A JACOBI WAVELET COLLOCATION METHOD FOR FRACTIONAL FISHER'S EQUATION IN TIME

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In this study, the Jacobi wavelet collocation method is studied to derive a solution of the time-fractional Fisher's equation in Caputo sense. Jacobi wavelets can be considered as a generalization of the wavelets since the Gegenbauer, and thus also Chebyshev and Legendre polynomials are a special type of the Jacobi polynomials. So, more accurate and fast convergence solutions can be possible for some kind of problems thanks to Jacobi wavelets. After applying the proposed method to the considered equation and discretizing the equation at the collocation points, an algebraic equation system is derived and solving the equation system is quite simple rather than solving a non-linear PDE. The obtained values of our method are checked against the other numerical and analytic solution of considered equation in the literature and the results are visualized by using graphics and tables so as to reveal whether the method is effectiveness or not. The obtained results evince that the wavelet method is quite proper because of its simple algorithm, high accuracy and less CPU time for solving the considered equation.

Key words: *time-fractional Fisher's equation, collocation method, fractional differential equation, Jacobi wavelet*

Introduction

Many scientists has focused on the fractional calculus which has numerous application fields in the many branches of science in the last few decades because the scientists assert that the description of properties of different materials and the some processes can be explained by fractional derivatives and integrals, so fractional differential or integral equations. Thanks to the constantly developing computer technology, it is possible to obtain numerical solutions of differential or integral equations with higher accuracy than before. Therefore, researchers are trying to obtain more accurate solutions of these equations than in the literature. Sometimes a small mistake we neglect can have big implications to end human life in the real world. In the literature, there are many proposed method for solving fractional differential equations (FDE) like finite difference [1, 2], Adomian decomposition [3, 4] and homotopy perturbation methods (HPM) [5, 6]. For more details about fractional calculus, see [7-11].

The solution of TFF equation is considered by few authors in the literature [12-15]. So we aim to solve TFF equation by using Jacobi wavelet collocation method in the research. Zhang *et al*. [12] solve the TFF equation using the HPM. Zhang *et al*. [12] also study TFF equation by local discontinuous Galerkin finite element method in another papers of them. An exact

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solution of the TFF equation which has a small delay is solved by the residual power series method [13, 14]. The paper [16] aims to investigate the symmetry properties and analytic solutions of the TFF equation by the help of the Lie symmetries. The TFF equation is given [12]:

$$D_t^{\alpha} v(x,t) = \rho v_{xx}(x,t) + \kappa v(x,t)\left[1 - v^{\sigma}(x,t)\right] = 0 \tag{1}$$

subject to the following initial condition:

$$v(x,0) = V_0(x) \tag{2}$$

and boundary conditions:

$$v(0,t) = V_1(t),\ v(1,t) = V_2(t) \tag{3}$$

where D_t^{α} represents the fractional derivative in time, κ – the parameter, the constant ρ denotes the diffusivity of population, $\sigma = 1, 2,...$ and $0 < \alpha < 1$. For $\alpha = 1$, the eq. (1) equals to the ordinary Fisher's equation introduced by Fisher [17] in 1937.

Wavelets have many a variety areas of science and engineering such as mathematics, signal processing, *etc*. Since the last three decades, various types of wavelets has been widely preferred to solve differential equations (ordinary, partial or fractional differential equations) because of its many advantages such as simple algorithm, high accuracy and less CPU time [18, 19]. On the other hand, Jacobi wavelets are quitely fresh topic which has been studied over the last half decade. In [20], Jacobi wavelets are defined and it is used for solving fractional integro-differential equations. Zaky *et al*. [21] studied Jacobi wavelet collocation method based on the operational matrix of integration of Jacobi wavelets. A new family of regularized Jacobi wavelets is defined in [22]. In this study, the Jacobi wavelet collocation method based on the operational matrix of derivatives of Jacobi wavelets is used to solve the fractional-fisher since Jacobi wavelets can be considered as a generalization of Gegenbauer (thus also Legendre and Chebyshev wavelets). When Jacobi wavelets are compared with these wavelets, one of its big advantages is that optimal values of the parameters ε, γ of Jacobi wavelets can be differently selected according to the types of the equation. So, getting more accurate and faster convergent solutions can be possible. The other one advantage and also the primary advantage of the method we used is that the method enables a simple procedure to convert the PDE with fractional derivative (in time or spatial) to a algebraic system that can be solved simply by many conventional methods.

Preliminaries

Block pulse functions

A complete set of orthogonal functions formed block pulse functions (BPF) can be represented on the interval [0, b) [23]:

$$b_i(x) = \begin{cases} 1, & \dfrac{i-1}{N} \le x < \dfrac{i}{N} b \\ 0, & \text{otherwise} \end{cases} \tag{4}$$

The fractional integral (in Riemann-Liouville sense)

The α order fractional integral operator I^{α} of a function $v(t)$ is given [7]:

$$\left(I^{\alpha} v\right)(t) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-k)^{\alpha-1} v(k)\,\mathrm{d}k, & \alpha \in \mathbb{R}^+ \\ v(t), & \alpha = 0 \end{cases} \tag{5}$$

where $\Gamma(.)$ is the gamma function.

The fractional derivative (in the Caputo sense)

The α order fractional derivative operator D_t^α of a function $v(t)$ is defined [24]:

$$\left(D_t^\alpha v\right)(t)=\begin{cases}\dfrac{1}{\Gamma(n-\alpha)}\displaystyle\int_0^t\frac{1}{(t-k)^{(\alpha-n+1)}}\frac{\mathrm{d}^n v(k)}{\mathrm{d}k^n}\mathrm{d}k, & n-1<\alpha\le n\\ \dfrac{\mathrm{d}^n v(k)}{\mathrm{d}k^n}, & \alpha=n\end{cases}\tag{6}$$

where α, $t>0$ and $n\in\mathbb{N}$. The useful relation between derivative and integral operator is given:

$$\left(I^\alpha D_t^\alpha v\right)(t)=v(t)-\sum_{k=0}^{n-1}v^{(k)}(0^+)\frac{t^k}{k!},\ t>0,\ n-1<\alpha\le n\tag{7}$$

Jacobi polynomials

The Jacobi polynomials $J_m^{(\varepsilon,\gamma)}(x)$ which was introduced by Carl Gustav Jacob Jacobi are defined via the following iterative formula for all $\varepsilon>-1$, $\gamma>-1$:

$$J_m^{(\varepsilon,\gamma)}(x)=\frac{(\varepsilon+\gamma+2m-1)\left[\varepsilon^2-\gamma^2+x(\varepsilon+\gamma+2m)(\varepsilon+\gamma+2m-2)\right]}{2m(\varepsilon+\gamma+2m-2)(\varepsilon+\gamma+m)}J_{m-1}^{(\varepsilon,\gamma)}(x)-$$
$$-\frac{(\varepsilon+m-1)(\gamma+m-1)(\varepsilon+\gamma+2m)}{m(\varepsilon+\gamma+2m-2)(\varepsilon+\gamma+m)}J_{m-2}^{(\varepsilon,\gamma)}(x)\tag{8}$$

where

$$J_0^{(\varepsilon,\gamma)}(x)=1$$
$$J_1^{(\varepsilon,\gamma)}(x)=\frac{\varepsilon+\gamma+2}{2}x+\frac{\varepsilon-\gamma}{2}\tag{9}$$

The weight function for Jacobi polynomials is given [22]:

$$\omega(x)=(1-x)^\varepsilon(1+x)^\gamma\tag{10}$$

Wavelets

The families of continuous wavelets are established from scaling and translation of a mother (or basic) wavelet $\psi(t)$:

$$\psi_{a,b}(t)=|a|^{-1/2}\psi\left(\frac{t-b}{a}\right),\quad a,b\in\mathbb{R},\quad a\ne0\tag{11}$$

in which a and b denote parameters of the scaling and translation, respectively. If a, b are restricted to $a=a_0^{-k}$, $b=nb_0a_0^{-k}$, in which k, $n\in\mathbb{N}$, and $1<a_0$, $0<b_0$, the following discrete wavelets $\psi_{k,n}(t)$ which constitute a basis for $L^2(\mathbb{R})$ are derived:

$$\psi_{k,n}(t)=\left|a_0\right|^{k/2}\psi\left(a_0^k t-nb_0\right)\tag{12}$$

If $a_0=2$ and $b_0=1$ are selected, $\psi_{k,n}(t)$ forms an orthonormal basis for $L^2(\mathbb{R})$:

$$\psi_{k,n}(t)=2^{k/2}\psi\left(2^k t-n\right)\tag{13}$$

The Jacobi wavelets are given [20, 21]:

$$\psi_{n,m}^{(\varepsilon,\gamma)}(x)=\begin{cases}\dfrac{2^{\frac{k+1}{2}}}{\sqrt{h_m^{(\varepsilon,\gamma)}}}J_m^{(\varepsilon,\gamma)}\left(2^{k+1}x-2n+1\right), & \dfrac{n-1}{2^k}\le x<\dfrac{n}{2^k}\\ 0, & \text{otherwise}\end{cases} \tag{14}$$

where

$$h_m^{(\varepsilon,\gamma)}=\frac{2^{\varepsilon+\gamma+1}\Gamma(\varepsilon+m+1)\Gamma(\gamma+m+1)}{(2m+1+\varepsilon+\gamma)m!\Gamma(\varepsilon+\gamma+m+1)} \tag{15}$$

where k, $M\in\mathbb{N}$, $n=1, 2,...,2^k$, are the number of decomposition levels, $m=0, 1, 2,..., M$ is the degree of the Jacobi polynomials, and Γ – the gamma function. The coefficient:

$$\frac{2^{\frac{k+1}{2}}}{\sqrt{h_m^{(\varepsilon,\gamma)}}}$$

in eq. (14) is for the normality. If $\varepsilon=\gamma$ in eq. (14), Gegenbauer wavelets are derived. In Gegenbauer wavelets, for $\gamma=0.5$, Legendre wavelets are derived and for $\gamma=0$, $\gamma=1$, Chebyshev wavelets of first kind and second kind are derived, respectively, γ is the parameter of Gegenbauer wavelets.

For the theorem of convergence of the Jacobi polynomials and Jacobi wavelets, please see [21, 25, 26].

Approximation the functions by wavelets

A function $g(x)\in L^2[0,1]$ can be written by Jacobi wavelets:

$$g(x)=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}g_{nm}\psi_{n,m}^{(\varepsilon,\gamma)}(x) \tag{16}$$

where

$$g_{nm}=\left\langle g(x),\psi_{n,m}^{(\varepsilon,\gamma)}(x)\right\rangle_{\omega_n}\quad\text{and}\quad\langle .,.\rangle_{\omega_n}$$

symbolizes the inner product. For simplicity, the series can be truncated as follows:

$$g(x)=\sum_{n=1}^{2^{(k-1)}}\sum_{m=0}^{M-1}g_{nm}\psi_{n,m}^{(\varepsilon,\gamma)}(x)=\mathbf{G}^T\mathbf{\Psi}(x) \tag{17}$$

where the superscript ^{T} is the transpose of the matrix and $\mathbf{G}$ and $\mathbf{\Psi}(x)$ are $N=2^{k-1}M$ matrices:

$$\mathbf{G}=\left[g_{10},g_{11},\ldots,g_{1M-1},g_{20},g_{21},\ldots,g_{2M-1},\ldots,g_{2^{k-1}0},g_{2^{k-1}1},\ldots,g_{2^{k-1}M-1}\right]^T$$

$$\mathbf{\Psi}(x)=\left[\psi_{10}^{(\varepsilon,\gamma)},\psi_{11}^{(\varepsilon,\gamma)},\ldots,{}_{1M-1}^{(\varepsilon,\gamma)},\psi_{20}^{(\varepsilon,\gamma)},\psi_{21}^{(\varepsilon,\gamma)},\ldots,\psi_{2M-1}^{(\varepsilon,\gamma)},\ldots,\psi_{2^{k-1}0}^{(\varepsilon,\gamma)},\psi_{2^{k-1}1}^{(\varepsilon,\gamma)},\ldots,\psi_{2^{k-1}M-1}^{(\varepsilon,\gamma)}\right]^T$$

The wavelet transform of the function $g(x)$ can be written as more compact representation:

$$g(x)=\sum_{i=1}^{N}g_i\psi_i(x)=\mathbf{G}^T\mathbf{\Psi}(x) \tag{18}$$

where the index i can be derived from $i = m + M(n-1) + 1$ and $N = 2^{k-1}M$, $\mathbf{G} = [g_1, g_2,...,g_N]^T$, $\mathbf{\Psi}(x) = [\psi_1, \psi_2,..., \psi_N]^T$. For $i = 1, 2,...,N$, the collocation points x_i are given:

$$x_i = \frac{2i-1}{2N},\ \left(i = 1,2,\ldots N\right) \tag{19}$$

Substituting the collocation points x_i into $\mathbf{\Psi}(x)$, we get the Jacobi wavelet matrix $\mathbf{\Phi}_{N\times N}$ as:

$$\mathbf{\Phi}_{N\times N} = \left[\mathbf{\Psi}\left(\frac{1}{2N}\right), \mathbf{\Psi}\left(\frac{3}{2N}\right),\ldots, \mathbf{\Psi}\left(\frac{2N-1}{2N}\right)\right] \tag{20}$$

A function $v(x, t) \in L^2([0, 1]\times[0, 1])$ can be expanded by Jacobi wavelets:

$$v(x,t) = \sum_{i=1}^{N}\sum_{j=1}^{N} v_{ij}\psi_i\left(x\right)\psi_j\left(t\right) = \mathbf{\Psi}^T\left(x\right)\mathbf{V}\mathbf{\Psi}\left(t\right) \tag{21}$$

where the elements v_{ij} of the matrix $\mathbf{V}$ can be calculated:

$$v_{ij} = \left\langle \psi_i(x), \left\langle v(x,t), \psi_j(t)\right\rangle_{\omega_n(t)}\right\rangle_{\omega_n(x)} \tag{22}$$

where the ω_n is the denotes the weight function for the orthogonality of Jacobi wavelets which are given by the formula $\omega_n(x) = \omega(2^{k+1}x - 2n + 1)$. For more details, see [22].

Operational matrix of integration

The n-times operational matrix $\mathbf{P}^n$ of integration of $\mathbf{\Psi}(x)$ can be expressed:

$$\underbrace{\int_0^t \ldots \int_0^t \mathbf{\Psi}(t)\mathrm{d}s\ldots\mathrm{d}s}_{n-\text{times}} \simeq \mathbf{P}^n\mathbf{\Psi}(t) \tag{23}$$

The fractional integration of the vector $\mathbf{\Psi}(x)$ is approximated [27]:

$$\left(I^\alpha\mathbf{\Psi}\right)(t) \simeq \mathbf{P}^\alpha\mathbf{\Psi}\left(t\right) \tag{24}$$

where $\mathbf{P}^\alpha$ is named the Jacobi wavelet operational matrix of fractional integration. The $\mathbf{P}^\alpha$ is defined:

$$\mathbf{P}^\alpha \cong \mathbf{P}^\alpha_{N\times N} = \mathbf{\Phi}\mathbf{P}^\alpha_B\mathbf{\Phi}^{-1} \tag{25}$$

where $\mathbf{P}^\alpha_B$, the BPF operational matrix of integration:

$$\mathbf{P}^\alpha_B = \frac{1}{N^\alpha}\frac{1}{\Gamma\left(\alpha+2\right)}\begin{bmatrix} 1 & Y_1 & Y_2 & \cdots & Y_{N-1} \\ 0 & 1 & Y_1 & \ldots & Y_{N-2} \\ 0 & 0 & 1 & \ldots & Y_{N-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{bmatrix} \tag{26}$$

where $Y_k = (k + 1)^{\alpha+1} - 2k^{\alpha+1} + (k - 1)^{\alpha+1}$ and $\mathbf{\Phi}$ is the wavelet matrix eq. (20).

Solution procedure

We consider the fractional-fisher with initial conditions given by eq. (2) and boundary conditions by eq. (3). Let' s assume:

$$\frac{\partial^{\alpha+2} v(x,t)}{\partial t^{\alpha} \partial x^2} = \mathbf{\Psi}^T(x)\mathbf{V}\mathbf{\Psi}(t) \tag{27}$$

in which $[v_{ij}]_{N\times N}$ are unknown matrices which should be determined. Integrating the eq. (27) order α times with respect to (w.r.t.) t and substituting the initial condition to it, we get:

$$\frac{\partial^2 v(x,t)}{\partial x^2} = \mathbf{\Psi}(x)^T \mathbf{V}\mathbf{P}^{\alpha}\mathbf{\Psi}(t) + V_0''(x) \tag{28}$$

Integrating the eq. (27) two times w.r.t. x, we get:

$$\frac{\partial^{\alpha} v(x,t)}{\partial t^{\alpha}} = \mathbf{\Psi}(x)^T \left(\mathbf{P}^2\right)^T \mathbf{V}\mathbf{\Psi}(t) + \left.\frac{\partial^{\alpha} v(x,t)}{\partial t^{\alpha}}\right|_{x=0} + x\left.\frac{\partial}{\partial x}\left[\frac{\partial^{\alpha} v(x,t)}{\partial t^{\alpha}}\right]\right|_{x=0} \tag{29}$$

Substituting $x = 1$ to the eq. (29), we have:

$$\left.\frac{\partial}{\partial x}\left[\frac{\partial^{\alpha} v(x,t)}{\partial t^{\alpha}}\right]\right|_{x=0} = \frac{\partial^{\alpha} V_2(t)}{\partial t^{\alpha}} - \frac{\partial^{\alpha} V_1(t)}{\partial t^{\alpha}} - \mathbf{\Psi}(1)^T \left(\mathbf{P}^2\right)^T \mathbf{V}\mathbf{\Psi}(t) \tag{30}$$

Substituting the eq. 30 to the eq. 29, we have:

$$\frac{\partial^{\alpha} v(x,t)}{\partial t^{\alpha}} = \mathbf{\Psi}(x)^T \left(\mathbf{P}^2\right)^T \mathbf{V}\mathbf{\Psi}(t) + \frac{\partial^{\alpha} V_1(t)}{\partial t^{\alpha}} + x\left[\frac{\partial^{\alpha} V_2(t)}{\partial t^{\alpha}} - \frac{\partial^{\alpha} V_1(t)}{\partial t^{\alpha}} - \mathbf{\Psi}(1)^T \left(\mathbf{P}^2\right)^T \mathbf{V}\mathbf{\Psi}(t)\right] \tag{31}$$

Integrating the eq. (31) order α times w.r.t. t, we derive:

$$v(x,t) = \mathbf{\Psi}(x)^T \left(\mathbf{P}^2\right)^T \mathbf{V}\mathbf{P}^{\alpha}\mathbf{\Psi}(t) - x\mathbf{\Psi}(1)^T \left(\mathbf{P}^2\right)^T \mathbf{V}\mathbf{P}^{\alpha}\mathbf{\Psi}(t) + R(x,t) \tag{32}$$

where

$$R(x,t) = V_0(x) + V_1(t) - V_1(0) + x\left[V_2(t) - V_2(0) - V_1(t) + V_1(0)\right] \tag{33}$$

Substituting the eqs. (28), (29), and (32) to he eq. (1), and substituting the collocation points x_i and t_i to the new equation, we get an algebraic equation system. After solving the system, we find the unknown matrix **V.** Substituting the matrix **V** to the eq. (32), we find the solution we are looking for.

Application

Some test examples are considered in order to confirm our theoretical statements which are given in the previous section. Numerical computations and simulations of the test examples are fulfilled by MAPLE.

Example 1. Let's deal with the eq. (1) for $\alpha = 1$, $\rho = 1$, $\kappa = 1$, and $\sigma = 1$. So, we have:

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + v(1 - v) \tag{34}$$

where $0 < x < 1$ and the initial and boundary conditions are given, respectively $v(x, 0) = \lambda$, $v(0, t) = v(1, t) = (\lambda e^t)/(1 - \lambda + \lambda e^t)^2$, λ is a constant.

The analytic solution $v(x, t) = (\lambda e^t)/(1 - \lambda + \lambda e^t)^2$ is in the reference [19]. Applying the our solution procedure on Section *Solution procedure* to eq. (34), we get the results in the tab. 1 and fig. 1. From the results, we deduce that our method is more accurate and faster convergence than the Haar wavelet method in [19].

Table 1. Numerical results for *Example 1* at the collocation points for $\lambda = 3$, $k = 2$, $M = 2$, $\varepsilon = 1$, and $\gamma = 3$

x	t	v_{exact}	$v_{\text{J.wavelet}}$	$\lvert v_{\text{J.wavelet}} - v_{\text{exact}} \rvert$
0.125	0.125	2.4291376128774	2.42913761288732	$9.92 \cdot 10^{-12}$
	0.375	1.8456751718109	1.84567517186078	$4.98 \cdot 10^{-11}$
	0.625	1.5548253634115	1.55482536339462	$1.67 \cdot 10^{-11}$
	0.875	1.3848651116036	1.38486511155072	$5.28 \cdot 10^{-10}$
0.375	0.125	2.4291376128774	2.42913761304581	$1.68 \cdot 10^{-10}$
	0.375	1.8456751718109	1.84567517202329	$2.12 \cdot 10^{-10}$
	0.625	1.5548253634115	1.55482536326964	$1.41 \cdot 10^{-10}$
	0.875	1.3848651116036	1.38486511144069	$1.62 \cdot 10^{-11}$
0.625	0.125	2.4291376128774	2.42913761280096	$7.64 \cdot 10^{-11}$
	0.375	1.8456751718109	1.84567517178089	$3.00 \cdot 10^{-11}$
	0.625	1.5548253634115	1.55482536346320	$5.18 \cdot 10^{-12}$
	0.875	1.3848651116036	1.38486511159793	$5.66 \cdot 10^{-11}$
0.875	0.125	2.4291376128774	2.42913761285192	$2.54 \cdot 10^{-11}$
	0.375	1.8456751718109	1.84567517180090	$1.00 \cdot 10^{-11}$
	0.625	1.5548253634115	1.55482536342867	$1.72 \cdot 10^{-11}$
	0.875	1.3848651116036	1.38486511160171	$1.88 \cdot 10^{-12}$

Example 2. We consider $\rho = 1$, $\sigma = 1$ in fractional-fisher, the equation is reduced:

$$\frac{\partial^{\alpha} v}{\partial t^{\alpha}} = \frac{\partial^2 v}{\partial x^2} + \kappa v(1 - v) \tag{35}$$

where $0 < x < 1$ and the initial and boundary conditions are given, respectively:

$$v(x,0) = \frac{1}{\left(1 + \mathrm{e}^{\sqrt{\frac{\kappa}{6}}x}\right)^2}, \quad v(0,t) = \frac{1}{\left(1 + \mathrm{e}^{-\frac{5}{6}\kappa t}\right)^2}, \quad v(1,t) = \frac{1}{\left(1 + \mathrm{e}^{\sqrt{\frac{\kappa}{6}} - \frac{5}{6}\kappa t}\right)^2}$$

For $\alpha = 1$, analytic solution of the equation:

$$v(x,t) = \frac{1}{\left(1 + e^{\sqrt{\frac{\kappa}{6}}x - \frac{5}{6}\kappa t}\right)^2}$$

in [19]. For $\kappa = 6$, applying the our solution procedure, we get the results in the fig. 2 and tabs. 2 and 3. When the tables and figure are analyzed, it is obvious that we obtain pretty better results than the results in both [28, 29].

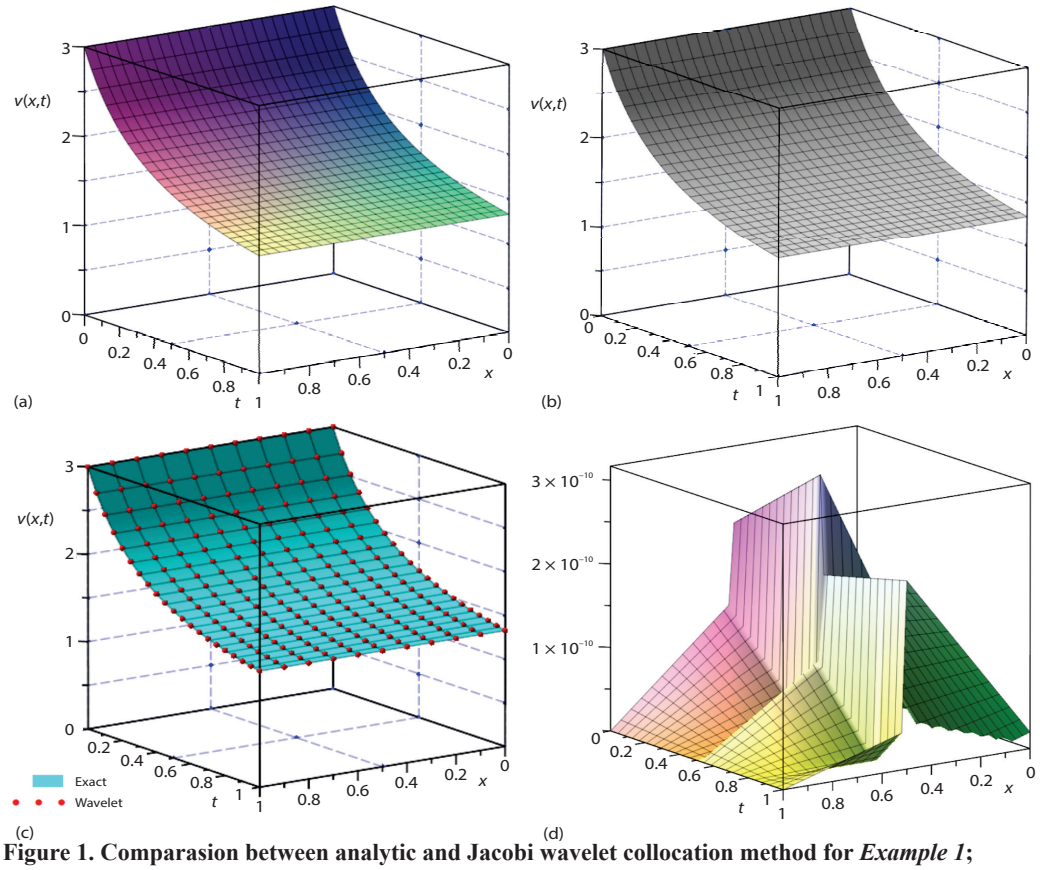

Figure 1. Comparasion between analytic and Jacobi wavelet collocation method for *Example 1*; (a) analytic solution, (b) wavelet solution, (c) both analytic and wavelet solution, and (d) error

Table 2. Comparison between our method and the reference [28] for $k = 2$, $M = 2$, $\varepsilon = 1$, and $\gamma = 3$

x	t	Analytic solution, v_{exact}	Wavelet solution, v_{wavelet}	Absolute error, $\lvert v_{\text{wawelet}} - v_{\text{exact}}\rvert$	[28], v_{HPTM}	Absolute error [28], $\lvert v_{HPTM} - v_{\text{exact}}\rvert$
0.3	0.10	0.302317425	0.302190142	0.000127283	0.304691131	0.002373706
	0.11	0.316042418	0.315734505	0.000307913	0.319292625	0.003250207
	0.12	0.329984205	0.329444603	0.000539602	0.334319781	0.004335576
	0.13	0.344120184	0.343302388	0.000817796	0.349777090	0.005656906
	0.14	0.358426914	0.357289416	0.001137500	0.365669045	0.007242131
0.4	0.10	0.275603147	0.276089876	0.000486729	0.276611064	0.001007917
	0.11	0.288830839	0.289218924	0.000388085	0.290266450	0.001435611
	0.12	0.302317425	0.302545020	0.000227596	0.304302372	0.001984947
	0.13	0.316042418	0.316052513	0.000010094	0.318718535	0.002676117
	0.14	0.329984205	0.329725172	0.000259033	0.333514645	0.003530440
0.5	0.10	0.250000000	0.240557428	0.009442570	0.249765515	0.000234485
	0.11	0.262653582	0.252704058	0.009949520	0.262435106	0.000218475
	0.12	0.275603147	0.265079052	0.010524100	0.275441031	0.000162116
	0.13	0.288830839	0.277669150	0.011161700	0.288778885	$5.19537 \cdot 10^{-5}$
	0.14	0.302317425	0.290460339	0.011857100	0.302444264	0.000126839

Table 3. Comparison between our method and the reference [29] for $k = 3$, $M = 2$, $t = 0.01$, $\varepsilon = 1$, and $\gamma = 1$

x	Analytic solution, v_{exact}	Wavelet solution, $v_{wavelet}$	Absolute error, $\lvert v_{wawelet} - v_{exact}\rvert$	[29]
0.00	0.262654	0.262655	$1.693 \cdot 10^{-6}$	0.2627
0.25	0.202649	0.201227	$1.422 \cdot 10^{-3}$	0.2027
0.50	0.151602	0.149684	$1.917 \cdot 10^{-3}$	0.1516
0.75	0.110099	0.110098	$1.697 \cdot 10^{-6}$	0.1101
1.00	0.077776	0.077776	$9 \cdot 10^{-11}$	0.0778

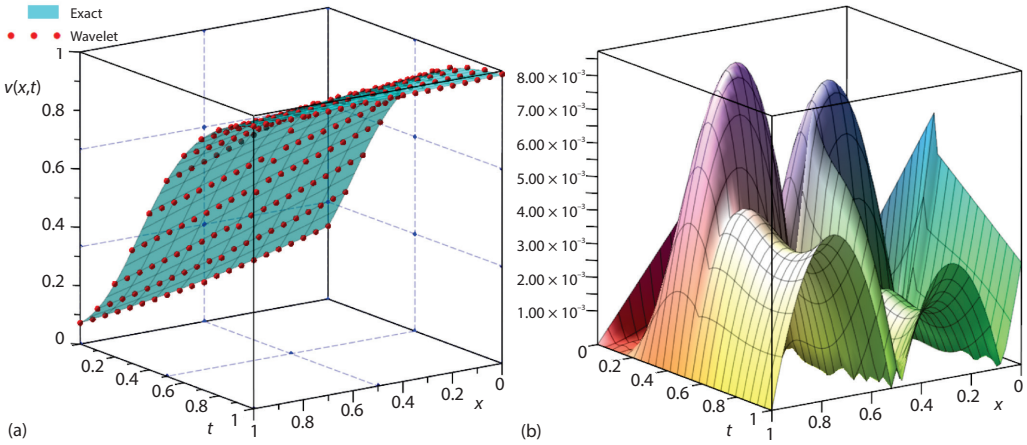

Figure 2. Comparison between analytic solution and Jacobi wavelet collocation method for *Example 2*: (a) both analytuc and wavelet solution, (b) error

For $\kappa = 6$ and different fractional α-values, the solutions of eq. (35) are compared in the fig. 3.

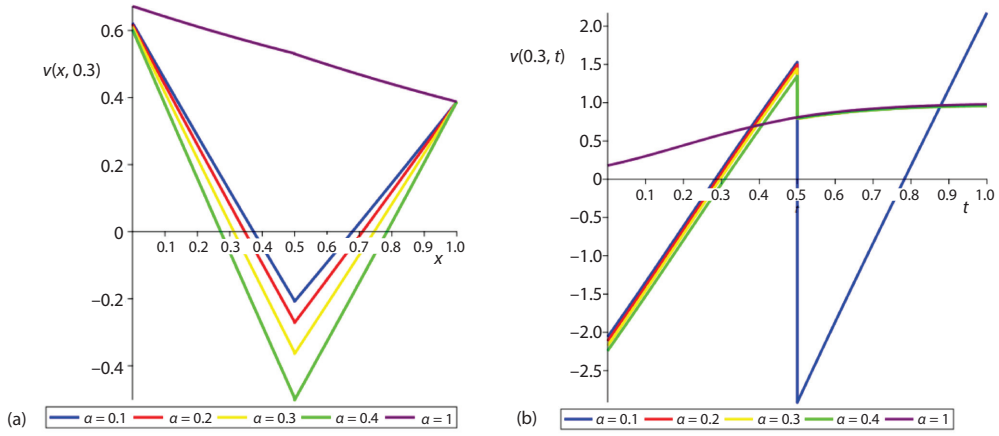

Figure 3. Comparision of solutions for different fractional α order for *Example 2*; (a) $v(x, t)$ for $t = 0.3$, $k = 2$, $M = 2$, $\varepsilon = 1$, and $\gamma = 3$, (b) $v(x, t)$ for $x = 0.3$, $k = 2$, $M = 2$, $\varepsilon = 1$, and $\gamma = 3$

Conclusion

In the study, a numerical solution of the TFF equation is derived by using Jacobi wavelet collocation method. One of advantages of Jacobi wavelets is that optimal values of the parameters ε, γ of Jacobi wavelets can be optimally selected according to the types of the equation in order to get more accurate and faster convergent solution. Finding the optimal parameters for a problem is a good topic for another works. Also, the method enables a simple procedure to convert a considered equation a algebraic system that we can solve simply. The procedure is tested on test examples to evidence the precision and effectiveness of the method. The numerical computations are done by using MAPLE software and comparatively shown in the tables and figures. The findings reveal that the presented method is extremely effective for TFF equation and it has high accuracy and less CPU time because of its simple algorithm.

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