NEW INSIGHT INTO THE FOURIER-LIKE AND DARCY-LIKE MODELS IN POROUS MEDIUM

by

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In this study, we propose the general calculus operators based on the Richardson scaling law and Korcak scaling law. The Richardson-scaling-law calculus is considered to investigate the Fourier-like law for the scaling-law flow of the heat in the heat-transfer process. The Korcak-scaling-law calculus is used to model the Darcy-like law for describing the scaling-law flow of the fluid in porous medium. The formulas are as the special cases of the topology calculus proposed for descriptions of the fractal scaling-law behaviors in nature phenomena.

Key words: scaling law, Richardson-scaling-law calculus, Fourier-like law, Korcak-scaling-law calculus, Darcy-like law, topology calculus

Introduction

The scaling law is a mathematical relationship, which is used to describe the complex behaviors in the nature phenomena, for instance, anomalous Hall effect [1], slow earthquakes [2], grey matter and white matter of cerebral cortex [3], nano-structured materials [4], turbulent shear flows [5], and human behavioral organization [6].

Let us recall the scaling laws as follows. The Mandelbrot scaling law, proposed by Mandelbrot in 1967, is presented as follows [7]:

\[
\phi(t) = \kappa t^{1-D}
\]

where \( \kappa \in (0, +\infty) \), \( t \in (0, +\infty) \), and \( D \in (0, +\infty) \) is the fractal dimension. The Richardson scaling law, coined by Richardson in 1926 [8], is given:

\[
\psi(t) = \kappa t^D
\]

where \( \kappa \in (0, +\infty) \), \( t \in (0, +\infty) \), and \( D \in (0, +\infty) \) is the scaling exponent. The Korcak scaling law, suggested by Korcak in 1938 [9]:

\[
\omega(t) = \kappa t^{-D}
\]

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where $\kappa \in (0, +\infty)$, $t \in (0, +\infty)$, and $D \in (0, +\infty)$ is the scaling exponent. The scaling law in life, presented by West et al. in 1999 [10], reads:

$$\varphi(t) = \kappa t^\beta + c$$

(5)

where $\kappa \in (0, +\infty)$ is the normalization constant, $t \in (0, +\infty)$ is the scaling exponent, $c \in (-\infty, +\infty)$ is the constant, and $t \in (-\infty, +\infty)$ is the radius. The topology calculus was proposed in [11] based on the Leibniz derivative [12], Stieltjes integral (or Stieltjes-Riemann integral) [13] and Riemann integral [14] (for more details, see [15]). The topology calculus was proposed in [15].

Due to the scaling-law behaviors in the temperature scaling law [16] and in the porous media [17], the main targets of the present paper are to propose the general calculus operators containing the Richardson scaling law and Korcak scaling law, and to consider the Fourier-like law for the scaling-law flow of the heat in the heat-transfer process and Darcy-like law for the scaling-law flow of the fluid in porous medium.

The general calculus operators involving the Richardson scaling law and Korcak scaling law

In this section, we propose the Richardson-scaling-law calculus and the Korcak-scaling-law calculus and discuss their properties based on the topology calculus.

Let $\mathbb{N}(\Phi)$ be the set of the continuous functions $\Phi(\varphi)$ in the domain $A$ and let $\mathcal{N}(\varphi)$ be the set of the continuous derivatives of the functions $\varphi(t)$ in the domain $B$.

Let $\Phi_{\varphi}(t) = (\Phi \circ \varphi)(t) = \Phi(\varphi(t))$.

Let us consider the sets of the composite functions, given:

$$\mathcal{S}(\Phi_{\varphi}) = \{ \Phi_{\varphi}(t) : \Phi_{\varphi}(t) = (\Phi \circ \varphi)(t), \Phi \in \mathbb{N}(\Phi), \Phi \in \mathcal{N}(\varphi), \varphi \in \mathcal{S}(\varphi) \}$$

(6)

The topology calculus

Let $\Phi_{\varphi} \in \mathcal{S}(\Phi_{\varphi})$, where $\varphi(t) = \kappa t^\beta + c$.

The topology derivative of the function $\Phi_{\varphi}(t)$ is defined as [15]:

$$T_{D_{t}}(\Phi_{\varphi}) = \frac{1}{(\kappa t^\beta + c)^{(1)}} \frac{d\Phi_{\varphi}(t)}{dt}$$

(7)

where $\kappa$ is the normalization constant, $\beta$ is the scaling exponent, $t$ is the radius, and $c$ is the moving term.

The topology partial derivatives of the function $\Phi_{\varphi} = \Phi_{\varphi}(x, y, z)$ are defined:

$$T_{\xi}^{(1)} \Phi_{\varphi} = \frac{1}{(\kappa x^\beta + c)^{(1)}} \frac{\partial \Phi_{\varphi}}{\partial x}, \quad T_{\eta}^{(1)} \Phi_{\varphi} = \frac{1}{(\kappa y^\beta + c)^{(1)}} \frac{\partial \Phi_{\varphi}}{\partial y}, \quad T_{\zeta}^{(1)} \Phi_{\varphi} = \frac{1}{(\kappa z^\beta + c)^{(1)}} \frac{\partial \Phi_{\varphi}}{\partial z}$$

$$T_{\xi}^{(2)} \Phi_{\varphi} = \frac{T_{\xi}^{(1)} \Phi_{\varphi}}{\partial \xi}, \quad T_{\eta}^{(2)} \Phi_{\varphi} = \frac{T_{\eta}^{(1)} \Phi_{\varphi}}{\partial \eta}, \quad T_{\zeta}^{(2)} \Phi_{\varphi} = \frac{T_{\zeta}^{(1)} \Phi_{\varphi}}{\partial \zeta}$$

$$T_{\xi} \xi \Phi_{\varphi} = \frac{T_{\xi} \xi \Phi_{\varphi}}{\partial \xi}, \quad T_{\eta} \eta \Phi_{\varphi} = \frac{T_{\eta} \eta \Phi_{\varphi}}{\partial \eta}, \quad T_{\zeta} \zeta \Phi_{\varphi} = \frac{T_{\zeta} \zeta \Phi_{\varphi}}{\partial \zeta}$$

(10)
The topology differential of the function $\Phi(t)$, denoted by $d\Phi(t)$, is given:

$$d\Phi(t) = (\kappa t^\beta + c) \frac{d}{dt} D_1^{(1)} \Phi(t) dt$$

(8)

Let $\Theta_\varphi \in \Re(\Phi_\varphi)$, where $\varphi(t) = \kappa t^\beta + c$.

The topology integral of the function $\Theta_\varphi(t)$ is defined [15]:

$$T_I^{(1)} \Theta_\varphi(t) = \Theta_\varphi(t) \int \kappa t^\beta + c dt$$

(9)

where $\kappa$ is the normalization constant, $\beta$ is the scaling exponent, $t$ is the radius, and $c$ is the moving term.

The indefinite topology integral of the function $\Theta_\varphi(t)$ is defined [15]:

$$T_I^{(1)} \Theta_\varphi(t) = \Theta_\varphi(t) \int \kappa t^\beta + c dt$$

(10)

where $\kappa$ is the normalization constant, $\beta$ is the scaling exponent, $t$ is the radius, and $c$ is the moving term.

Let $\Theta_\varphi \in \Re(\Phi_\varphi)$ and $\Pi_\varphi \in \Re(\Phi_\varphi)$.

The properties of the topology calculus can be given:

(A1) The sum and difference rules for the topology derivative:

$$T D_1^{(1)} \left[ \Theta_\varphi(t) \pm \Pi_\varphi(t) \right] = T D_1^{(1)} \Theta_\varphi(t) \pm T D_1^{(1)} \Pi_\varphi(t)$$

(11)

(A2) The constant multiple rule for the topology derivative:

$$T D_1^{(1)} \left[ C \Theta_\varphi(t) \right] = C T D_1^{(1)} \Theta_\varphi(t)$$

(12)

where $C$ is a constant;

(A3) The product rule for the topology derivative [15]:

$$T D_1^{(1)} \left[ \Theta_\varphi(t) \cdot \Pi_\varphi(t) \right] = \Pi_\varphi(t) T D_1^{(1)} \Theta_\varphi(t) + \Theta_\varphi(t) T D_1^{(1)} \Pi_\varphi(t)$$

(13)

(A4) The quotient rule for the topology derivative [15]:

$$T D_1^{(1)} \left[ \frac{\Theta_\varphi(t)}{\Pi_\varphi(t)} \right] = \frac{\Pi_\varphi(t) T D_1^{(1)} \Theta_\varphi(t) - \Theta_\varphi(t) T D_1^{(1)} \Pi_\varphi(t)}{\Pi_\varphi(t) \cdot \Pi_\varphi(t)}$$

(14)

where $\Pi_\varphi(t) \neq 0$.

(A5) The chain rule for the topology derivative:

$$T D_1^{(1)} \left[ w(\Theta_\varphi(t)) \right] = w'(\Theta_\varphi(t)) \cdot T D_1^{(1)} \Theta_\varphi(t)$$

(15)

where $w^{(1)}(\varphi) = dw(\varphi)/d\varphi$ exists.

(A6) The first fundamental theorem of the topology integral:

$$\Theta_\varphi(t) - \Theta_\varphi(a) = T D_1^{(1)} \left[ T I_1^{(1)} \Theta_\varphi(t) \right]$$

(16)
(A7) The mean value theorem for the topology integral:

\[ T \int_a^t \Theta_\phi(t) \, dt = \Theta_\phi(l)[\varphi(t) - \varphi(a)] \]  

where \( a < l < t \).

(A8) The second fundamental theorem of the topology integral:

\[ \Theta_\phi(t) = T D_t^{(1)} \left[ \int_a^t \Theta_\phi(t) \, dt \right] \]  

(A9) The net change theorem for the topology integral:

\[ \Theta_\phi(b) - \Theta_\phi(a) = T \int_a^b \left[ T D_t^{(1)} \Theta_\phi(t) \right] \]

(A10) The integration by parts for the topology integral [15]:

\[ T \int_a^b \left[ \Theta_\phi(t) T D_t^{(1)} \Pi_\phi(t) \right] = \Theta_\phi(t) \cdot \Pi_\phi(t) - \Theta_\phi(a) \cdot \Pi_\phi(a) - T \int_a^b \left[ \Theta_\phi(t) T D_t^{(1)} \Pi_\phi(t) \right] \]

(A11) The topology integral for the composite function:

\[ \int_a^b \left[ w(\Theta_\phi(t)) \right] \, dt = \int_a^b \left[ w^{(1)}(\Theta_\phi) \cdot T D_t^{(1)} \Theta_\phi(x) \right] \, dt \]

(A12) The second fundamental theorem of the topology integral:

\[ \Theta_\phi(t) = T D_t^{(1)} \left[ T I_t^{(1)} \Theta_\phi(t) \right] \]

(A13) The net change theorem for the topology integral:

\[ T I_t^{(1)} \left[ T D_t^{(1)} \Theta_\phi(t) \right] = \Theta_\phi(t) + C \]

(A14) The integration by parts for the topology integral [15]:

\[ T I_t^{(1)} \left[ \Theta_\phi(t) T D_t^{(1)} \Pi_\phi(t) \right] = \Theta_\phi(t) \cdot \Pi_\phi(t) - T I_t^{(1)} \left[ \Theta_\phi(t) T D_t^{(1)} \Pi_\phi(t) \right] \]

(A15) The topology integral for the composite function:

\[ \int w^{(1)}(\Theta_\phi) \cdot T D_t^{(1)} \Theta_\phi(t) \, dt = w(\Theta_\phi(t)) + C \]

where \( C \) is the constant.

The Richardson-scaling-law calculus

Let \( \Phi_\psi \in \mathfrak{H}(\Phi_\psi) \), where \( \psi(t) = \kappa t^D \).

The Richardson-scaling-law derivative of the function \( \Phi_\psi(t) \) is defined:

\[ RSL D_t^{(1)} \Phi_\psi(t) = \frac{t^{1-D}}{D\kappa} \frac{d\Phi_\psi(t)}{dr} \]

where \( \kappa \) is the normalization constant, \( D \) is the scaling exponent, and \( t \) is the radius.

The Richardson-scaling-law partial derivatives of the function \( \Phi_\psi = \Phi_\psi(x, y, z) \) are defined:

\[ RSL \frac{\partial \Phi_\psi}{\partial x} = \frac{t^{1-D}}{D\kappa} \frac{\partial \Phi_\psi}{\partial x} \], \[ RSL \frac{\partial \Phi_\psi}{\partial y} = \frac{t^{1-D}}{D\kappa} \frac{\partial \Phi_\psi}{\partial y} \], \[ RSL \frac{\partial \Phi_\psi}{\partial z} = \frac{t^{1-D}}{D\kappa} \frac{\partial \Phi_\psi}{\partial z} \]
\[ RSL \gamma_x(1) \left[ \frac{RSL \gamma_x(1)}{RSL \gamma_x(1)} \Phi_\psi \right] = RSL \gamma_x(2) \Phi_\psi, \quad RSL \gamma_z(1) \left[ \frac{RSL \gamma_z(2)}{RSL \gamma_z(1)} \Phi_\psi \right] = RSL \gamma_z(2) \Phi_\psi \]

\[ RSL \gamma_y(1) \left[ \frac{RSL \gamma_y(1)}{RSL \gamma_y(1)} \Phi_\psi \right] = RSL \gamma_y(2) \Phi_\psi, \quad RSL \gamma_z(1) \left[ \frac{RSL \gamma_z(1)}{RSL \gamma_z(1)} \Phi_\psi \right] = RSL \gamma_z(2) \Phi_\psi \]

\[ RSL \gamma_z(1) \left[ \frac{RSL \gamma_z(2)}{RSL \gamma_z(1)} \Phi_\psi \right] = RSL \gamma_z(2) \Phi_\psi, \quad RSL \gamma_z(1) \left[ \frac{RSL \gamma_z(1)}{RSL \gamma_z(1)} \Phi_\psi \right] = RSL \gamma_z(2) \Phi_\psi \]

\[ RSL \gamma_z(1) \left[ \frac{RSL \gamma_z(2)}{RSL \gamma_z(1)} \Phi_\psi \right] = RSL \gamma_z(2) \Phi_\psi \]

The Richardson-scaling-law differential of the function \( \Phi_\psi(t) \), denoted by \( d\Phi_\psi(t) \), is given:

\[ d\Phi_\psi(t) = D\kappa t^{D-1} RSL D_t(1) \Phi_\psi(t) dt \]  
(27)

Let \( \Theta_\psi \in \mathfrak{R}(\Theta_\psi) \), where \( \psi(t) = \kappa t^D \).

The Richardson-scaling-law integral of the function \( \Theta_\psi(t) \) is defined:

\[ RSL \int_0^1 \Theta_\psi(t) D\kappa t^{D-1} dt \]
(28)

where \( \kappa \) is the normalization constant, \( D \) is the scaling exponent, and \( t \) is the radius.

The indefinite Richardson-scaling-law integral of the function \( \Theta_\psi(t) \) is defined:

\[ RSL \int_0^1 \Theta_\psi(t) D\kappa t^{D-1} dt \]
(29)

where \( \kappa \) is the normalization constant, \( D \) is the scaling exponent, and \( t \) is the radius.

Let \( \Theta_\psi \in \mathfrak{R}(\Theta_\psi) \) and \( \Pi_\psi \in \mathfrak{R}(\Pi_\psi) \).

The properties of the Richardson-scaling-law calculus can be given:

(B1) The sum and difference rules for the Richardson-scaling-law derivative:

\[ RSL D_t(1) \left[ \Theta_\psi(t) \pm \Pi_\psi(t) \right] = RSL D_t(1) \Theta_\psi(t) \pm RSL D_t(1) \Pi_\psi(t) \]
(30)

(B2) The constant multiple rule for the Richardson-scaling-law derivative:

\[ RSL D_t(1) \left[ C \Theta_\psi(t) \right] = C RSL D_t(1) \Theta_\psi(t) \]
(31)

where \( C \) is a constant;

(B3) The product rule for the Richardson-scaling-law derivative [15]:

\[ RSL D_t(1) \left[ \Theta_\psi(t) \cdot \Pi_\psi(t) \right] = \Pi_\psi(t) RSL D_t(1) \Theta_\psi(t) + \Theta_\psi(t) RSL D_t(1) \Pi_\psi(t) \]
(32)

(B4) The quotient rule for the Richardson-scaling-law derivative:

\[ RSL D_t(1) \left[ \frac{\Theta_\psi(t)}{\Pi_\psi(t)} \right] = \frac{\Pi_\psi(t) RSL D_t(1) \Theta_\psi(t) - \Theta_\psi(t) RSL D_t(1) \Pi_\psi(t)}{\Pi_\psi(t) \cdot \Pi_\psi(t)} \]
(14)

where \( \Pi_\psi(t) \neq 0 \).
B5) The chain rule for the Richardson-scaling-law derivative:

\[ RSL_{D}^{(1)} \left\{ w \left[ \Theta_{\psi} (t) \right] \right\} = w^{(1)} \left( \Theta_{\psi} \right) \cdot RSL_{D}^{(1)} \Theta_{\psi} (t) \]  

(33)

where \( w^{(1)} \left( \Theta_{\psi} \right) = dw \left( \Theta_{\psi} \right) / d\Theta_{\psi} \) exists.

B6) The first fundamental theorem of the Richardson-scaling-law integral:

\[ \Theta_{\psi} (t) - \Theta_{\psi} (a) = RSL_{a}^{(1)} \left[ RSL_{D}^{(1)} \Theta_{\psi} (t) \right] \]  

(34)

B7) The mean value theorem for the Richardson-scaling-law integral:

\[ RSL_{a}^{(1)} \Theta_{\psi} (t) = \Theta_{\psi} (l) \left[ \psi (t) - \psi (a) \right] \]  

(35)

where \( a < l < t \).

B8) The second fundamental theorem of the Richardson-scaling-law integral:

\[ \Theta_{\psi} (t) = RSL_{D}^{(1)} \left[ RSL_{a}^{(1)} \Theta_{\psi} (t) \right] \]  

(36)

B9) The net change theorem for the Richardson-scaling-law integral:

\[ \Theta_{\psi} (b) - \Theta_{\psi} (a) = RSL_{a}^{(1)} \left[ RSL_{D}^{(1)} \Theta_{\psi} (t) \right] \]  

(37)

B10) The integration by parts for the Richardson-scaling-law integral:

\[ \int_{a}^{b} RSL_{D}^{(1)} \left\{ w \left[ \Theta_{\psi} (t) \right] \right\} dt = \int_{a}^{b} w^{(1)} \left( \Theta_{\psi} \right) \cdot RSL_{D}^{(1)} \Theta_{\psi} (t) dt \]  

(39)

B11) The Richardson-scaling-law integral for the composite function:

\[ \int_{a}^{b} RSL_{D}^{(1)} \left\{ \Theta_{\psi} (t) \cdot RSL_{D}^{(1)} \Pi_{\psi} (t) \right\} = \Theta_{\psi} (t) \cdot \Pi_{\psi} (t) - \Theta_{\psi} (a) \cdot \Pi_{\psi} (a) - RSL_{a}^{(1)} \left[ \Theta_{\psi} (t) \cdot RSL_{D}^{(1)} \Pi_{\psi} (t) \right] \]  

(38)

B12) The second fundamental theorem of the Richardson-scaling-law integral:

\[ \Theta_{\psi} (t) = RSL_{D}^{(1)} \left[ RSL_{a}^{(1)} \Theta_{\psi} (t) \right] \]  

(40)

B13) The net change theorem for the Richardson-scaling-law integral:

\[ RSL_{a}^{(1)} \left[ RSL_{D}^{(1)} \Theta_{\psi} (t) \right] = \Theta_{\psi} (t) + C \]  

(41)

B14) The integration by parts for the Richardson-scaling-law integral:

\[ \int_{a}^{b} RSL_{D}^{(1)} \left\{ \Theta_{\psi} (t) \cdot RSL_{D}^{(1)} \Pi_{\psi} (t) \right\} = \Theta_{\psi} (t) \cdot \Pi_{\psi} (t) - RSL_{a}^{(1)} \left[ \Theta_{\psi} (t) \cdot RSL_{D}^{(1)} \Pi_{\psi} (t) \right] \]  

(42)

B15) The Richardson-scaling-law integral for the composite function:

\[ \int w^{(1)} \left( \Theta_{\psi} \right) \cdot RSL_{D}^{(1)} \Theta_{\psi} (t) dt = w \left[ \Theta_{\psi} (t) \right] + C \]  

(43)

where \( C \) is the constant.

The basic formulas for the Richardson-scaling-law calculus can be given:

\[ RSL_{D}^{(1)} 1 = 0, \quad RSL_{D}^{(1)} (kt^{D}) = 1, \quad RSL_{D}^{(1)} (\ln(kt^{D})) = n(kt^{D})^{-1} \]  

(44a,b,c)

\[ RSL_{D}^{(1)} e^{\kappa t^{D}} = e^{\kappa t^{D}}, \quad RSL_{D}^{(1)} \ln(kt^{D}) = \frac{1}{\kappa t^{D}}, \quad RSL_{D}^{(1)} s^{\kappa t^{D}} = (\ln s)^{n}(\kappa t^{D})^{n} \]  

(45a,b,c)
\[
\begin{align*}
RSL D_i^{(1)} (\log_s (\kappa t^D)) &= \frac{1}{\kappa t^D} \ln s, \\
RSL D_i^{(1)} e^{\Theta_{\psi}(t)} &= e^{\Theta_{\psi}(t)} RSL D_i^{(1)} \Theta_{\psi}(t) \quad (46a,b) \\
RSL D_i^{(1)} \ln \Theta_{\psi}(t) &= \frac{RSL D_i^{(1)} \Theta_{\psi}(t)}{\Theta_{\psi}(t)}, \\
RSL D_i^{(1)} \log_s \Theta_{\psi}(t) &= \frac{RSL D_i^{(1)} \Theta_{\psi}(t)}{\ln s \Theta_{\psi}(t)} \quad (47a,b) \\
RSL D_i^{(1)} s^{\Theta_{\psi}(t)} &= \left[ \ln s \right] s^{\Theta_{\psi}(t)}, \\
RSL I_t^{(1)} = \frac{1}{\kappa t^D} + C, \quad RSL I_t^{(1)} \Theta_{\psi}(t) = \log_s \Theta_{\psi}(t) + C \quad (48a,b) \\
RSL I_t^{(1)} \left[ (\ln s) s^{\Theta_{\psi}(t)} \right] &= \left[ \ln (\kappa t^D) \right] + C, \quad RSL I_t^{(1)} \left[ \frac{1}{\kappa t^D} \right] = \log_s (\kappa t^D) + C \quad (49a,b) \\
RSL I_t^{(1)} \left[ (\ln s) s^{\Theta_{\psi}(t)} \right] &= s^{\Theta_{\psi}(t)} + C, \quad RSL I_t^{(1)} \left[ e^{\Theta_{\psi}(t)} \right] RSL D_i^{(1)} \Theta_{\psi}(t) = e^{\Theta_{\psi}(t)} + C \quad (50a,b) \\
RSL I_t^{(1)} \left[ \Theta_{\psi}(t) \right] RSL I_t^{(1)} \Theta_{\psi}(t) = \Theta_{\psi}(t) + C, \quad RSL I_t^{(1)} \left[ \frac{RSL D_i^{(1)} \Theta_{\psi}(t)}{\Theta_{\psi}(t)} \right] = \ln \Theta_{\psi}(t) + C \quad (52a,b) \\
RSL I_t^{(1)} (e^{\Theta_{\psi}(t)}) &= e^{\Theta_{\psi}(t)} + C, \quad RSL I_t^{(1)} \left[ (\ln s) \right] s^{\Theta_{\psi}(t)} \right] = s^{\Theta_{\psi}(t)} + C \quad (53a,b) \\
\end{align*}
\]

where \( C \) is the constant and \( e^{\Theta_{\psi}(t)} \) is the Kohlrausch-Williams-Watts function [11,15].

**The Korcak-scaling-law calculus**

Let \( \Phi_\omega \in \Re (\Phi_\omega) \), where \( \omega(t) = \kappa t^D \).

The Korcak-scaling-law derivative of the function \( \Phi_\omega(t) \) is defined:

\[
KSL D_i^{(1)} \Phi_\omega(t) = -\frac{\kappa t^D}{D\kappa} \frac{\partial \Phi_\omega(t)}{\partial t} \quad (54)
\]

where \( \kappa \) is the normalization constant, \( D \) is the scaling exponent, and \( t \) is the radius.

The Korcak-scaling-law partial derivatives of the function \( \Phi_\omega = \Phi_\omega(x,y,z) \) are defined:

\[
\begin{align*}
KSL \partial_x^{(1)} \Phi_\omega &= -\frac{\partial_x^{(1)}}{D\kappa} \frac{\partial \Phi_\omega}{\partial x}, \\
KSL \partial_y^{(1)} \Phi_\omega &= -\frac{\partial_y^{(1)}}{D\kappa} \frac{\partial \Phi_\omega}{\partial y}, \\
KSL \partial_z^{(1)} \Phi_\omega &= -\frac{\partial_z^{(1)}}{D\kappa} \frac{\partial \Phi_\omega}{\partial z} \\
KSL \partial_x^{(2)} \Phi_\omega &= KSL \partial_y^{(2)} \Phi_\omega = KSL \partial_z^{(2)} \Phi_\omega \\
KSL \partial_x^{(1)} \partial_y^{(1)} \Phi_\omega &= KSL \partial_y^{(1)} \partial_x^{(1)} \Phi_\omega, \\
KSL \partial_x^{(1)} \partial_z^{(1)} \Phi_\omega &= KSL \partial_z^{(1)} \partial_x^{(1)} \Phi_\omega, \\
KSL \partial_y^{(1)} \partial_z^{(1)} \Phi_\omega &= KSL \partial_z^{(1)} \partial_y^{(1)} \Phi_\omega \\
KSL \partial_x^{(2)} \Phi_\omega &= KSL \partial_y^{(2)} \Phi_\omega = KSL \partial_z^{(2)} \Phi_\omega.
\end{align*}
\]

and

\[
\begin{align*}
KSL \partial_x^{(1)} \partial_y^{(2)} \Phi_\omega &= KSL \partial_y^{(2)} \partial_x^{(1)} \Phi_\omega, \\
KSL \partial_x^{(1)} \partial_z^{(2)} \Phi_\omega &= KSL \partial_z^{(2)} \partial_x^{(1)} \Phi_\omega, \\
KSL \partial_y^{(1)} \partial_z^{(2)} \Phi_\omega &= KSL \partial_z^{(2)} \partial_y^{(1)} \Phi_\omega
\end{align*}
\]
The Korcak-scaling-law differential of the function $\Phi_\omega(t)$, denoted by $d\Phi_\omega(t)$, is given:

$$d\Phi_\omega(t) = -DKT^{-(D+1)} KSL D_t^{(1)} \Phi_\omega(t) \, dt$$  (55)

Let $\Theta_\omega \in \mathcal{R}(\Phi_\omega)$, where $\omega(t) = \kappa t^D$.

The Korcak-scaling-law integral of the function $\Theta_\omega(t)$ is defined:

$$KSL \int_a^t \Theta_\omega(t) \, dK T^{-(D+1)} = -\int_a^t \Theta_\omega(t) \, dK T^{-(D+1)} \, dt$$  (56)

where $\kappa$ is the normalization constant, $D$ is the scaling exponent, and $t$ is the radius.

The indefinite Korcak-scaling-law integral of the function $\Theta_\omega(t)$ is defined:

$$KSL \int_0^t \Theta_\omega(t) \, dK T^{-(D+1)} = -\int_0^t \Theta_\omega(t) \, dK T^{-(D+1)} \, dt$$  (57)

where $\kappa$ is the normalization constant, $D$ is the scaling exponent, and $t$ is the radius.

Let $\Theta_\omega \in \mathcal{R}(\Phi_\omega)$ and $\Pi_\omega \in \mathcal{R}(\Phi_\omega)$.

The properties of the Korcak-scaling-law calculus can be given:

(C1) The sum and difference rules for the Korcak-scaling-law derivative:

$$KSL D_t^{(1)} [\Theta_\omega(t) \pm \Pi_\omega(t)] = KSL D_t^{(1)} \Theta_\omega(t) \pm KSL D_t^{(1)} \Pi_\omega(t)$$  (58)

(C2) The constant multiple rule for the Korcak-scaling-law derivative:

$$KSL D_t^{(1)} [C \Theta_\omega(t)] = C KSL D_t^{(1)} \Theta_\omega(t)$$  (59)

where $C$ is a constant;

(C3) The product rule for the Korcak-scaling-law derivative [15]:

$$KSL D_t^{(1)} [\Theta_\omega(t) \cdot \Pi_\omega(t)] = \Pi_\omega(t) KSL D_t^{(1)} \Theta_\omega(t) + \Theta_\omega(t) KSL D_t^{(1)} \Pi_\omega(t)$$  (60)

(C4) The quotient rule for the Korcak-scaling-law derivative:

$$KSL D_t^{(1)} \left[ \frac{\Theta_\omega(t)}{\Pi_\omega(t)} \right] = \frac{\Pi_\omega(t) KSL D_t^{(1)} \Theta_\omega(t) - \Theta_\omega(t) KSL D_t^{(1)} \Pi_\omega(t)}{\Pi_\omega(t) \cdot \Pi_\omega(t)}$$  (61)

where $\Pi_\omega(t) \neq 0$.

(C5) The chain rule for the Korcak-scaling-law derivative:

$$KSL D_t^{(1)} \left[ w(\Theta_\omega(t)) \right] = w^{(1)}(\Theta_\omega) \cdot KSL D_t^{(1)} \Theta_\omega(t)$$  (62)

where $w^{(1)}(\Theta_\omega) = dw(\Theta_\omega)/d\Theta_\omega$ exists.

(C6) The first fundamental theorem of the Korcak-scaling-law integral:

$$\Theta_\omega(t) - \Theta_\omega(a) = KSL \int_a^t \Theta_\omega(t) \, dt$$  (63)

(C7) The mean value theorem for the Korcak-scaling-law integral:

$$KSL \int_a^t \Theta_\omega(t) \, dt = \Theta_\omega(l) [\omega(t) - \omega(a)]$$  (64)

where $a < l < t$.

(C8) The second fundamental theorem of the Korcak-scaling-law integral:

$$\Theta_\omega(t) = KSL \int_a^t \Theta_\omega(t) \, dt$$  (65)
(C9) The net change theorem for the Korcak-scaling-law integral:
\[ \Theta_{\alpha}(b) - \Theta_{\alpha}(a) = \frac{KSL D_t^{(1)}}{\kappa} \int_a^b \left[ \frac{KSL D_t^{(1)}}{\kappa} \Theta_{\alpha}(t) \right] \] (66)

(C10) The integration by parts for the Korcak-scaling-law integral:
\[ \frac{KSL D_t^{(1)}}{\kappa} \left[ \Theta_{\alpha}(t) KSL D_t^{(1)} \Pi_{\alpha}(t) - \Theta_{\alpha}(a) \cdot \Pi_{\alpha}(a) - \frac{KSL D_t^{(1)}}{\kappa} \int_a^b \left[ \Theta_{\alpha}(t) KSL D_t^{(1)} \Pi_{\alpha}(t) \right] \right] \] (67)

(C11) The Korcak-scaling-law integral for the composite function:
\[ \int_a^b \left[ w[\Theta_{\alpha}(t)] \right] dt = \int_a^b \left[ w^{(1)}(\Theta_{\alpha}) \cdot KSL D_t^{(1)} \Theta_{\alpha}(t) \right] dt \] (68)

(C12) The second fundamental theorem of the Korcak-scaling-law integral:
\[ \Theta_{\alpha}(t) = \frac{KSL D_t^{(1)}}{\kappa} \left[ \frac{KSL D_t^{(1)}}{\kappa} \Theta_{\alpha}(t) \right] \] (69)

(C13) The net change theorem for the Korcak-scaling-law integral:
\[ \Theta_{\alpha}(t) = \frac{KSL D_t^{(1)}}{\kappa} \left[ \frac{KSL D_t^{(1)}}{\kappa} \Theta_{\alpha}(t) \right] = \Theta_{\alpha}(t) + C \] (70)

(C14) The integration by parts for the Korcak-scaling-law integral:
\[ \frac{KSL D_t^{(1)}}{\kappa} \left[ \Theta_{\alpha}(t) KSL D_t^{(1)} \Pi_{\alpha}(t) - \Theta_{\alpha}(a) \cdot \Pi_{\alpha}(a) - \frac{KSL D_t^{(1)}}{\kappa} \int_a^b \left[ \Theta_{\alpha}(t) KSL D_t^{(1)} \Pi_{\alpha}(t) \right] \right] \] (71)

(C15) The Korcak-scaling-law integral for the composite function:
\[ \int_a^b \left[ w^{(1)}(\Theta_{\alpha}) \cdot KSL D_t^{(1)} \Theta_{\alpha}(t) \right] dt = w[\Theta_{\alpha}(t)] + C \] (72)

where \( C \) is the constant.

The basic formulas for the Korcak-scaling-law calculus can be presented as follows:
\[ KSL D_t^{(1)} 1 = 0, \quad KSL D_t^{(1)} (\kappa T - D) = 1, \quad KSL D_t^{(1)} (\kappa T - D)^n = n(\kappa T - D)^{n-1} \] (73a,b,c)

\[ KSL D_t^{(1)} e^{-\kappa T - D} = e^{-\kappa T - D}, \quad KSL D_t^{(1)} \ln(\kappa T - D) = \frac{1}{\kappa T - D}, \quad KSL D_t^{(1)} s^{-\kappa T - D} = (\ln s)s^{-\kappa T - D} \] (74a,b,c)

\[ KSL D_t^{(1)} \log_s(\kappa T - D) = \frac{1}{\kappa T - D} \ln s, \quad KSL D_t^{(1)} e^{\Theta_{\alpha}(t)} = e^{\Theta_{\alpha}(t)} KSL D_t^{(1)} \Theta_{\alpha}(t) \] (75a,b)

\[ KSL D_t^{(1)} \ln \Theta_{\alpha}(t) = \frac{KSL D_t^{(1)} \Theta_{\alpha}(t)}{\Theta_{\alpha}(t)}, \quad KSL D_t^{(1)} \log_s \Theta_{\alpha}(t) = \frac{KSL D_t^{(1)} \Theta_{\alpha}(t)}{(\ln s) \Theta_{\alpha}(t)} \] (76a,b)

\[ KSL D_t^{(1)} \Theta_{\alpha}(t) = \left[ (\ln s) s^{\Theta_{\alpha}(t)} \right] KSL D_t^{(1)} \Theta_{\alpha}(t), \quad KSL D_t^{(1)} 1 = \kappa T - D + C \] (77a,b)

\[ KSL D_t^{(1)} \left[ n(\kappa T - D)^{n-1} \right] = (\kappa T - D)^n + C, \quad KSL D_t^{(1)} \left[ \frac{D_t^{(1)} \Theta_{\alpha}(t)}{(\ln s) \Theta_{\alpha}(t)} \right] = \log s \Theta_{\alpha}(t) + C \] (78a,b)

\[ KSL D_t^{(1)} \left[ \frac{1}{(\kappa T - D)} \right] = \ln(\kappa T - D) + C, \quad KSL D_t^{(1)} \left[ \frac{1}{(\ln s)} \frac{1}{(\kappa T - D)} \right] = \log s(\kappa T - D) + C \] (79a,b)

\[ KSL D_t^{(1)} \left[ (\ln s) s^{-\kappa T - D} \right] = s^{-\kappa T - D} + C, \quad KSL D_t^{(1)} \left[ e^{\Theta_{\alpha}(t)} KSL D_t^{(1)} \Theta_{\alpha}(t) \right] = e^{\Theta_{\alpha}(t)} + C \] (80a,b)
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\[ \begin{aligned}
KSL I^1_{t(t)} & \left[ \frac{\Theta_{\sigma}(t)}{\Theta_{\sigma}(t)} \right] = \ln \Theta_{\sigma}(t) + C, \\
KSL I^1_{t(t)} & \left( e^{\sigma - \rho} \right) = e^{\sigma - \rho} + C, \\
KSL I^1_{t(t)} & \left( \ln s \right) s^{\Theta_{\rho}(t)} = s^{\Theta_{\rho}(t)} + C
\end{aligned} \] (81a,b)

where C is the constant and \( e^{\sigma - \rho} \) is the Kohlrausch-Williams-Watts function [11, 15].

Applications

In this section, we propose the Fourier-like law for the scaling-law flow in the heat-transfer process and the Darcy-like law for the scaling-law flow of the fluid in porous medium.

The Fourier-like law for the scaling-law flow

The Fourier-like law for the scaling-law flow in the heat-transfer process can be defined:

\[ q(x, y, z, t) = -\alpha RSL \nabla T(x, y, z, t) \]

\[ = -\alpha C D^{-1} \frac{\partial T(x, y, z, t)}{\partial x} - j\alpha C D^{-1} \frac{\partial T(x, y, z, t)}{\partial y} - k\alpha C D^{-1} \frac{\partial T(x, y, z, t)}{\partial z} \] (83)

where \( T(x, y, z, t) \) is the temperature field in the unit volume at the Cartesian co-ordinates \( x, y \) and \( z \) and at the time \( t \), \( q(x, y, z, t) \) is the vector of the local heat flux density, \( i, j, k \) denote the unit vectors in the Cartesian co-ordinate system, \( \kappa \) is the normalization constant, \( D \) is the scaling exponent, \( \alpha \) is the material conductivity, and the Richardson-scaling-law gradient in a Cartesian co-ordinate system is defined:

\[ RSL \nabla D = i(\kappa C D^{-1}) \frac{\partial}{\partial x} + j(\kappa C D^{-1}) \frac{\partial}{\partial y} + k(\kappa C D^{-1}) \frac{\partial}{\partial z} \] (84)

which is connected with the Laplace-like operator, represented:

\[ RSL \Delta_D = RSL \nabla D \cdot RSL \nabla D = \left( \kappa C D^{-1} \frac{\partial}{\partial x} \right)^2 + \left( \kappa C D^{-1} \frac{\partial}{\partial y} \right)^2 + \left( \kappa C D^{-1} \frac{\partial}{\partial z} \right)^2 \] (85)

which is connected the Laplace operator [18] when \( D = 1 \).

In 1-D case, the Fourier-like law for the scaling-law flow in the heat-transfer process reads:

\[ q(x, t) = -\alpha C D^{-1} \frac{\partial T(x, t)}{\partial x} \] (86)

where \( T(x, t) \) is the temperature field, \( q(x, t) \) is the local heat flux density and \( \alpha \) is the material conductivity.

When \( D = 1 \), the Fourier-like law for the scaling-law flow of the heat is the Fourier law for the flow of the heat [19].

The Darcy-like law for the scaling-law flow of the fluid

The Darcy-like law for the scaling-law flow of the fluid in porous medium can be defined:

\[ \Theta(x, y, z, t) = \lambda KSL \nabla \Xi(x, y, z, t) = i\lambda (\kappa C D^{-1}) \frac{\partial \Xi(x, y, z, t)}{\partial x} \]

\[ + j\lambda (\kappa C D^{-1}) \frac{\partial \Xi(x, y, z, t)}{\partial y} + k\lambda (\kappa C D^{-1}) \frac{\partial \Xi(x, y, z, t)}{\partial z} \] (87)
where \( \Theta(x, y, z, t) \) is the specific discharge, \( \Xi(x, y, z, t) \) is the hydraulic head, \( \kappa \) is the normalization constant, \( D \) is the scaling exponent, \( \lambda \) is the hydraulic conductivity, the Korcak-scaling-law gradient in a Cartesian co-ordinate system is defined:

\[
KSL \nabla_D = 2 \left[ -\kappa D x^{-(D+1)} \frac{\partial}{\partial x} \right] + \left[ -\kappa D y^{-(D+1)} \frac{\partial}{\partial y} \right] + \left[ -\kappa D z^{-(D+1)} \frac{\partial}{\partial z} \right] \tag{88}
\]

which is connected with the Laplace-like operator, given:

\[
KSL \Delta_D = KSL \nabla_D^2 = KSL \nabla_D \cdot KSL \nabla_D
\]

\[
= \left[ \kappa D x^{-(D+1)} \frac{\partial}{\partial x} \right]^2 + \left[ \kappa D y^{-(D+1)} \frac{\partial}{\partial y} \right]^2 + \left[ \kappa D z^{-(D+1)} \frac{\partial}{\partial z} \right]^2 \tag{89}
\]

which is connected the Laplace operator [18] when \( D = -1 \).

In 1-D case, the Darcy-like law for the scaling-law flow of the fluid in porous medium can be expressed:

\[
\Theta(x,t) = \lambda \kappa D x^{-(D+1)} \frac{\partial \Xi(x,t)}{\partial x} \tag{90}
\]

where \( \Xi(x,t) \) is the hydraulic head, \( \Theta(x,t) \) is the specific discharge, \( \lambda \) is the hydraulic conductivity, \( \kappa \) is the normalization constant, and \( D \) is the scaling exponent.

When \( D = 1 \), the Darcy-like law for the scaling-law flow of the fluid is the Darcy law for the flow of the fluid [20].

Conclusion

In the present work, we proposed the Richardson-scaling-law calculus and Korcak-scaling-law calculus for the first time. Based on the results for the Richardson-scaling-law gradient and the Korcak-scaling-law gradient, we considered the Fourier-like law for the scaling-law flow of the heat and the Darcy-like law for describing the scaling-law flow of the fluid, respectively. The obtained results are as mathematical tools proposed for descriptions of the fractal scaling-law phenomena in applied sciences.

Acknowledgment

This work is supported by the Yue-Qi Scholar of the China University of Mining and Technology (No. 102504180004).

Nomenclature

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t )</td>
<td>time, [s]</td>
</tr>
<tr>
<td>( x, y, z )</td>
<td>co-ordinates, [m]</td>
</tr>
<tr>
<td>( q(x, y, z, t) )</td>
<td>local heat flux density, [W]</td>
</tr>
<tr>
<td>( T(x, y, z, t) )</td>
<td>temperature field, [K]</td>
</tr>
<tr>
<td>( \kappa )</td>
<td>hydraulic conductivity, [ms(^{-1})]</td>
</tr>
<tr>
<td>( \lambda )</td>
<td>hydraulic conductivity, [ms(^{-1})]</td>
</tr>
<tr>
<td>( \Theta(x,t) )</td>
<td>specific discharge, [ms(^{-1})]</td>
</tr>
<tr>
<td>( \Xi(x,t) )</td>
<td>hydraulic head, [m]</td>
</tr>
</tbody>
</table>

References

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