THE VECTOR POWER-LAW CALCULUS WITH APPLICATIONS IN POWER-LAW FLUID FLOW

by

Xiao-Jun YANG\textsuperscript{a,b,c}

\textsuperscript{a} State Key Laboratory for Geo-Mechanics and Deep Underground Engineering, China University of Mining and Technology, Xuzhou, China
\textsuperscript{b} School of Mechanics and Civil Engineering, China University of Mining and Technology, Xuzhou, China
\textsuperscript{c} College of Mathematics, China University of Mining and Technology, Xuzhou, China

Original scientific paper
https://doi.org/10.2298/TSCI2006289Y

In this article, based on the Leibniz derivative and Stieltjes-Riemann integral, we suggest the vector power-law calculus to consider the conservations of the mass and angular momentums for the power-law fluid.

Key words: power-law fluid, Leibniz derivative, Stieltjes-Riemann integral, vector power-law calculus

Introduction

Fractals are beautiful mathematical constructs, described by the scaling law [1], which is a mathematical relationship to the complex behaviors in the nature phenomena [2], where this description is related to the theory of dynamical systems [3] in condensed matter problems [4], and in the statistical mechanics of disordered systems [5].

In 1967, Mandelbrot structured the Mandelbrot scaling law, given [6]:

$$\phi(t) = \kappa t^{1-D}$$

where $\kappa \in (0, +\infty)$ is the normalization constant, $t \in (0, +\infty)$ is the radius or scale, and $D \in (0, +\infty)$ is the fractal dimension.

The theory of the functions related to fractals [7] has been developed in the different descriptions for the scaling law, which is considered by the measures [8], which is relevant to Hausdorff measure [9], where this quantitation is related to the power-law. The Hausdorff derivative in the Hausdorff space was proposed in [10]. The fractal derivative in the scaling law was considered in [11]. The metric derivative in the metric space was proposed in [11]. The vector calculus based on the Riemann-Liouville fractional derivative was proposed in [12] and further developed in [13, 14].

The scaling-law calculus is one of the hot topics on the theory of the calculus with respect to monotone functions, which includes the Leibniz derivative [15] and Stieltjes-Riemann integral [16] based on the Riemann work [17]. The power-law calculus was proposed in 2019 [18] and further developed in 2020 [19] based on the Leibniz derivative [15] and

Author’s e-mail: dyangxiaojun@163.com
Stieltjes-Riemann integral [16]. Moreover, the power-law derivative is equal to the Hausdorff derivative without the strict proof that was proposed in [20, 21].

The target of the paper is to propose the theory of the vector power-law calculus based on the calculus with respect to monotone functions, to give the generalized integral transforms, and to present the novel application to the power-law fluid-flow.

The results on the calculus with respect to monotone functions

In this section, we introduce the power-law differential calculus and power-law integral calculus.

Let \( \Phi_\omega(t) = (\Phi \circ \omega)(t) = \Phi[\omega(t)] \), where \( \omega(t) \) is the monotone function, e.g., \( \omega^{(1)}(t) = d\omega(t)/dt > 0 \). Let \( \mathcal{A}(\Phi) \) be the set of the continuous functions \( \Phi(\omega) \) in the domain \( \mathbb{N} \). Let \( \mathcal{M}(\omega) \) be the set of the continuous derivatives of the functions \( \omega(t) \) in the domain \( \mathcal{I} \). Let \( \Phi_\varphi(t) = (\Phi \circ \varphi)(t) = \Phi[\varphi(t)] \).

Let us consider the sets of the composite functions, given as below:

\[
\mathcal{N}(\Phi_\omega) = \{ \Phi_\omega(t) : \Phi_\omega(t) \in \mathcal{M}(\omega), \omega \in \mathcal{A}(\omega) \} \tag{2}
\]

The calculus with respect to monotone functions

Let \( \Phi_\omega \in \mathcal{N}(\Phi_\omega) \). The Leibniz derivative of the function \( \Phi_\omega(t) \) is defined [15, 18, 20]:

\[
L_{D}^{(1)} \Phi_\omega(t) = \frac{1}{\omega^{(1)}(t)} \frac{d\Phi_\omega(t)}{dt} \tag{3}
\]

The geometric interpretations of the Leibniz derivative is the rate of change of the functional \( \Phi(\omega) \) with the function \( \omega(t) \) in the independent variable \( t \) [18, 19]. It is to say, the slope of the functional \( \Phi(\omega) \) with the function \( \omega(t) \) in the independent variable \( t \) [18, 19].

Let \( \Phi_\omega \in \mathcal{N}(\Phi_\omega) \). The Stieltjes-Riemann integral of the function \( \Theta_\omega(t) \) is defined [16, 18, 19]:

\[
L_{a}^{(0)} \Theta_\omega(t) = \int_{a}^{b} \Theta_\omega(t) \omega^{(1)}(t) dt \tag{4}
\]

Similarly, the geometric interpretations of the Stieltjes-Riemann integral is the area enclosed by the integrand function \( \Phi(\omega) \) and the function \( \omega(t) \) in the independent variable \( t \in [a, b] \) [18, 19].

The power-law calculus

Let \( \omega(t) = t^{D} \), where \( D \) is the fractal dimension. Let \( \Phi_\omega \in \mathcal{N}(\Phi_\omega) \). The power-law derivative of the function \( \Phi_\omega(t) \) is defined [18, 20]:

\[
PL_{D}^{(1)} \Phi_\omega(t) = \frac{t^{1-D}}{D} \frac{d\Phi_\omega(t)}{dt} \tag{5}
\]

It is not difficult to show that the geometric interpretations of the power-law derivative is the rate of change of the functional \( \Phi(\omega) \) with the function \( \omega(t) = t^{D} \) in the independent variable \( t \) [21].

Let \( \Phi_\omega \in \mathcal{N}(\Phi_\omega) \). The power-law differential of the function \( \Phi_\omega(t) \), denoted by \( d\Phi_\omega(t) \), is given:

\[
d\Phi_\omega(t) = D t^{D-1} PL_{D}^{(1)} \Phi_\omega(t) dt \tag{6}
\]
Let $\Phi_{\omega} \in \Phi$. The power-law integral of the function $\Theta_{\omega}(t)$ is defined [18, 19]:

$$
\int_{a}^{b} [\Theta_{\omega}(t)]^{D_{\omega}-1} dt
$$

(7)

Similarly, it is shown that the geometric interpretations of the power-law integral is the area enclosed by the integrand function $\Phi(\omega)$ and the function $\omega(t)=t^{D}$ in the independent variable $t \in [a, b]$ [21].

Let $\Theta_{\omega} \in \Phi$. The indefinite power-law integral of the function $\Theta_{\omega}(t)$ is defined:

$$
\int [\Theta_{\omega}(t)]^{D_{\omega}-1} dt
$$

(8)

Let $\Theta_{\omega} \in \Phi$ and $\Pi_{\omega} \in \Pi$. The properties of the power-law differential calculus used in this paper can be presented as follows [18,19]:

(Y1) The product rule for the power-law derivative [15]:

$$
\frac{d}{dt} [\Theta_{\omega}(t) \cdot \Pi_{\omega}(t)] = \Theta_{\omega}(t) \frac{d}{dt} \Theta_{\omega}(t) + \Pi_{\omega}(t) \frac{d}{dt} \Pi_{\omega}(t)
$$

(9)

(Y2) The chain rule for the power-law derivative:

$$
\frac{d}{dt} \{ \Theta_{\omega}(t) \} = \frac{d}{dt} \Theta_{\omega}(t)
$$

(10)

where $w^{(l)}(\Theta_{\omega}) = dw(\Theta_{\omega})/d\Theta_{\omega}$ exists.

The power-law partial derivatives, power-law gradients and directional power-law derivative

Let us consider the power-law co-ordinate system, given as $\{i, j, k\} = \{x^{D_{1}}, y^{D_{2}}, z^{D_{3}}\}$, where $D_{1}$, $D_{2}$, and $D_{3}$ are the fractal dimensions, and $i$, $j$, and $k$ are the unit vector in the Cartesian co-ordinate system.

Let us consider the function, defined by

$$
\psi_{\omega}(x, y, z) = \psi(x^{D_{1}}, y^{D_{2}}, z^{D_{3}}).
$$

The power-law partial derivatives of the function $\psi_{\omega}(x, y, z)$ are defined:

$$
\frac{\partial \psi_{\omega}}{\partial x} = \left( \frac{1}{D_{1}} \frac{\partial}{\partial x} \right) \psi_{\omega}, ~ \frac{\partial \psi_{\omega}}{\partial y} = \left( \frac{1}{D_{2}} \frac{\partial}{\partial y} \right) \psi_{\omega}, ~ \frac{\partial \psi_{\omega}}{\partial z} = \left( \frac{1}{D_{3}} \frac{\partial}{\partial z} \right) \psi_{\omega}, \quad (11a,b,c)
$$

where

$$
\psi = \psi_{\omega}(x^{D_{1}}, y^{D_{2}}, z^{D_{3}}).
$$

The total power-law differential of the function $\psi_{\omega}(x, y, z)$ is defined:

$$
d\psi_{\omega} = [D_{x} x^{D_{1}} \frac{\partial \psi_{\omega}}{\partial x} dx] + [D_{y} y^{D_{2}} \frac{\partial \psi_{\omega}}{\partial y} dy] + [D_{z} z^{D_{3}} \frac{\partial \psi_{\omega}}{\partial z} dz]
$$

(12)

Thus, the power-law derivative with respect to the time $t$ is given:

$$
\frac{d\psi_{\omega}}{dt} = [D_{x} x^{D_{1}} \frac{\partial \psi_{\omega}}{\partial x} \frac{dx}{dt}] + [D_{y} y^{D_{2}} \frac{\partial \psi_{\omega}}{\partial y} \frac{dy}{dt}] + [D_{z} z^{D_{3}} \frac{\partial \psi_{\omega}}{\partial z} \frac{dz}{dt}]
$$

(13)

The power-law gradient of first type in the Cartesian co-ordinate system is defined:

$$
\nabla^{(D_{1}, D_{2}, D_{3})} = i(D_{x} x^{D_{1}}) \frac{\partial \psi_{\omega}}{\partial x} + j(D_{y} y^{D_{2}}) \frac{\partial \psi_{\omega}}{\partial y} + k(D_{z} z^{D_{3}}) \frac{\partial \psi_{\omega}}{\partial z}
$$

(14)

which deduces that the power-law gradient of second type in the Cartesian co-ordinate system:

$$
\nabla^{(D)} = i(D_{x} x^{D_{1}}) \frac{\partial \psi_{\omega}}{\partial x} + j(D_{y} y^{D_{2}}) \frac{\partial \psi_{\omega}}{\partial y} + k(D_{z} z^{D_{3}}) \frac{\partial \psi_{\omega}}{\partial z}
$$

(15)
The power-law gradient of first type of the scalar field $\psi_\alpha = \psi_\alpha(x,y,z)$ reads:
\[
\nabla^{(\alpha_1,..\alpha_n)}\psi_\alpha = i(D_{x_1}D_{x_1}^{-1})^{PL} \partial_{x_1}^{(l)}\psi_\alpha + j(D_{y_2}D_{y_2}^{-1})^{PL} \partial_{y_2}^{(l)}\psi_\alpha + k(D_{z_3}D_{z_3}^{-1})^{PL} \partial_{z_3}^{(l)}\psi_\alpha
\]
(16)

Similarly, the power-law gradient of second type of the scalar field $\psi$ can be written:
\[
\nabla^{(D)}\psi = i(D_{x_1}D_{x_1}^{-1})^{PL} \partial_{x_1}^{(l)}\psi + j(D_{y_2}D_{y_2}^{-1})^{PL} \partial_{y_2}^{(l)}\psi + k(D_{z_3}D_{z_3}^{-1})^{PL} \partial_{z_3}^{(l)}\psi
\]
(17)

From eqs. (16) and (17) we have that:
\[
d\psi_\alpha = \nabla^{(\alpha_1,..\alpha_n)}\psi_\alpha \cdot dr = \nabla^{(\alpha_1,..\alpha_n)}\psi_\alpha \cdot n dr \quad \text{and} \quad d\psi = \nabla^{(D)}\psi \cdot dr = \nabla^{(D)}\psi \cdot n dr\]
(18a,b)

where $dr = n dr = idx + jdy + kdz$, where $n$ is the unit normal, and $dr$ is a distance measured along the normal $n$.

The directional power-law derivative of the function $\psi_\alpha = \psi_\alpha(x,y,z)$, denoted by $\nabla^{(\alpha_1,..\alpha_n)}_n\psi_\alpha$, is defined:
\[
\frac{d\psi_\alpha}{dr} = \nabla^{(\alpha_1,..\alpha_n)}_n\psi_\alpha \cdot n = \partial_\alpha^{(\alpha_1,..\alpha_n)}_n\psi_\alpha
\]
(19)

which leads to:
\[
\frac{d\psi}{dr} = \nabla^{(D)}_n\psi \cdot n = \partial^{(D)}_n\psi
\]
(20)

where $d\psi_\alpha/dr$ and $d\psi/dr$ are the rates of changes of $\psi_\alpha$ and $\psi$ along the normal $n$, respectively.

The power-law Laplace-like operator of first type, denoted as $\nabla^{(\alpha_1,..\alpha_n)}\otimes\nabla^{(\alpha_1,..\alpha_n)} = \nabla^{(2\alpha_1,..2\alpha_n)}$, of the scalar field $\psi_\alpha$ is defined:
\[
\nabla^{(2\alpha_1,..2\alpha_n)}\psi_\alpha = \left[(D_{x_1}D_{x_1}^{-1})^{PL} \partial_{x_1}^{(l)}\right]^2\psi_\alpha + \left[(D_{y_2}D_{y_2}^{-1})^{PL} \partial_{y_2}^{(l)}\right]^2\psi_\alpha + \left[(D_{z_3}D_{z_3}^{-1})^{PL} \partial_{z_3}^{(l)}\right]^2\psi_\alpha
\]
(21)

In a similar way, the power-law Laplace-like operator of second type, denoted as $\nabla^{(2D)} = \nabla^{(D)}_n\otimes\nabla^{(D)}_n$, of the scalar field $\psi$ is defined:
\[
\nabla^{(2D)}\psi = \left[(D_{x_1}D_{x_1}^{-1})^{PL} \partial_{x_1}^{(l)}\right]^2\psi + \left[(D_{y_2}D_{y_2}^{-1})^{PL} \partial_{y_2}^{(l)}\right]^2\psi + \left[(D_{z_3}D_{z_3}^{-1})^{PL} \partial_{z_3}^{(l)}\right]^2\psi
\]
(22)

The properties for the power-law gradient of first type read:
\[
\nabla^{(\alpha_1,..\alpha_n)}_n\psi_\alpha \otimes \nabla^{(\alpha_1,..\alpha_n)}_n\psi_\alpha = \nabla^{(2\alpha_1,..2\alpha_n)}_n\psi_\alpha
\]
(23)

\[
\nabla^{(\alpha_1,..\alpha_n)}_n(\psi_\alpha \Theta_\alpha) = \psi_\alpha \nabla^{(\alpha_1,..\alpha_n)}_n\Theta_\alpha + \Theta_\alpha \nabla^{(\alpha_1,..\alpha_n)}_n\psi_\alpha
\]
(24)

\[
\nabla^{(\alpha_1,..\alpha_n)}(\Theta_\alpha \nabla^{(\alpha_1,..\alpha_n)}_n\psi_\alpha) = \Theta_\alpha \nabla^{(2\alpha_1,..2\alpha_n)}_n\psi_\alpha + \nabla^{(\alpha_1,..\alpha_n)}_n\psi_\alpha \otimes \nabla^{(\alpha_1,..\alpha_n)}_n\Theta_\alpha
\]
(25)

where $\psi_\alpha$ and $\Theta_\alpha$ are the scalar fields.

The properties for the power-law gradient of second type can be given:
\[
\nabla^{(D)}_n\otimes\nabla^{(D)}_n\psi = \nabla^{(2D)}_n\psi, \quad \nabla^{(D)}(\psi \Theta) = \psi \nabla^{(D)}\Theta + \Theta \nabla^{(D)}\psi
\]
(26a,b)

\[
\nabla^{(D)}(\Theta \nabla^{(D)}_n\psi) = \Theta \nabla^{(2D)}_n\psi + \nabla^{(D)}_n\psi \otimes \nabla^{(D)}_n\Theta
\]
(27)

where $\psi$ and $\Theta$ are the scalar fields.
Theory of the vector power-law calculus

The element of the vector line \( \ell = \ell_\omega(x,y,z) = \ell(x^{0_1},y^{0_2},z^{0_3}) \) is given:

\[
dl = m dl = i(D_x x^{0_1})dx + j(D_y y^{0_2})dy + k(D_z z^{0_3})dz
\]  

and

\[
d\ell = |dl| = \sqrt{(D_x x^{0_1})^2 + (D_y y^{0_2})^2 + (D_z z^{0_3})^2}
\]  

where \( m \) is the vector with \( |m| = 1 \).

The arc length \( \ell = \int_a^b dl \) from \( t = a \) to \( t = b \) is given:

\[
\ell = \int_a^b \sqrt{(D_x x^{0_1})^2 + (D_y y^{0_2})^2 + (D_z z^{0_3})^2} \, dt
\]  

The line power-law integral of the vector field

The line power-law integral of the vector field \( \mathbf{T} = \mathbf{T}_\omega(x,y,z) \) along the curve \( L(x,y,z) = L(x^{0_1},y^{0_2},z^{0_3}) \), denoted by \( \mathfrak{B} \), is defined:

\[
\mathfrak{B} = \int_{L(x,y,z)} T_\omega(x,y,z) \, dl
\]  

where \( \mathbf{T} = \mathbf{T}_\omega(x,y,z) = T_i(x^{0_1},y^{0_2},z^{0_3}) = T_iT_j + T_k \), and the element of the vector line is:

\[
dl = i(D_x x^{0_1})dx + j(D_y y^{0_2})dy + k(D_z z^{0_3})dz = i d\ell(x) + j d\zeta(y) + k d\zeta(z)
\]  

With use of eq. (6.1), we get:

\[
\int L(x,y,z) T \cdot dl = \int L(x,y,z) T_\omega(x,y,z) \, dl = \int_{L(t)} T_\omega \, dl
\]  

where \( dl/dt = i(D_x x^{0_1})dx/dt + j(D_y y^{0_2})dy/dt + k(D_z z^{0_3})dz/dt \).

Therefore, by using eqs. (32), (31) can be presented as follows:

\[
\int L(x,y,z) T \cdot dl = \int L(x,y,z) T_\omega \, dl = \int L(t) T_\omega \, dl
\]  

The vector field \( \mathbf{T} = \mathbf{T}_\omega(x,y,z) \) in \( L(x,y,z) = L(x^{0_1},y^{0_2},z^{0_3}) \) is said to be conservative if:

\[
\oint_{L(x,y,z)} T \cdot dl = 0
\]

The double power-law integral of the scalar field

The double power-law integral of the scalar field \( \Theta_\omega(x,y) = \Theta(x^{0_1},y^{0_2}) \) on the region \( S(x,y) = S(x^{0_1},y^{0_2}) \), denoted by \( A(\Theta) \), is defined:

\[
A(\Theta) = \iint_{S(x,y)} \Theta_\omega(x,y) \, dS
\]  

where \( dS = (D_x x^{0_1})(D_y y^{0_2}) \, dx \, dy \).
When $x = \ell(x)$ and $y = \zeta(y)$, we have:
\[
dS = (D_1 x^{D_1-1})(D_2 y^{D_2-1})\,dx\,dy = d\ell(x) d\zeta(y)
\]  
(37)

It is shown from eqs. (36) and (37) that:
\[
\begin{align*}
\oint S_{(x,y)} \Theta_\omega(x,y) dS &= \oint_{c}^{d} \oint_{a}^{b} \Theta_\omega(x,y)(D_1 x^{D_1-1})\,dx \left((D_2 y^{D_2-1})\,dy\right) \\
&= \oint_{a}^{b} \oint_{c}^{d} \Theta_\omega(x,y)(D_2 y^{D_2-1})\,dy \left((D_1 x^{D_1-1})\,dx\right)
\end{align*}
\]
(38)

where $x \in [a,b]$ and $y \in [c,d]$.

The volume power-law integral of the scalar field

The volume power-law integral of the scalar field $\Theta_\omega(x,y,z) = \Theta(x^{D_1}, y^{D_2}, z^{D_3})$ is defined:
\[
V(\Theta) = \iiint_{\Omega(x,y,z)} \Theta_\omega(x,y,z) dV
\]
(39)

with
\[
dV = (D_1 x^{D_1-1})(D_2 y^{D_2-1})(D_3 z^{D_3-1})\,dx\,dy\,dz = d\ell(x) d\zeta(y) d\zeta(z)
\]
where $\ell(x) = x^{D_1}$, $\zeta(y) = y^{D_2}$, and $\zeta(z) = z^{D_3}$.

Thus, we have:
\[
\begin{align*}
\iiint_{\Omega(x,y,z)} \Theta_\omega(x,y,z) dV &= \iiint_{\Omega(x,y,z)} \Theta_\omega(x,y,z)(D_1 x^{D_1-1})\,dx \\
&= \iiint_{\Omega(x,y,z)} \Theta_\omega(x,y,z)(D_2 y^{D_2-1})\,dy \\
&= \iiint_{\Omega(x,y,z)} \Theta_\omega(x,y,z)(D_3 z^{D_3-1})\,dz
\end{align*}
\]
(40)

where $x \in [a,b]$, $y \in [c,d]$, and $z \in [\alpha,\beta]$.

The surface power-law integral of the vector field

The surface power-law integral of the vector field $\Psi_\omega(x,y,z) = \Psi(x^{D_1}, y^{D_2}, z^{D_3})$ is defined:
\[
\begin{align*}
\iint_{S_{(x,y,z)}} \Psi_\omega(x,y,z) \cdot dS &= \iint_{S_{(x,y,z)}} \Psi_\omega(x,y,z) \cdot n dS \\
\end{align*}
\]
(41)

where $n = dS/dS$ is the unit normal vector to the surface $S(x,y,z) = S(x^{D_1}, y^{D_2}, z^{D_3})$.

Let us consider that $n = dS/dS = dS/dS$, $dS = |dS|$, and
\[
\begin{align*}
\iint_{S_{(x,y,z)}} \Psi_\omega(x,y,z) \cdot dS &= \iint_{S_{(x,y,z)}} \Psi_\omega(x,y,z) \cdot n dS \\
&= d\zeta(y) d\zeta(z) i + d\ell(x) d\zeta(z) j + d\ell(x) d\zeta(y) k \\
&= i(D_2 y^{D_2-1})(D_3 z^{D_3-1})\,dy\,dz + j(D_1 x^{D_1-1})(D_3 z^{D_3-1})\,dx\,dz + k(D_1 x^{D_1-1})(D_2 y^{D_2-1})\,dx\,dy
\end{align*}
\]
(42)
where \( \frac{d\zeta(y)}{dz}(y) \frac{d\zeta(z)}{dz} = (D_1 y^{(1-D_1)})(D_2 y^{(1-D_2)}) \frac{dy}{dz} \), \( \frac{d\zeta(x)}{dz}(x) \frac{d\zeta(z)}{dz} = (D_1 x^{(1-D_1)})(D_2 x^{(1-D_2)}) \frac{dx}{dy} \), and \( \frac{d\zeta(x)}{dz}(x) \frac{d\zeta(y)}{dy} = (D_1 x^{(1-D_1)})(D_2 z^{(1-D_2)}) \frac{dx}{dz} \).

Thus, we may have from eqs. (41) and (42) that:

\[
\int\int_{S(x,y,z)} \psi_m(x,y,z) \, dS = \int\int_{S(x,y,z)} \psi_m \frac{d\zeta(y)}{dz}(y) \frac{d\zeta(z)}{dz} + \psi_y \frac{d\zeta(x)}{dz}(x) \frac{d\zeta(z)}{dz} + \psi_z \frac{d\zeta(x)}{dz}(x) \frac{d\zeta(y)}{dy} \tag{43}
\]

where \( \psi = \psi_m(x,y,z) = \psi(x^{D_1}, y^{D_2}, z^{D_3}) = iy_x + jy_y + ky_z \).

The flux of the vector field \( \psi = \psi_m(x,y,z) \) across the surface \( dS \), denoted by \( \Phi \), is defined:

\[
\Phi = \oint_{S(x,y,z)} \psi \cdot dS \tag{44}
\]

The power-law divergence of the vector field

The power-law divergence of the vector field \( \psi \) is defined:

\[
\nabla^{(D_1,D_2,D_3)} \psi = \lim_{\Delta V_m \rightarrow 0} \frac{1}{\Delta V} \oint_{\Delta S_m(x,y,z)} \psi \cdot dS \tag{45}
\]

where the volume \( V \) is divided into a large number of small subvolumes \( \Delta V_m \) with surfaces \( \Delta S_m(x,y,z) \), \( \psi \) is a continuously differentiable vector field, and \( dS \) is an element of the surface \( S(x,y,z) \) bounding the solid \( \Omega(x,y,z) \).

With use of (14), (45) can be written:

\[
\nabla^{(D_1,D_2,D_3)} \psi = i(D_1 x^{D_1-1})PL \frac{d\zeta(x)}{dx} + j(D_2 y^{D_2-1})PL \frac{d\zeta(y)}{dy} + k(D_3 z^{D_3-1})PL \frac{d\zeta(z)}{dz} \tag{46}
\]

where \( \psi = \psi_m(x,y,z) = \psi(x^{D_1}, y^{D_2}, z^{D_3}) = iy_x + jy_y + ky_z \).

The power-law curl of the vector field

The power-law curl of the vector field \( T \) is defined:

\[
\nabla^{(D_1,D_2,D_3)} \times T = \lim_{\Delta S_m(x,y,z) \rightarrow 0} \frac{1}{\Delta S_m(x,y,z)} \oint_{\Delta S_m(x,y,z)} T_m(x,y,z) \cdot dl \tag{47}
\]

where \( T = T_m(x,y,z) = T(x^{D_1}, y^{D_2}, z^{D_3}) = T_x i + T_y j + T_z k \) be a continuously differentiable vector field, \( dl \) – the element of the vector line, \( \Delta S_m(x,y,z) \) is a small surface element perpendicular to \( n \), \( \Delta L_m(x,y,z) \) – the closed curve of the boundary of \( \Delta S_m(x,y,z) \), and \( n \) are oriented in a positive sense.

Similarly, eq. (47) can be represented:

\[
\nabla^{(D_1,D_2,D_3)} \times T = \left( \begin{array}{ccc}
(D_1 x^{D_1-1})PL \frac{d\zeta(x)}{dx} & (D_2 y^{D_2-1})PL \frac{d\zeta(y)}{dy} & (D_3 z^{D_3-1})PL \frac{d\zeta(z)}{dz}
\end{array} \right) \tag{48}
\]

where \( T = T_m(x,y,z) = T(x^{D_1}, y^{D_2}, z^{D_3}) = T_x i + T_y j + T_z k \).

The Gauss-like theorem

From the definition of eq. (45), we present the Gauss-like theorem as follows.
Let us consider that:
\[
\iiint_{\Omega(x,y,z)} \nabla \cdot \boldsymbol{\psi} \, dV = \iint_{S(x,y,z)} \boldsymbol{\psi} \cdot \mathbf{n} \, dS
\]  
(49)

where \( \boldsymbol{\psi} \) is a continuously differentiable vector field, \( dV \) denotes an element of volume \( \Omega(x,y,z) \), \( \mathbf{n} \) is the unit outward normal to \( S(x,y,z) \), and \( dS \) is an element of the surface area of the surface \( S(x,y,z) \) bounding the solid \( \Omega(x,y,z) \).

Taking \( dS = n \, dS \), we have from eq. (49) that:
\[
\iiint_{\Omega(x,y,z)} \nabla \cdot \boldsymbol{\psi} \, dV = \iint_{S(x,y,z)} \boldsymbol{\psi} \cdot \mathbf{n} \, dS \quad \text{(50a,b)}
\]

It is illustrated that eq. (50b) is the case of eq. (50a) when \( D_1 = D_2 = D_3 = D \).

From the definition of eq. (48), we present the Stokes-like theorem as follows.

\textbf{The Stokes-like theorem}

Let us consider that:
\[
\oint_{L(x,y,z)} \boldsymbol{\psi} \cdot d\mathbf{l} = \oint_{L(x,y,z)} T_x (D_1 x^{D_1-1}) \, dx + T_y (D_2 y^{D_2-1}) \, dy + T_z (D_3 z^{D_3-1}) \, dz
\]
(51)

where \( \boldsymbol{\psi} \) is a constant vector field, \( S(x,y,z) \) denotes an open, two-sided curve surface, \( L(x,y,z) \) represents the closed contour bounding \( S \), and \( d\mathbf{l} \) denotes the element of the vector line.

Taking \( dS = n \, dS \), we show from eq. (51) that:
\[
\iint_{S(x,y,z)} \left[ \nabla \times \boldsymbol{\psi} \right] \cdot \mathbf{n} \, dS = \oint_{L(x,y,z)} \boldsymbol{\psi} \cdot d\mathbf{l} \quad \text{(52a,b)}
\]

It is shown that eq. (52b) is the case of eq. (52a) when \( D_1 = D_2 = D_3 = D \).

\textbf{The Green-like theorem}

The Green-like theorem states:
\[
\oint_{L(x,y)} T_x (D_1 x^{D_1-1}) \, dx + T_y (D_2 y^{D_2-1}) \, dy = 
\]
\[
= \iint_{S(x,y)} \left[ x^{D_1-1} y^{D_2-1} \right] T_x \, dx \, dy 
\]
(53)

where \( S(x,y) \) is the domain bounded by the contour \( L(x,y) \), and \( T = T_x i + T_y j \).

When \( D_1 = D_2 = D_3 = D \), we have that:
\[
\oint_{L(x,y,z)} \boldsymbol{\psi} \cdot d\mathbf{l} = D \iint_{S(x,y)} x^{D_1-1} y^{D_2-1} \, dx \, dy
\]
(54)
The Green-like identities

Taking $\Phi = \Theta V^{(1)}$ as $\Phi$, we have that:

$$V^{(1,1,1)} [\Theta \Phi^{(1,1,1)}] = \Theta V^{(2,2,2)} \Phi + V^{(1,1,1)} \Phi \cdot V^{(1,1,1)} \Theta$$

(55)

and

$$V^{(1,1,1)} [\Phi \Theta^{(1,1,1)}] = \Phi V^{(2,2,2)} \Theta + V^{(1,1,1)} \Phi \cdot V^{(1,1,1)} \Theta$$

(56)

where $\Phi = \Phi_n(x, y, z) = \Phi(x^{D_1}, y^{D_1}, z^{D_1})$ and $\Theta = \Theta_n(x, y, z) = \Theta(x^{D_2}, y^{D_2}, z^{D_2})$ are the scalar fields.

With the use of eq. (49), the Green-like identity of first type can be given:

$$\int \int \int_{\Omega(x,y,z)} V^{(1)} [\Theta \Phi^{(1,1,1)}] dV = \int \int_{S(x,y,z)} \Theta \Phi^{(1)} dS$$

(57)

In a similar way, we have that:

$$\int \int \int_{\Omega(x,y,z)} V^{(1)} [\Phi \Theta^{(1,1,1)}] dV = \int \int_{S(x,y,z)} \Phi \Theta^{(1)} dS$$

(58)

which reduces to the Green-like identity of second type, given:

$$\int \int \int_{\Omega(x,y,z)} V^{(1)} [\Theta \Phi^{(1,1,1)} - \Phi \Theta^{(1,1,1)}] dV = \int \int_{S(x,y,z)} \Theta \Phi^{(1)} - \Phi \Theta^{(1)} dS$$

(59)

Taking $D_1 = D_2 = D_3 = D$, we have from eqs. (57) and (59) that:

$$\int \int \int_{\Omega(x,y,z)} V^{(1)} [\Theta \Phi^{(1,1,1)} - \Phi \Theta^{(1,1,1)}] dV = \int \int_{S(x,y,z)} \Theta \Phi^{(1)} dS$$

(60)

and

$$\int \int \int_{\Omega(x,y,z)} V^{(1)} [\Theta \Phi^{(1,1,1)} - \Phi \Theta^{(1,1,1)}] dV = \int \int_{S(x,y,z)} \Theta \Phi^{(1)} - \Phi \Theta^{(1)} dS$$

(61)

Taking $D_1 = D_2 = D_3 = D = 1$, the Gauss-like, Stokes-like and Green-like theorems and Green-like identities become the Gauss [22], Stokes [23], Green theorems and Green identities [24], respectively.

Applied to describe the power-law fluid flow

Let us consider the power-law co-ordinate system, given as $t^{D_0}, x^{D_1}, y^{D_1}, z^{D_1} = t^D + \lambda^D x^{D_1} + \mu^D y^{D_1} + \kappa^D z^{D_1}$, where $D_0$, $D_1$, $D_2$, and $D_3$ are the fractal dimensions, and $\lambda$, $\mu$, and $\kappa$ are the unit vector in the Cartesian co-ordinate system.

The material power-law derivative of the power-law fluid field.

Let $\Phi = \Phi(x^D, y^D, z^D)$ be the power-law type fluid field.
The total power-law differential of the power-law type scalar field is given:

\[
\begin{align*}
\frac{d\Phi}{dt} &= \left[ D_1 x^{D_1-1} \ell_1^{(1)} \Phi \right] dx + \left[ D_2 y^{D_2-1} \ell_2^{(1)} \Phi \right] dy + \\
&\quad + \left[ D_3 z^{D_3-1} \ell_3^{(1)} \Phi \right] dz + \left[ D_4 t^{D_4-1} \ell_4^{(1)} \Phi \right] dt
\end{align*}
\]

which leads to:

\[
\begin{align*}
\frac{d\Phi}{dt} &= \left[ D_1 x^{D_1-1} \ell_1^{(1)} \Phi \right] \frac{\partial x}{\partial t} + \left[ D_2 y^{D_2-1} \ell_2^{(1)} \Phi \right] \frac{\partial y}{\partial t} + \\
&\quad + \left[ D_3 z^{D_3-1} \ell_3^{(1)} \Phi \right] \frac{\partial z}{\partial t} + D_4 t^{D_4-1} \ell_4^{(1)} \Phi
\end{align*}
\]

The material power-law derivative

The material power-law derivative of the power-law fluid density \( \phi \) is defined:

\[
\frac{D\phi}{Dt} = D_4 t^{D_4-1} \ell_4^{(1)} \phi + \mathbf{v} \cdot \nabla^{(D_1, D_2, D_3)} \phi
\]

where \( \mathbf{v} = (\partial x/\partial t, \partial y/\partial t, \partial z/\partial t) = i \nu_x + j \nu_y + k \nu_z \) is the velocity vector.

For \( D_4 = 1 \) the material power-law-space derivative of the power-law fluid density, reads:

\[
\frac{D\phi}{Dt} = \frac{\partial \phi}{\partial t} + \mathbf{v} \cdot \nabla \phi
\]

which, by using \( D_1 = D_2 = D_3 = D \), leads to:

\[
\frac{D\phi}{Dt} = \frac{\partial \phi}{\partial t} + \mathbf{v} \cdot \nabla (D) \phi
\]

For \( D_1 = D_2 = D_3 = 1 \) the material power-law-time derivative of the power-law fluid density, denoted as, can be given:

\[
\frac{D\phi}{Dt} = D_4 t^{D_4-1} \ell_4^{(1)} \phi + \mathbf{v} \cdot \nabla \phi
\]

It is not difficult to show that the Stokes material derivative, proposed by Stokes in 1845 to consider the velocity [25] and further developed in 1851 [26], is one of the special cases of eqs. (64)-(67) when \( D_1 = D_2 = D_3 = D_4 = 1 \), and it illustrates the relationship among the change in Lagrangian co-ordinate \( X = (t, X, Y, Z) \), Eulerian co-ordinate \( x = (t, x, y, z) \), and Eulerian-like co-ordinate \( \mathbf{x} = (t^{D_1}, x^{D_1}, y^{D_1}, z^{D_1}) \) for any fluid field.

The transport theorem for the power-law fluid

From eq. (64) we have that the transport theorem for the power-law fluid, e.g.:

\[
\frac{D}{Dt} \int \int \int_{\Omega(t)} \Xi dV = \int \int \int_{\Omega(t)} \left[ D_4 t^{D_4-1} \ell_4^{(1)} \Xi + \mathbf{v} \cdot \nabla^{(D_1, D_2, D_3)} \Xi \right] dV
\]

which, by using eq. (52a), leads to:

\[
\frac{D}{Dt} \int \int \int_{\Omega(t)} \Xi dV = \int \int \int_{\Omega(t)} D_4 t^{D_4-1} \ell_4^{(1)} \Xi dV + \int_{\Sigma(t)} \Xi \cdot \mathbf{n} dS
\]
since

$$\iiint_{\Omega(t)} \mathbf{v} \cdot \nabla^{(D_1,D_2,D_3)} \mathbf{\Xi} \, dV = \iiint_{\Omega(t)} \mathbf{\Xi} \cdot (\mathbf{v} \cdot \mathbf{n}) \, dS = \iiint_{\Omega(t)} \mathbf{\Xi} \cdot \mathbf{v} \cdot dS$$

(70)

where $\mathbf{S}(t)$ is the surface of $\Omega(t)$, $\mathbf{n}$ is the unit normal to the surface, $\mathbf{v}$ is the velocity vector, and $\mathbf{\Xi} = \Xi(t,x,y,z) = \Xi(t^0,t^1,t^2,t^3)$ is the power-law fluid.

Here, the Reynolds transport theorem, proposed by in 1903 Reynolds [27], is the special case of eqs. (68) and (69) when $D_1 = D_2 = D_3 = D_0 = 1$.

Let us define the mass of the power-law fluid is defined:

$$\iiint_{\Omega(t)} \mathbf{3} \, dV = \mathbf{S}$$

(71)

where $\mathbf{3} = \mathbf{3}_m(t,x,y,z) = \mathbf{3}(t^0,x^0,y^0,z^0)$ and $\mathbf{S} = \mathbf{S}_m(t,x,y,z) = \mathbf{S}(t^0,x^0,y^0,z^0)$.

The conservation of the mass of the power-law fluid is given:

$$D_t t^{D-1} \mathbf{c}_i^{(i)} \mathbf{3} + \mathbf{v} \cdot \nabla^{(D_1,D_2,D_3)} \mathbf{3} = 0 \quad \text{and} \quad D_t t^{D-1} PL \mathbf{c}_i^{(i)} \mathbf{3} + \mathbf{v} \cdot \nabla^{(D_1,D_2,D_3)} \mathbf{3} \cdot (\mathbf{3} \mathbf{v}) = 0$$

(72a,b)

since $\mathbf{v}$ is the velocity vector (a constant vector), and there is from eqs. (68) and (71):

$$\frac{D}{Dt} \iiint_{\Omega(t)} \mathbf{3} \, dV = \iiint_{\Omega(t)} \left[ D_t t^{D-1} PL \mathbf{c}_i^{(i)} \mathbf{3} + \mathbf{v} \cdot \nabla^{(D_1,D_2,D_3)} \mathbf{3} \right] \, dV = 0$$

(73)

Here, the conservation of the mass of the fluid without power-law, proposed by Euler in 1757 [28], is the special case of eqs. (72b) and (73), and proposed by Lagrange in 1781 [29], is the special case of eq. (72b), where $D_1 = D_2 = D_3 = D_0 = 1$.

Let us consider the velocity gradient tensor for the power-law fluid, defined:

$$\nabla^{(D_1,D_2,D_3)} \mathbf{v} = \frac{1}{2} (\zeta + \tau) + \frac{1}{2} (\zeta - \tau) = \mathbf{h} + \frac{1}{2} (\zeta - \tau) \quad \text{and} \quad \zeta = \nabla^{(D_1,D_2,D_3)} \mathbf{v}$$

(74a,b)

where the strain tensor for the power-law fluid is defined as $\mathbf{h} = (\zeta + \tau)/2$ with velocity gradient $\zeta = \nabla^{(D_1,D_2,D_3)} \mathbf{v}$ and $\tau = \mathbf{v} \cdot \nabla^{(D_1,D_2,D_3)} \mathbf{v}$.

The stress tensor for the power-law fluid is defined:

$$\mathbf{T} = -p \mathbf{I} + 2 \beta \mathbf{h}$$

(75)

where $\beta$ are the shear moduli of viscosity, and $\mathbf{I}$ is the unit tensor.

It is noted that the strain tensor is the special case, proposed by Cauchy in [30, 31], and the Stokes decompose term [25] is the special case of eq. (74), the stress tensor, proposed by Stokes in 1845 [25], is the special case of eq. (75) where $D_1 = D_2 = D_3 = D_0 = 1$.

The conservation of the momentums for the power-law fluid

The conservation of the linear and angular momentums for the power-law fluid is:

$$\frac{D}{Dt} \iiint_{\Omega(t)} \mathbf{3} \, dV = \iiint_{\Omega(t)} \mathbf{b} \, dV + \iiint_{\Sigma(t)} \mathbf{T} \cdot dS$$

(76)

where $\mathbf{b}$ is the specific body force.

Thus, we have:

$$D_t t^{D-1} PL \mathbf{c}_i^{(i)} (\mathbf{3} \mathbf{v}) + \mathbf{v} \cdot \nabla^{(D_1,D_2,D_3)} (\mathbf{3} \mathbf{v}) = \nabla^{(D_1,D_2,D_3)} \mathbf{v} \cdot \mathbf{T} + \mathbf{b}$$

(77)
since

\[ \iiint_{\Omega(t)} \left[ D_{st}^{D_0^{-1} PL} \mathcal{C}^{(1)}(\mathbf{j}) + \mathbf{v} \cdot \nabla^{(D_1,D_2,D_3)}(\mathbf{j}) - \mathbf{b} - \nabla^{(D_1,D_2,D_3)} \cdot \mathbf{T} \right] dV = 0 \]  

(78)

where

\[ \frac{D}{Dt} \iiint_{\Omega(t)} \mathbf{j} dV = \iiint_{\Omega(t)} \left[ D_{st}^{D_0^{-1} PL} \mathcal{C}^{(1)}(\mathbf{j}) + \mathbf{v} \cdot \nabla^{(D_1,D_2,D_3)}(\mathbf{j}) \right] dV \]  

(79)

and

\[ \iiint_{S(t)} \mathbf{T} \cdot dS = \iiint_{\Omega(t)} \nabla^{(D_1,D_2,D_3)} \cdot \mathbf{J} dV \]  

(80)

From eqs. (74a), (74b) and (77) we have:

\[ \nabla^{(D_1,D_2,D_3)} \cdot \mathbf{T} = -\nabla^{(D_1,D_2,D_3)} p + \beta \nabla^{(2D_1,2D_2,2D_3)} \mathbf{v} \]  

(81)

such that

\[ \mathcal{J} \left[ D_{st}^{D_0^{-1} PL} \mathcal{C}^{(1)}(\mathbf{j}) + \mathbf{v} \cdot \nabla^{(D_1,D_2,D_3)}(\mathbf{j}) \right] = -\nabla^{(D_1,D_2,D_3)} p + \beta \nabla^{(2D_1,2D_2,2D_3)} \mathbf{v} + \mathbf{b} \]  

(82)

From eqs. (74a) and (82) we have for \( \beta = 0 \):

\[ \mathcal{J} \left[ D_{st}^{D_0^{-1} PL} \mathcal{C}^{(1)}(\mathbf{j}) + \mathbf{v} \cdot \nabla^{(D_1,D_2,D_3)}(\mathbf{j}) \right] = -\nabla^{(D_1,D_2,D_3)} p + \beta \nabla^{(2D_1,2D_2,2D_3)} \mathbf{v} + \mathbf{b} \quad \text{and} \quad \nabla^{(D_1,D_2,D_3)} \cdot \mathbf{v} = 0 \]  

(83a,b)

Here, the Navier-Stokes equations for the fluid, proposed by Navier in 1822 [32] and by Stokes in 1845 [25] are the special cases of eqs. (74b) and (77), and the Euler equations for the fluid, proposed by Euler in 1757 [28], are the special cases of eqs. (83a) and (83b), where \( D_1 = D_2 = D_3 = D_0 = 1 \).

Similarly, from eq. (67) we have that:

\[ \frac{D\mathbf{v}}{Dt} = D_{st}^{D_0^{-1} PL} \mathcal{C}^{(1)}(\mathbf{j}) + \mathbf{v} \cdot \nabla \mathbf{v} \]  

(84)

Here, the Stokes formula for the fluid, proposed by Stokes in 1845 [25], is the special case of eq. (84) for \( D_1 = D_2 = D_3 = D_0 = 1 \).

Conclusion

In the present work, we have proposed the theory of the vector power-law calculus based on the Leibniz, Stieltjes, and Riemann tasks. The Navier-Stokes-like and Euler-like equations for the power-law fluid were presented based on the conservations of the mass and angular momentums for the power-law fluid. The proposed results are proposed as an advanced mathematical tool for decryptions for the power-law physical phenomenon.

Acknowledgment

This work is supported by the Yue-Qi Scholar of the China University of Mining and Technology (No. 102504180004).

Nomenclature

\[ \mathbf{b} \quad \text{– specific body force, [Nm}^{-3}\text{]} \]

Greek symbol

\[ \mathbf{v} \quad \text{– velocity vector, [ms}^{-1}\text{]} \]

\[ t \quad \text{– time, [s]} \]

\[ x, y, z \quad \text{– co-ordinates, [m]} \]
References

[12] He, J. H., Fractal Calculus and Its Geometrical Explanation, Results in Physics, 10 (2018), 2, pp. 272-276
[15] Leibniz, G. W., Memoir Using the Chain Rule, 1676
[29] Lagrange, J. L., Mémoire sur la Théorie du Mouvement des Fluides, Académie de Berlin, Mémoires, 4 (1781), 1781, pp. 695-748