THE ANALYTIC SOLUTIONS FOR THE UNSTEADY ROTATING FLOWS OF THE GENERALIZED MAXWELL FLUID BETWEEN COAXIAL CYLINDERS

by

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Original scientific paper
https://doi.org/10.2298/TSCI2006041W

In this paper, we consider the unsteady rotating flow of the generalized Maxwell fluid with fractional derivative model between two infinite straight circular cylinders, where the flow is due to an infinite straight circular cylinder rotating and oscillating pressure gradient. The velocity field is determined by means of the combine of the Laplace and finite Hankel transforms. The analytic solutions of the velocity and the shear stress are presented by series form in terms of the generalized G and R functions. The similar solutions can be also obtained for ordinary Maxwell and Newtonian fluids as limiting cases.

Key words: *generalized Maxwell fluid, fractional derivative, finite Hankel transform, oscillating pressure gradient, analytic solutions*

Introduction

As we know that the generalized Maxwell fluid is a subclass of non-Newtonian fluids, which does not obey the Newtonian postulate that the stress tensor is directly proportional to the rate of deformation tensor [1]. Numerous traditional integer order differential models can not describe the response characteristics of these non-Newtonian fluids. In order to describe the rheological properties of wide classes of materials more clearly and deeply, the rheological constitutive equations with fractional derivatives have been introduced (see [2-8] and the reference therein). Furthermore, it has been shown that the constitutive equations with fractional derivatives are also linked to the molecular theories [2]. The modified viscoelastic models were provided to describe the behavior for Xanthan gum in [9]. The fractional derivative models of non-Newtonian fluids were produced by replacing the time derivative of an integer order by the Riemann-Liouville fractional differential operator [10-15].

Based on the previous statements, the aim of the paper is to interest into the torsional oscillatory motion of the generalized Maxwell fluid between two infinite coaxial circular cylinders, both of them oscillating around their common axis with the given angular frequencies.

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Torsional oscillations between two cylinders

Constitutive equations

Let us consider the classical differential model, given as:

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S} \text{ and } \mathbf{S} + \lambda[\mathbf{S} + (\mathbf{V}\cdot\nabla)\mathbf{S} - \mathbf{LS} - \mathbf{SL}^T] = \mu\{\mathbf{A} + \lambda_r[\mathbf{A} + (\mathbf{V}\cdot\nabla)\mathbf{A} - \mathbf{LA} - \mathbf{AL}^T]\}$$

where $\mathbf{T}$ is the Cauchy stress tensor, $-p\mathbf{I}$ – the indeterminate spherical stress, $\mathbf{S}$ – the extra stress tensor, $\mathbf{V}$ – the velocity vector, $\mathbf{L}$ – the velocity gradient, $\mathbf{A} = \mathbf{L} + \mathbf{L}^T$ – the Rivlin-Ericksen tensor, μ – the dynamic viscosity, λ and λ_r are the relax and delay constants, ∇ – the gradient operator, and T – the transpose operation.

It leads to the Newtonian fluids when $\lambda = \lambda_r = 0$ and the Maxwell fluids when $\lambda_r = 0$, respectively.

Here, we now replace the classical differential operator by the Riemann-Liouville fractional derivative operator and consider the following constitutive equations of the incompressible generalized Maxwell fluid, *e. g.*,

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S} \quad \text{and} \quad \mathbf{S} + \lambda[D_t^\alpha \mathbf{S} + (\mathbf{V}\cdot\nabla)\mathbf{S} - \mathbf{LS} - \mathbf{SL}^T] = \mu\mathbf{A} \tag{1}$$

where $D_t^\alpha f(t)$ is denoted as the Riemann-Liouville fractional derivative of the function $f(t)$, defined as [11, 15, 16]:

$$D_0^\alpha x(t) = \frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t (t-s)^{-\alpha} x(s)\mathrm{d}s\,, \quad 0<\alpha<1$$

and $\Gamma(\cdot)$ is the Gamma function [15, 16].

Mathematical formulation of problem and governing equation

In the cylindrical co-ordinates (r,θ,z), the axial couette flow velocity is given:

$$\mathbf{V} = u(r,t)e_\theta$$

where e_θ is the unit vector in the θ-axis.

At the moment $t = 0$, then we have that:

$$\mathbf{V}(r,0) = \mathbf{0} \quad \text{and} \quad \mathbf{S}(r,0) = \mathbf{0}$$

Thus, it follows that:

$$(1 + \lambda D_t^\alpha)\tau = \mu\left(\frac{\partial}{\partial r} - \frac{1}{r}\right)u \tag{2}$$

since $\mathbf{S}_{rr} = \mathbf{S}_{rz} = \mathbf{S}_{\theta z} = \mathbf{S}_{\theta\theta} = 0$, and $\tau(r,t) = \mathbf{S}_{r\theta}(r,t)$ is the shear stresses, where λ is the material constant.

Considering the pressure gradient and ignoring body forces in the axial direction, the balance of linear momentum reads:

$$\rho\frac{\partial u}{\partial t} = -\frac{1}{r}\frac{\partial P}{\partial\theta} + \frac{1}{r^2}\frac{\partial(r^2\tau)}{\partial r} \tag{3}$$

where ρ is the density of the fluid.

We consider an incompressible generalized Maxwell fluid at rest in the annular region between two infinite circular cylinders of the radius R_1 and R_2 $(R_2 > R_1)$.

When $t=0^{+}$, the inner cylinder is suddenly moved with a time-dependent pressure gradient in the θ-axial direction, *e. g.*:

$$\frac{\partial P}{\partial \theta}=-\rho P_0 \sin(\omega t)$$

The associated initial and boundary conditions are:

$$u(r,0)=\frac{\partial u(r,0)}{\partial t}=0 \tag{4}$$

$$u(R_1,t)=\omega_1 \sin\beta_1 t \quad \text{and} \quad u(R_2,t)=\omega_2 \sin\beta_2 t$$

where $\omega_1, \omega_2, \beta_1, \text{and}\, \beta_2$ are the constants, $t>0$ and $r\in[R_{1,}R_2]$.

Calculation of the velocity field and the shear stress

Making use of eqs. (2)-(4) and taking the Laplace transform and inverse transform of u, we have that:

$$u(r,s)=\mathrm{L}^{-1}[\overline{u}(r,s);t]=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\overline{u}(r,s)\mathrm{e}^{st}\mathrm{d}s$$

and

$$\mathrm{L}[D_t^{\alpha}u(t),s]=s^{\alpha}\mathrm{L}u(t)-D^{\alpha-1}u(0)$$

Taking Laplace transform of eq. (2), we obtain:

$$\overline{\tau}_{r\theta}=\frac{\mu\left(\dfrac{\partial\overline{u}}{\partial r}-\dfrac{\overline{u}}{r}\right)}{1+\lambda s^{\alpha}},\quad \frac{\partial\overline{\tau}}{\partial r}=\frac{\mu}{1+\lambda s^{\alpha}}\left(\frac{\partial^2\overline{u}}{\partial r^2}+\frac{1}{r^2}\overline{u}-\frac{1}{r}\frac{\partial\overline{u}}{\partial r}\right) \tag{5}$$

Then,

$$\rho s\overline{u}=\frac{\rho P_0}{r}\frac{\omega}{\omega^2+s^2}+\frac{\mu}{1+\lambda s^{\alpha}}\left(\frac{\partial^2\overline{u}}{\partial r^2}+\frac{1}{r}\frac{\partial\overline{u}}{\partial r}-\frac{1}{r^2}\overline{u}\right)$$

and

$$\frac{\partial^2\overline{u}}{\partial r^2}+\frac{1}{r}\frac{\partial\overline{u}}{\partial r}-\frac{1}{r^2}\overline{u}+\frac{\rho P_0(1+\lambda s^{\alpha})}{\mu r}\frac{\omega}{\omega^2+s^2}-\frac{\rho s(1+\lambda s^{\alpha})}{\mu}\overline{u}=0 \tag{6}$$

The finite Hankel transforms of $\overline{u}(r,s)$ can be given:

$$\overline{u}_H=\int_{R_1}^{R_2} r\overline{u}(r,s)B_1(rr_n)\,\mathrm{d}r, n=1,2,\cdots$$

and r_n are the positive roots of the transcendental equation, *e. g.*:

$$B_1(R_1 r_n)=J_1(R_1 r r_n)Y_1(R_2 r_n)-J_1(R_2 r_n)Y_1(R_1 r_n)$$

where $J_1(\cdot)$ and $Y_1(\cdot)$ are the Bessel functions of order zero of the first and second kind.

Applying the finite Hankel transform in eq. (6) and taking into account the conditions in eq. (5), we find that:

$$\frac{\rho P_0(1+\lambda s^\alpha)}{\mu}\int_{R_1}^{R_2} r\frac{1}{r}\frac{\omega}{\omega^2+s^2}B_1(rr_n)\mathrm{d}r = \frac{\rho P_0\omega(1+\lambda s^\alpha)}{\mu(\omega^2+s^2)}\frac{\overline{B}(R_1r_n)-\overline{B}(R_2r_n)}{r_n} \tag{7}$$

where $\overline{B}(rr_n) = J_0(rr_n)Y_1(R_2r_n) - J_1(R_2r_n)Y_0(rr_n)$.

Since:

$$\int_{R_1}^{R_2} r\left(\frac{\partial^2\overline{u}}{\partial r^2}+\frac{1}{r}\frac{\partial\overline{u}}{\partial r}-\frac{1}{r^2}\overline{u}\right)B_1(rr_n)\,\mathrm{d}r = -\overline{u}rr_n\overline{B}_1(rr_n)\Big|_{R_1}^{R_2} +$$

$$+\int_{R_1}^{R_2}\overline{u}r_n\overline{B}_1(rr_n)\,\mathrm{d}r + \int_{R_1}^{R_2}\overline{u}rr_n\frac{\partial}{\partial r}\left[\overline{B}_1(rr_n)\right]\mathrm{d}r - \int_{R_1}^{R_2}\overline{u}r_n\overline{B}_1(rr_n)\,\mathrm{d}r$$

and

$$\frac{\partial}{\partial r}\left[\overline{B}_1(rr_n)\right] = \frac{\partial}{\partial r}\left[J_0(rr_n)Y_1(R_2r_n) - J_1(R_2r_n)Y_0(rr_n)\right] = -r_nB_1(rr_n)$$

$$J_n(x)Y_{n+1}(x) - J_{n+1}(x)Y_n(x) = -\frac{2}{\pi x}$$

and

$$J_1(R_1r_n)Y_1(R_2r_n) - J_1(R_2r_n)Y_1(R_1r_n) = 0$$

we have that:

$$\int_{R_1}^{R_2} r\left(\frac{\partial^2\overline{u}}{\partial r^2}+\frac{1}{r}\frac{\partial\overline{u}}{\partial r}-\frac{1}{r^2}\overline{u}\right)B_1(rr_n)\,\mathrm{d}r = \frac{2}{\pi}\frac{\omega_2\beta_2}{(s^2+\beta_2^2)} - \frac{2}{\pi}\frac{\omega_1\beta_1}{(s^2+\beta_1^2)}\frac{J_1(R_2r_n)}{J_1(R_1r_n)} - r_n^2\overline{u}_H \tag{8}$$

and

$$\int_{R_1}^{R_2} r\frac{\rho s(1+\lambda s^\alpha)}{\mu}\overline{u}B_1(rr_n)\,\mathrm{d}r = \frac{\rho s(1+\lambda s^\alpha)}{\mu}\overline{u}_H \tag{9}$$

With use of eqs. (7)-(9), we have that:

$$\frac{\rho P_0\omega(1+\lambda s^\alpha)}{\mu(\omega^2+s^2)}\frac{\overline{B}(R_1r_n)-\overline{B}(R_2r_n)}{r_n} + \frac{2}{\pi}\frac{\omega_2\beta_2}{(s^2+\beta_2^2)} - \frac{2}{\pi}\frac{\omega_1\beta_1}{(s^2+\beta_1^2)}\frac{J_1(R_2r_n)}{J_1(R_1r_n)} -$$

$$-r_n^2\overline{u}_H - \frac{\rho s(1+\lambda s^\alpha)}{\mu}\overline{u}_H = 0$$

$$\overline{u}_H = \frac{2}{\pi}\frac{\omega_2\beta_2}{(s^2+\beta_2^2)r_n^2} - \frac{2}{\pi}\frac{\omega_1\beta_1}{(s^2+\beta_1^2)r_n^2}\frac{J_1(R_2r_n)}{J_1(R_1r_n)} - \left(\frac{2}{\pi}\frac{\omega_2\beta_2\rho s(1+\lambda s^\alpha)}{(s^2+\beta_2^2)r_n^2(\rho s+\rho\lambda s^{\alpha+1}+\mu r_n^2)} -\right.$$

$$\left.-\frac{2}{\pi}\frac{\omega_1\beta_1\rho s(1+\lambda s^\alpha)}{(s^2+\beta_1^2)r_n^2(\rho s+\rho\lambda s^{\alpha+1}+\mu r_n^2)}\frac{J_1(R_2r_n)}{J_1(R_1r_n)}\right) +$$

$$+\frac{\rho P_0\omega(1+\lambda s^\alpha)}{(\rho s+\rho\lambda s^{\alpha+1}+\mu r_n^2)(\omega^2+s^2)}\frac{\overline{B}(R_1r_n)-\overline{B}(R_2r_n)}{r_n}$$

and

$$\overline{B}(R_1r_n)-\overline{B}(R_2r_n)=J_0(R_1r_n)Y_1(R_2r_n)-J_1(R_2r_n)Y_0(R_1r_n)-$$
$$-J_0(R_2r_n)Y_1(R_2r_n)+J_1(R_2r_n)Y_0(R_2r_n)$$

By means of the inverse Hankel transform formula [1], we get:

$$\overline{u}=\frac{\pi^2}{2}\sum_{n=1}^{\infty}\frac{r_n^2J_1^2(R_1r_n)B_1(rr_n)}{J_1^2(R_1r_n)-J_1^2(R_2r_n)}\overline{u}_H$$

If $a(r)=[AR_1(R_2^2-r^2)+BR_2(r^2-R_1^2)]/[(R_2^2-R_1^2)r]$, then we have:

$$a_n=\int_{R_1}^{R_2}ra(r)B_1(rr_n)\mathrm{d}r=-\int_{R_1}^{R_2}\frac{AR_1(r^2-R_2^2)}{R_2^2-R_1^2}B_1(rr_n)\mathrm{d}r+\int_{R_1}^{R_2}\frac{BR_2(r^2-R_1^2)}{R_2^2-R_1^2}B_1(rr_n)\mathrm{d}r$$

$$\int_{R_1}^{R_2}r^2B_1(rr_n)\mathrm{d}r=\left[R_2^2J_2\left(R_2r_n\right)-R_1^2J_2\left(R_1r_n\right)\right]\frac{Y_1(R_2r_n)}{r_n}-\frac{J_1(R_2r_n)}{r_n}\left[R_2^2Y_2\left(R_2r_n\right)-R_1^2Y_2\left(R_1r_n\right)\right]$$

$$\int_{R_1}^{R_2}\frac{AR_1(r^2-R_2^2)}{R_2^2-R_1^2}B_1(rr_n)\mathrm{d}r=\frac{2}{\pi}\frac{A}{r_n^2}\frac{J_1(R_2r_n)}{J_1(R_1r_n)}\quad\text{and}\quad\int_{R_1}^{R_2}\frac{BR_2(r^2-R_1^2)}{R_2^2-R_1^2}B_1(rr_n)\mathrm{d}r=\frac{2}{\pi}\frac{B}{r_n^2}$$

It is not difficult to find that:

$$a_n=\int_{R_1}^{R_2}ra(r)B_1(rr_n)\mathrm{d}r=\frac{2}{\pi}\frac{B}{r_n^2}-\frac{2}{\pi}\frac{A}{r_n^2}\frac{J_1(R_2r_n)}{J_1(R_1r_n)},\quad A=\frac{\omega_1\beta_1}{s^2+\beta_1^2}\quad\text{and}\quad B=\frac{\omega_2\beta_2}{s^2+\beta_2^2}$$

Thus, we can obtain:

$$\overline{u}=\frac{\frac{\omega_1\beta_1}{s^2+\beta_1^2}R_1(R_2^2-r^2)+\frac{\omega_2\beta_2}{s^2+\beta_2^2}R_2(r^2-R_1^2)}{(R_2^2-R_1^2)r}-\pi\sum_{n=1}^{\infty}\frac{J_1^2(R_1r_n)B_1(rr_n)}{J_1^2(R_1r_n)-J_1^2(R_2r_n)}\cdot$$
$$\cdot\frac{\omega_2\beta_2\rho s(1+\lambda s^\alpha)}{(s^2+\beta_2^2)(\rho s+\rho\lambda s^{\alpha+1}+\mu r_n^2)}+\pi\sum_{n=1}^{\infty}\frac{J_1^2(R_1r_n)B_1(rr_n)}{J_1^2(R_1r_n)-J_1^2(R_2r_n)}\frac{\omega_1\beta_1\rho s(1+\lambda s^\alpha)}{(s^2+\beta_1^2)(\rho s+\rho\lambda s^{\alpha+1}+\mu r_n^2)}\cdot$$
$$\cdot\frac{J_1(R_2r_n)}{J_1(R_1r_n)}+\frac{\pi^2}{2}\sum_{n=1}^{\infty}\frac{r_nJ_1^2(R_1r_n)B_1(rr_n)}{J_1^2(R_1r_n)-J_1^2(R_2r_n)}\frac{\rho P_0(\lambda s^\alpha+1)\omega[\overline{B}(R_1r_n)-\overline{B}(R_2r_n)]}{(s^2+\omega^2)(\rho s+\rho\lambda s^{\alpha+1}+\mu r_n^2)}\tag{10}$$

Let us consider Laplace transform of the function $G_{a,b,c}(d,t)$ and $E_{\alpha,\beta}(z)$, *e. g.*:

$$G_{a,b,c}(d,t)=\mathrm{L}^{-1}\left[s^b(s^a-d)^{-c}\right]\quad\text{and}\quad t^{\alpha n+\beta-1}E_{\alpha,\beta}^{(n)}(-ct^\alpha)=\mathrm{L}^{-1}\left[n!s^{\alpha-\beta}(s^\alpha+c)^{-n-1}\right]$$

where

$$G_{a,b,c}(d,t)=\sum_{j=0}^{\infty}\frac{(c)_jd^jt^{(j+c)a-b-1}}{j!\Gamma[(j+c)a-b]}$$

is the generalized G function [11], $(c)_j$ is the Pochhammer polynomial [5], and $E_{\alpha,\beta}(z)$ is the generalized Mittag-Leffler function [12, 15, 16].

Here, we present the velocity field, given as:

$$u=\frac{R_1(R_2^2-r^2)}{(R_2^2-R_1^2)r}\omega_1\sin\beta_1 t+\frac{R_2(r^2-R_1^2)}{(R_2^2-R_1^2)r}\omega_2\sin\beta_2 t-\pi\sum_{n=1}^{\infty}\frac{J_1^{\,2}(R_1r_n)B_1(rr_n)\omega_2}{J_1^2(R_1r_n)-J_1^2(R_2r_n)}\cdot$$
$$\cdot\sum_{k=0}^{\infty}\left(-\frac{\mu r_n^2}{\lambda\rho}\right)^k\int_0^t\sin\left[\beta_2(t-\tau)\right]G_{\alpha,-k,k}(-\lambda^{-1},\tau)\,\mathrm{d}\tau+$$
$$+\pi\sum_{n=1}^{\infty}\frac{J_1^{\,2}(R_1r_n)B_1(rr_n)\omega_1}{J_1^2(R_1r_n)-J_1^2(R_2r_n)}\sum_{k=0}^{\infty}\left(-\frac{\mu r_n^2}{\lambda\rho}\right)^k\frac{J_1(R_2r_n)}{J_1(R_1r_n)}\cdot$$
$$\cdot\int_0^t\sin\left[\beta_1(t-\tau)\right]G_{\alpha,-(k+1),k}(-\lambda^{-1},\tau)\,\mathrm{d}\tau+\frac{\pi^2}{2}\sum_{n=1}^{\infty}\frac{r_nJ_1^2(R_1r_n)B_1(rr_n)P_0[\bar{B}(R_1r_n)-\bar{B}(R_2r_n)]}{J_1^2(R_1r_n)-J_1^2(R_2r_n)}\cdot$$
$$\cdot\sum_{k=0}^{\infty}\left(-\frac{\mu r_n^2}{\lambda\rho}\right)^k\int_0^t\sin\left[\omega(t-\tau)\right]G_{\alpha,-(k+1),k}(-\lambda^{-1},\tau)\,\mathrm{d}\tau$$

and obtain that

$$u(R_1,t)=w_1\sin\beta_1 t$$

since $B(R_1r_n)=0$.

With aid of eqs. (5) and (10), we have that:

$$\bar{\tau}=\frac{\mu}{1+\lambda s^{\alpha}}\left(\frac{\partial}{\partial r}-\frac{1}{r}\right)\bar{u}=\frac{\mu}{1+\lambda s^{\alpha}}\frac{2R_1R_2}{(R_2^2-R_1^2)r^2}\left(\frac{R_1\omega_2\beta_2}{s^2+\beta_2^2}-\frac{R_2\omega_1\beta_2}{s^2+\beta_1^2}\right)+$$
$$+\mu\pi K\sum_{n=1}^{\infty}\frac{J_1(R_1r_n)\left[\dfrac{2}{r}B_1(rr_n)-r_n\bar{B}(rr_n)\right]}{J_1^2(R_1r_n)-J_1^2(R_2r_n)}-\frac{\mu\pi^2}{2}\cdot$$
$$\cdot\sum_{n=1}^{\infty}\frac{r_nJ_1^2(R_1r_n)\left[\dfrac{2}{r}B_1(rr_n)-r_n\bar{B}(rr_n)\right]}{J_1^2(R_1r_n)-J_1^2(R_2r_n)}\frac{\omega\rho P_0[\bar{B}(R_1r_n)-\bar{B}(R_2r_n)]}{(s^2+\omega^2)(\rho s+\rho\lambda s^{\alpha+1}+\mu r_n^2)}$$

where

$$K=\frac{\omega_2\beta_2J_1(R_1r_n)\rho s}{(s^2+\beta_2^2)(\rho s+\rho\lambda s^{\alpha+1}+\mu r_n^2)}-\frac{\omega_1\beta_1J_1(R_2r_n)\rho s}{(s^2+\beta_1^2)(\rho s+\rho\lambda s^{\alpha+1}+\mu r_n^2)}$$

In a similar way, we have that:

$$\tau(r,t)=\frac{2\mu R_1R_2}{\lambda(R_2^2-R_1^2)r^2}\int_0^t\left\{R_1\omega_2\sin\left[\beta_2(t-\tau)\right]-R_2\omega_1\sin\left[\beta_1(t-\tau)\right]\right\}R_{\alpha,0}(-\lambda^{-1},0,\tau)\,\mathrm{d}\tau-$$
$$-\frac{\mu\pi}{\lambda}\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\frac{J_1(R_1r_n)\left[\dfrac{2}{r}B_1(rr_n)-r_n\bar{B}(rr_n)\right]}{J_1^2(R_1r_n)-J_1^2(R_2r_n)}\left(-\frac{\mu r_n^2}{\lambda\rho}\right)^k\cdot$$
$$\cdot\int_0^t\left\{\omega_1J_1(R_2r_n)\sin\left[\beta_1(t-\tau)\right]-\omega_2J_1(R_1r_n)\sin\left[\beta_2(t-\tau)\right]\right\}G_{\alpha,-k,k+1}(-\lambda^{-1},\tau)\,\mathrm{d}\tau-$$

$$-\frac{\mu\pi^2}{2\lambda}\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\frac{r_n J_1^2(R_1r_n)\left[\frac{2}{r}B_1(rr_n)-r_n\bar{B}(rr_n)\right]}{J_1^2(R_1r_n)-J_1^2(R_2r_n)}P_0[\bar{B}(R_1r_n)-\bar{B}(R_2r_n)]\cdot$$
$$\cdot\left(-\frac{\mu r_n^2}{\lambda\rho}\right)^k\int_0^t\sin[\omega(t-\tau)]G_{\alpha,-(k+1),k+1}(-\lambda^{-1},\tau)\,\mathrm{d}\tau$$

Limiting case

Making $\alpha\to 1$ into eq. (2), we obtain:

$$u=\frac{R_1(R_2^2-r^2)}{(R_2^2-R_1^2)r}\omega_1\sin\beta_1 t+\frac{R_2(r^2-R_1^2)}{(R_2^2-R_1^2)r}\omega_2\sin\beta_2 t-$$
$$-\pi\sum_{n=1}^{\infty}\frac{J_1^2(R_1r_n)B_1(rr_n)\omega_2}{J_1^2(R_1r_n)-J_1^2(R_2r_n)}\sum_{k=0}^{\infty}\left(-\frac{\mu r_n^2}{\lambda\rho}\right)^k\int_0^t\sin\beta_2(t-\tau)G_{1,-k,k}(-\lambda^{-1},\tau)\,\mathrm{d}\tau+$$
$$+\pi\sum_{n=1}^{\infty}\frac{J_1^2(R_1r_n)B_1(rr_n)\omega_1}{J_1^2(R_1r_n)-J_1^2(R_2r_n)}\sum_{k=0}^{\infty}\left(-\frac{\mu r_n^2}{\lambda\rho}\right)^k\frac{J_1(R_2r_n)}{J_1(R_1r_n)}\int_0^t\sin\beta_1(t-\tau)G_{1,-(k+1),k}(-\lambda^{-1},\tau)\,\mathrm{d}\tau+$$
$$+\frac{\pi^2}{2}\sum_{n=1}^{\infty}\frac{r_nJ_1^2(R_1r_n)B_1(rr_n)P_0[\bar{B}(R_1r_n)-\bar{B}(R_2r_n)]}{J_1^2(R_1r_n)-J_1^2(R_2r_n)}\sum_{k=0}^{\infty}\left(-\frac{\mu r_n^2}{\lambda\rho}\right)^k\int_0^t\sin\omega(t-\tau)G_{1,-(k+1),k}(-\lambda^{-1},\tau)\,\mathrm{d}\tau$$

and

$$\tau(r,t)=\frac{2\mu R_1R_2}{\lambda(R_2^2-R_1^2)r^2}\int_0^t[R_1\omega_2\sin\beta_2(t-\tau)-R_2\omega_1\sin\beta_1(t-\tau)]R_{1,0}(-\lambda^{-1},0,\tau)\,\mathrm{d}\tau-$$
$$-\frac{\mu\pi}{\lambda}\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\frac{J_1(R_1r_n)\left[\frac{2}{r}B_1(rr_n)-r_n\bar{B}(rr_n)\right]}{J_1^2(R_1r_n)-J_1^2(R_2r_n)}\left(-\frac{\mu r_n^2}{\lambda\rho}\right)^k\cdot$$
$$\cdot\int_0^t[\omega_1J_1(R_2r_n)\sin\beta_1(t-\tau)-\omega_2J_1(R_1r_n)\sin\beta_2(t-\tau)]G_{1,-k,k+1}(-\lambda^{-1},\tau)\,\mathrm{d}\tau-$$
$$-\frac{\mu\pi^2}{2\lambda}\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\frac{r_nJ_1^2(R_1r_n)\left[\frac{2}{r}B_1(rr_n)-r_n\bar{B}(rr_n)\right]}{J_1^2(R_1r_n)-J_1^2(R_2r_n)}P_0[\bar{B}(R_1r_n)-\bar{B}(R_2r_n)]\cdot$$
$$\cdot\left(-\frac{\mu r_n^2}{\lambda\rho}\right)^k\int_0^t\sin\omega(t-\tau)G_{1,-(k+1),k+1}(-\lambda^{-1},\tau)\,\mathrm{d}\tau$$

Conclusion

In this work, we have obtained the analytic solutions of the velocity and the shear stress presented by integral and series form in terms of the generalized G and R functions for a generalized Maxwell fluid between coaxial cylinders. Moreover, for $\alpha\to 1$, it is found that the analytic velocity solutions correspond to the standard Maxwell fluid in the same motion.

Acknowledgment

The research is supported by the National Natural Science Foundation of China (No. 12001064), the Natural Science Foundation of Hunan Province (No. 2019JJ50659), the Hunan Provincial Education Department Project (No.20B006), the Double First-class International Cooperation Expansion Project (No. 2019IC39), and the Hunan Provincial Key Laboratory of Mathematical Modeling and Analysis in Engineering (No. 2017TP1017, Changsha University of Science and Technology).

Nomenclature

$i(x,t)$ – current on the cable, [A]
P – pressure, [Pa]
t – time, [s]
$u(r,t)$ – velocity, [ms^{-1}]

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Paper submitted: August 3, 2019
Paper revised: January 15, 2020
Paper accepted: January 20, 2020