APPROXIMATE ANALYTICAL SOLUTION FOR MODIFIED KORTEWEG-de VRIES EQUATION WITH LOCAL FRACTIONAL DERIVATIVE VIA NEW ITERATIVE METHOD

by

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In this paper, we obtain the approximate analytical solution of variable coefficients modified Korteweg-de Vries equation with local fractional derivative by using new iterative method.

Key words: *modified Korteweg-de Vries equation, variable coefficients, local fractional derivative, new iterative method*

Introduction

In this paper, we consider the following variable coefficients modified Korteweg-de Vries equation (MKdV) with local fractional derivative in fractal media:

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}}+N(x,t,u)\frac{\partial^{\alpha} u(x,t)}{\partial x^{\alpha}}+c(x,t)\frac{\partial^{3\alpha} u(x,t)}{\partial x^{3\alpha}}=0, \quad 0<\alpha\leq 1 \tag{1}$$

subject to the initial condition:

$$u(x,0)=\varphi(x^{\alpha}) \tag{2}$$

where $N(x,t,u)=a(x,t)u^2+b(x,t)u$, $a(x,t)$, $b(x,t)$, and $c(x,t)$ are given functions:

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}}=\lim_{t_1\to t}\frac{\Delta^{\alpha}[u(x,t_1)-u(x,t)]}{(t_1-t)^{\alpha}} \tag{3}$$

and

$$\frac{\partial^{\alpha} u(x,t)}{\partial x^{\alpha}}=\lim_{x_1\to x}\frac{\Delta^{\alpha}[u(x_1,t)-u(x,t)]}{(x_1-x)^{\alpha}} \tag{4}$$

with

$$\Delta^{\alpha}[u(x,t_1)-u(x,t)]\cong\Gamma(\alpha+1)[u(x,t_1)-u(x,t)]$$

and

$$\Delta^{\alpha}[u(x_1,t)-u(x,t)]\cong\Gamma(\alpha+1)[u(x_1,t)-u(x,t)]$$

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The differential equations with local fractional derivative have recently proved to be suitable tools for modeling many non-differentiable phenomena. Many physics problems in fractal media lead to non-linear models involving local fractional derivatives [1-6]. For example, eq. (1) arises in the mathematical models of the various non-linear phenomena, such as heat conduction in fractal porous media [7-15]. The problem of eq. (1) is often difficult or impossible to solve analytically it. In recent years, some analytical methods for solving local fractional differential equations have been studied by many authors [16-19]. The new iterative method (NIM) was proposed first by Daftardar-Gejji and Jafari in [20-22] and has proven useful for searching approximate analytical solutions of the non-linear local fractional differential equation.

To solve the problem (1)-(2) analytically by NIM, we recall the definition of local fractional integral operator.

The local fractional integral operator is the inverse operation of the local fractional derivative [23]:

$$ {}_aI_b^{(\alpha)} f(x) = \frac{1}{\Gamma(1+\alpha)} \int_a^b f(x)(\mathrm{d}x)^\alpha = \lim_{\Delta t \to 0} \sum_{j=0}^{N-1} f(t_j)(\Delta t_j)^\alpha \tag{5} $$

where a partition of the interval $[a,b]$ is denoted as (t_j, t_{j+1}), $t_0 = a$ and $t_N = b$ with $\Delta t_j = t_{j+1} - t_j$ and $\Delta t = \max\{\Delta t_0, \Delta t_1, \cdots\cdots, \Delta t_N\}$.

The following formulas of local fractional derivative hold true:

$$ \frac{\mathrm{d}^\alpha (x^{n\alpha})}{\mathrm{d}x^\alpha} = \frac{\Gamma(1+n\alpha)x^{(n-1)\alpha}}{\Gamma[1+(n-1)\alpha]} \tag{6} $$

$$ \frac{1}{\Gamma(1+\alpha)} \int_a^b x^{n\alpha} (\mathrm{d}x)^\alpha = \frac{\Gamma(1+n\alpha)[b^{(n+1)\alpha} - a^{(n+1)\alpha}]}{\Gamma[1+(n+1)\alpha]} \tag{7} $$

$$ \frac{1}{\Gamma(1+\alpha)} \int_a^b E_\alpha(x^\alpha)(\mathrm{d}x)^\alpha = E_\alpha(b^\alpha) - E_\alpha(a^\alpha) \tag{8} $$

where

$$ E_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{x^{(n\alpha)}}{\Gamma(1+n\alpha)}, 0 < \alpha \le 1 $$

The new iterative method

To illustrate the NIM [20-22], we consider the following general function equation:

$$ u = P(u) + \Omega(u) + f \tag{9} $$

where P is a linear operator, Ω is a non-linear operator from a Banach space $\mathrm{M} \to \mathrm{M}$, and f is a known function.

Suppose that the solution u of eq. (9) having the following form:

$$ u = \sum_{l=0}^{\infty} u_l \tag{10} $$

Then the non-linear operator Ω can be decomposed as:

$$\Omega\left(\sum_{i=0}^{\infty}u_i\right)=\Omega(u_0)+\sum_{i=1}^{\infty}\left[\Omega\left(\sum_{j=0}^{i}u_j\right)-\Omega\left(\sum_{j=0}^{i-1}u_j\right)\right] \tag{11}$$

From eqs. (10) and (11), eq. (9) is equivalent to:

$$\sum_{i=0}^{\infty}u_i=f+\sum_{i=0}^{\infty}P(u_i)+\Omega(u_0)+\sum_{i=1}^{\infty}\left[\Omega\left(\sum_{j=0}^{i}u_j\right)-\Omega\left(\sum_{j=0}^{i-1}u_j\right)\right] \tag{12}$$

Define the recurrence relation:

$$u_0=f(x),\quad u_1=P(u_0)+G_0\quad\text{and}\quad u_l=P(u_l)+G_l,\quad l=1,2,\cdots \tag{13}$$

where

$$G_0=\Omega(u_0) \tag{14}$$

$$G_l=\Omega\left(\sum_{i=0}^{l}u_i\right)-\Omega\left(\sum_{i=0}^{l-1}u_i\right),\quad l=1,2,\cdots \tag{15}$$

Then we obtain the k-term approximate solution of eq. (9) as follows:

$$u=u_0+u_1+,\cdots,+u_{k-1} \tag{16}$$

Solution of the problem (1)-(2)

Now, we derive the main algorithms of the NIM for solving the variable coefficients MKdV equation with local fractional derivative.

Consider the following variable coefficients modified MKdV equation with local fractional derivative:

$$\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}}+N(x,t,u)\frac{\partial^{\alpha}u(x,t)}{\partial x^{\alpha}}+c(x,t)\frac{\partial^{3\alpha}u(x,t)}{\partial x^{3\alpha}}=0 \tag{17}$$

Subject to the initial condition:

$$u(x,0)=\varphi(x^{\alpha})$$

To apply NIM, we rewrite the eq. (17) as:

$$u(x,t)=u(x,0)+{}_0I_t^{(\alpha)}\left[-N(x,t,u)\frac{\partial^{\alpha}u}{\partial x^{\alpha}}-c(x,t)\frac{\partial^{3\alpha}u}{\partial x^{3\alpha}}\right] \tag{18}$$

Suppose that the solution of eq. (15) takes the form:

$$u(x,t)=\sum_{k=0}^{\infty}u_k(x,t) \tag{19}$$

and the non-linear term in eq. (17) is decomposed:

$$N(x,t,u)\frac{\partial^{\alpha}u}{\partial x^{\alpha}}=\sum_{k=0}^{\infty}D_k \tag{20}$$

where

$$D_0 = N(x,t,u_0)\frac{\partial^\alpha u_0}{\partial x^\alpha}$$

and

$$D_k = N\left(x,t,\sum_{i=0}^{k} u_i\right)\frac{\partial^\alpha}{\partial x^\alpha}\left(\sum_{i=0}^{k} u_i\right) - N\left(x,t,\sum_{i=0}^{k-1} u_i\right)\frac{\partial^\alpha}{\partial x^\alpha}\left(\sum_{i=0}^{k-1} u_i\right), \quad k=1,2,\cdots$$

Thus, according to eq. (13), approximate solution can be obtained:

$$\begin{gathered} u_0(x,t) = u(x,0) \\ u_1(x,t) = -{}_0I_t^{(\alpha)}\left[D_0 + c(x,t)\frac{\partial^{3\alpha}u_0}{\partial x^{3\alpha}}\right] \\ \cdots \\ u_{k+1}(x,t) = -{}_0I_t^{\alpha}\left[D_k + c(x,t)\frac{\partial^{3\alpha}u}{\partial x^{3\alpha}}\right], \quad k=1,2,\ \cdots \end{gathered} \tag{21}$$

To illustrate the procedure and to test its effectiveness, we consider eq. (1) in the form:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + \left(\frac{48t^\alpha}{23}u^2 - \frac{96t^\alpha}{23}u\right)\frac{\partial^\alpha u(x,t)}{\partial x^\alpha} + \frac{2t^\alpha}{23}\frac{\partial^{3\alpha}u(x,t)}{\partial x^{3\alpha}} = 0 \tag{22}$$

with the initial condition:

$$u(x,0) = 1 + \frac{1}{2\cosh_\alpha(x^\alpha)} \tag{23}$$

By the aforementioned algorithms, we obtain:

$$\begin{gathered} u_0(x,t) = \frac{2\cosh_\alpha(x^\alpha)+1}{2\cosh_\alpha(x^\alpha)} \\ u_1(x,t) = -\frac{\sinh_\alpha(x^\alpha)}{\cosh_\alpha^2(x^\alpha)}\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} \\ u_2(x,t) = \frac{3[\cosh_\alpha(2x^\alpha)-3]}{\cosh_\alpha^3(x^\alpha)}\frac{t^{4\alpha}}{\Gamma(1+4\alpha)} \\ u_3(x,t) = -\frac{30\sinh_\alpha(x^\alpha)[\cosh_\alpha(2x^\alpha)-11]}{\cosh_\alpha^4(x^\alpha)}\frac{t^{6\alpha}}{\Gamma(1+6\alpha)} \end{gathered}$$

Thus, the $4-term$ approximate solution of (22) is given by:

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t)$$

Conclusion

This work presented the application of NIM to the variable coefficients MKDV equation with local fractional derivative. The NIM gives approximate analytical solutions of the equation in fractal media. Our example shows that the NIM is an efficient and reliable algorithm.

Nomenclature

t – time, [s]
$u(x, t)$ – wave speed, [ms^{-1}]
x – space co-ordinates, [m]

Greek symbol

α – fractal dimension, [-]

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