left(t\right)=tr_{SU(2)}\left\{\left|\Psi\left(t\right)><\Psi\left(t\right)\right|\right\} \tag{16}$$

$$S_{AB}\left(t\right)=-\sum_{j=1}^{4}\lambda_j(t)\ln\left[\lambda_j\left(t\right)\right] \tag{17}$$

Here we use the fidelity as a quantifier of the system state entanglement during the time evolution.

Fidelity

The fidelity is an essential component of quantum information, in characterising quantum phase transitions (QPT) is well known. As a distance measure, the fidelity is used to identify and explain the closeness of two given quantum states. Here, we define the fidelity of the present system in terms of the initial and final state:

$$\zeta\left(t\right)=\left|\langle\Psi\left(0\right)\right|\Psi\left(t\right)\rangle|^2 \tag{18}$$

By using the same conditions as the population inversion, for large values of the parameter j adjust (j = 3) and in absence of the classical external terms, in this case, the fidelity function demonstrates regular behaviour and fluctuates between 0 and 0.5. There are many points approaching to sudden death and rebirth for large values of j, see fig. 2(a). By adding the external field terms, the fidelity death periods are shorted, and the maximum values decrease and reach the perfect 0.08, see fig. 2(c). In the second case, for small values of j adjust (j = 3)

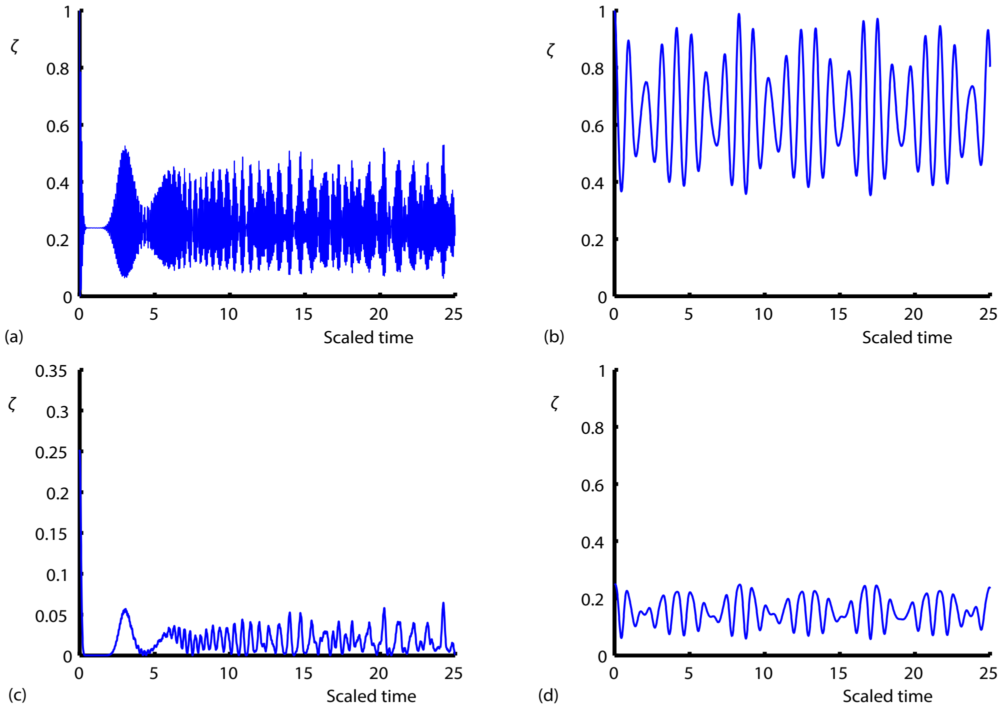

Figure 2. System fidelity $\zeta(t)$ against the scaled time for $\Theta = \pi/4$ and $\Delta = 5$ where the classical field is ignored (a) and (b) and considered (c) and (d); the number of levels: $J = 60$ (a) and (c), and $J = 3$ (b) and (d)

and in absence of the external field terms, the regular oscillations in the aforementioned case become chaotic oscillations between 0.4 and 1. Figure 2(b) shows that the periods of fidelity death appear during the collapse periods, while the maxima happen during the revivals periods of the atomic inversion. Figure 2(d) shows the case where we insert the external terms, the situation has changed dramatically so that the fidelity break down and the maxima decrease considerably as the time increases.

Atom-atom entanglement

Quantum correlations, especially entanglement, have potential applications in information sciences, quantum algorithms and fuzzy systems [34-36]. In this article, we utilize concurrence to estimate the qubit-qubit non-local correlation or entanglement. In terms of the reduced density matrix for TQ A and B concurrence has been defined [37]:

$$C_{AB}(t) = \max\{\mu_4 - \mu_3 - \mu_2 - \mu_1, 0\} \tag{19}$$

where μ_j (j = 1, 2, 3, 4) is the eigenvalues of the square roots of the density matrix $R = \rho_{AB}(\sigma_y \otimes \sigma_y)\,\rho^*_{AB}(\sigma_y \otimes \sigma_y)$ and σ_y is the Pauli matrix, and ρ^*_{AB} – the complex conjugate of ρ_{AB}.

We use the conditions mentioned in atomic inversion study the entanglement between the two atoms. For large values of levels (J = 60) and after excluding the external field, we notice that the function $C_{AB}(t)$ fluctuates between 0 and 0.4, as it reaches the maximum values at the middle of the collapse region while it reaches the smallest values at the end of the collapse region. After that the function decreases slightly and then fluctuates chaotic, as shown in Figure (3a). When we reduce the number of levels (J = 3), we notice that the regular fluctuation in the preceding state becomes chaotic and the entanglement between the two atoms reach separation

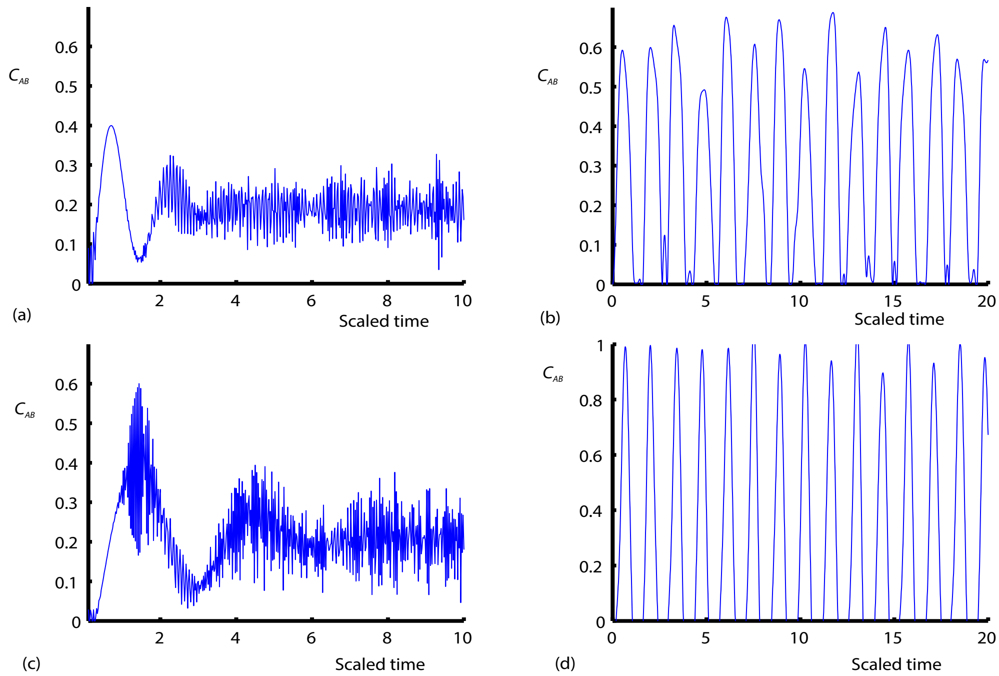

Figure 3. Atom-Atom entanglement measured by the concurrence $C_{AB}(t)$ against the scaled time for $\Theta = \pi/4$ and $\Delta = 5$ where the classical field is ignored (a) and (b) and considered (c) and (d); also, the number of level is $J = 60$ (a) and (c) and $J = 3$ (b) and (d)

at many points, as is evident in the fig. 3(b). When we do the role of the external field, we put $\vartheta_1 = \vartheta_2 = \pi/4$, and in the case of the number of large levels, we find that the maximum values increase and the smallest values decrease. This means that the amplitude of the vibrations has increased after taking into account the external field as seen in the fig. 3(c). For the small values of the number of levels, we find that the function $C_{AB}(t)$ fluctuates and reaches the maximum and minimum values on a regular basis. Thus, entanglement is weak in this case, see fig. 3(d).

Second-order correlation function

The statistical properties of a system through the interaction time could be quantified using the second-order correlation function. The statistical properties of the system exchange between Poissonian, sub-Poissonian, and super-Poissonian according to the value of the second-order correlation function. This section explores the behaviour of the correlation function. We use the normalized second-order correlation function $g^2(t)$ to estimate the coherence behaviour and the classical and non-classical behaviour. The normalized second-order correlation function is defined for the system:

$$g^2(t) = \frac{\langle \hat{J}_+^2 \hat{J}_-^2 \rangle}{\langle \hat{J}_+ \hat{J}_- \rangle^2} \tag{20}$$

In fig. (4) plots the function $g^2(t)$ for the same conditions as the previous sections. For large values of the parameter ($J = 60$) and in the absence of external field terms, the correlation function $g^2(t)$ has oscillation with small amplitude, which reflects that the periods of classical

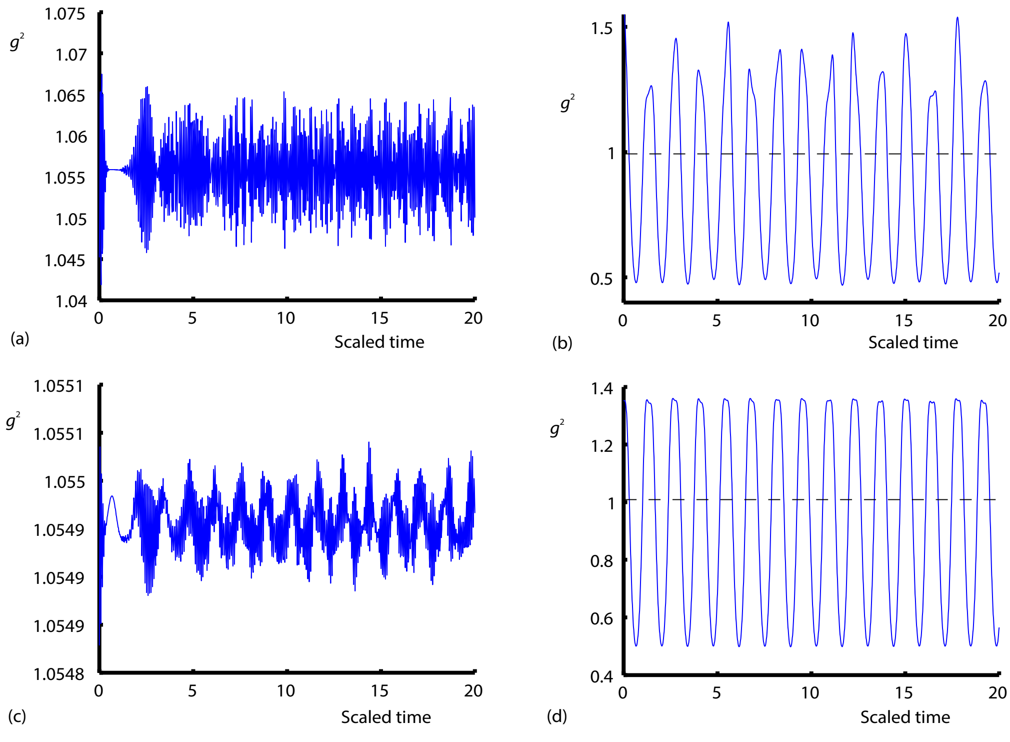

Figure 4. Non-classical properties quantified the second-order correlation function g^2(t) against the scaled time for $\Theta = \pi/4$ and $\Delta = 5$ where the classical field is ignored (a) and (b) and considered (c) and (d); also, the number of level is $J = 60$ (a) and (b), and $J = 3$ (b) and (d)

behaviour as shown in fig. 4(a). For decreasing the parameter number of levels ($J = 3$), the correlation function demonstrates regular fluctuations between the classical and non-classical behaviour, see fig. 4(b). When considering the external field in the presence of the large numbers in this ($J = 60$), in this case, the classical behaviour becomes the dominant behaviour in the system and the amplitude of oscillations is increasing compared to the previous case fig. 4(c). For small values of the number of levels, we notice that the correlation function regular fluctuates between the classical and non-classical behavior and the amplitude of fluctuation increase as seen in fig. 4(d).

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Conclusion

In this article, a model has been studied that contains the interaction of three atoms, two of which are two-level atoms and the other is multi-level. Moreover, a strong external electromagnetic field was added. Transformations of atomic operators were used to obtain the general solution of the Schrödinger differential equations. The collapse and revival phenomena are determined for the large values of the number of levels, while they completely disappeared for the small values of the number of levels. Distortions also occurred to the atomic population after the addition of the external classical field of interaction and generate a strong interaction between the parts of the quantum system. The entanglement between the parts of the system is quantified using fidelity and concurrence, the two measures have great similarity in results. For large values of levels generate a strong entanglement while for small values of the number of levels generate a weak entanglement, while adding the external field increases entanglement intensity during most interaction periods. Moreover, the type of distribution for this model is estimated through the correlation function. For small values of the number of levels the non-classical distribution appeared (sub-Poissonian distribution), but for large values of the number of levels the distribution appeared classical (super-Poissonian distribution), by including the external classical field, the sub-Poissonian distribution periods decreased.

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