POLYNOMIAL CHARACTERISTIC METHOD
An Easy Approach to Lie Symmetry

by

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Along the approach to Lie symmetry, it is always needed to solve an over-determined system, which is difficult and complex if not impossible. Here we suggest a new polynomial characteristic method combined with Lie algorithm to complete symmetry classification for a class of perturbed equations. A differential polynomial characteristic set algorithm is proposed to decompose the determining equations into a series of equations easy to be solved.

Key words: *Lie algorithm, differential polynomial characteristic set algorithm, symmetry classification*

Introduction

At present, symmetry of a PDE is widely used in thermal science, mechanics, mathematics and physics, because it can reveal important information on its solution structure [1-3], its conservation laws [4-7], and variational formulations [8-19]. However, it requires to solve an over-determined linear PDE, the solution process is extremely difficult and time-consuming. To solve the problem, this paper proposes a polynomial characteristic method, and a differential polynomial characteristic set algorithm is suggested to decompose the determining equations into a series of equations easy to be solved.

Lie algorithm

Consider the following PDE:

$$G(x,u,\partial u,\cdots,\partial^J u)=0 \tag{1}$$

Definition 1. The following transformation:

$$\begin{cases} x^* = A(x,u,\varepsilon) \\ u^* = B(x,u,\varepsilon) \end{cases} \tag{2}$$

is a point symmetry admitted by PDE (1), if its *k*-th extension leaves eq. (1) invariant.

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Theorem 1. Let:

$$X = C_i(x,u)\frac{\partial}{\partial x_i} + D(x,u)\frac{\partial}{\partial u} \tag{3}$$

be the infinitesimal generator of eq. (2). Let the following expression:

$$X^{(J)} = C_i(x,u)\frac{\partial}{\partial x_i} + D(x,u)\frac{\partial}{\partial u} + D_i^{(1)}(x,u,\partial u)\frac{\partial}{\partial u_i} + \cdots + D_{i_1 i_2 \cdots i_J}^{(J)}(x,u,\partial u,\cdots,\partial^J u)\frac{\partial}{\partial u_{i_1 i_2 \cdots i_J}} \tag{4}$$

be the J-th-extended infinitesimal generator of eq. (3), then eq. (2) is a point symmetry admitted by eq. (1), if:

$$X^{(J)}G(x,u,\partial u,\cdots,\partial^J u) = 0, \quad \text{when} \quad G(x,u,\partial u,\cdots,\partial^J u) = 0 \tag{5}$$

The following PDE:

$$\Delta = \Delta(x,u,C_i,D) \tag{6}$$

is called as the determining equation (DTE), which is obtained from eq. (5). Consequently, the problem of computing symmetry of eq. (2) is reduced to finding the corresponding eq. (3), which is equivalent to solve eq. (6) for C_i and D. Generally, eq. (6) is hard to solve, in this article, a differential polynomial characteristic set algorithm is suggested to solve this problem.

Differential polynomial characteristic set algorithm

Suppose *DTE* and initials and separate products for differential polynomials (*IP*) are differential polynomial systems (DPS), we frequently use the following notations [20]:

$$Zero(DTE) = \text{all solutions of } DTE = 0$$

$$Zero\frac{DTE}{IP} = \text{all solutions of } DTE = 0 \text{ such that } IP \neq 0$$

$$Zero(DTE, IP) = \text{all solutions of } DTE = 0 \text{ such that } IP = 0$$

where *IP* denotes the initials and separate products for differential polynomials, called *IP* products;

$$\mathrm{P}remd\frac{DTE}{DCS} = \text{the set of pseudo-remainders of } DTE \text{ w.r.t chain } DCS$$

Theorem 2. Let DPS be a finite differential polynomials system, then the differential polynomial characteristic set algorithm yields a finite number characteristic set DCS_k of DPS with initials and separate products IP_k such that the following zero decomposition:

$$Zero(DPS) = \sum_k Zero\frac{DCS_k}{IP_k}$$

is held.

The algorithm to determine characteristic set DCS listed below.

Input. A differential polynomial system.
Output. A differential characteristic set *DCS*.
Start. Let $i = 0$;
Step 1. Select a basic set DBS_i from DPS_i;
Step 2. Compute all the nonzero coherent dps of DBS_i, and put them in set from IT'_i;
Step 3. $\forall IP$ (non-zero coherent differential polynomial) $\in IT_i^{'}$, compute $\mathrm{P}remd(IP/DBS_i)\setminus\{0\}$, and put the results in set IT_i;
Step 4. Compute $R_i = \mathrm{P}remd((DPS_i \setminus DBS_i)/DBS_i)\setminus\{0\}$, let $RJ_i = IT_i \cup R_i$;
Step 5. If $RJ_i = \phi$ (Empty), then $DCS = DBS_i$, and stop; else $i = i+1$, and $DPS_i = DPS_{i-1} \cup RJ_{i-1}$, go to *Step 1*.

Symmetry classification based on characteristic set algorithm

Consider the following non-linear wave equations [21]:

$$u_{tt} + \varepsilon u_t = [G(u)u_x]_x \tag{7}$$

suppose the infinitesimal generator admitted by eq. (7) is:

$$X = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \eta(x,t,u)\frac{\partial}{\partial u} \tag{8}$$

from eq. (5), the determining equations for *X* are:

$$DTE = \begin{cases} \xi_t = 0, \quad \xi_u = 0, \quad \tau_u = 0, \quad \tau_x = 0, \quad \eta_{uu} = 0 \\ \varepsilon\eta_t + \eta_{tt} - G(u)\eta_{xx} = 0 \\ 2\eta_{tu} + \varepsilon\tau_t - \tau_{tt} = 0 \\ 2G(u)\xi_x - 2G(u)\tau_t - \eta G'(u) = 0 \\ G(u)\xi_{xx} - 2G(u)\eta_{xu} - 2\eta_x G'(u) = 0 \\ 2\xi_x G'(u) - \eta_u G'(u) - 2\tau_t G'(u) - \eta G''(u) = 0 \end{cases} \tag{9}$$

taking left hand of each equation in *DTE*, we have:

$$DPS = \begin{cases} \xi_t, \quad \xi_u, \quad \tau_u, \quad \tau_x, \quad \eta_{uu} \\ \varepsilon\eta_t + \eta_{tt} - G(u)\eta_{xx} \\ 2\eta_{tu} + \varepsilon\tau_t - \tau_{tt} \\ 2G(u)\xi_x - 2G(u)\tau_t - \eta G'(u) \\ G(u)\xi_{xx} - 2G(u)\eta_{xu} - 2G_x f'(u) \\ 2\xi_x G'(u) - \eta_u G'(u) - 2\tau_t G'(u) - \eta G''(u) \end{cases} \tag{10}$$

Step 1. Compute $Zero(DPS)$

According to the order $x \prec t \prec u \prec \xi \prec \tau \prec \eta$, using differential polynomial characteristic set algorithm, we obtain characteristic set:

$$DCS_1 = \{\xi_x, \xi_t, \xi_u, \eta, \tau_x, \tau_t, \tau_u\}$$

with *IP* products $I_1 \neq 0$, and:

$$I_1 = G'(u)^2 G''(u) - 2G(u)G''(u)^2 + G(u)G'(u)G'''(u)$$

then, we get:

$$Zero(DPS) = Zero\frac{DCS_1}{I_1} + Zero(\{DPS, I_1\})$$

in which:

$$Zero\frac{DCS_1}{I_1} = \{\xi = c_1, \eta = 0, \tau = c_2\}$$

where c_1 and c_2 are arbitrary constants.

Step 2. $I_1 = 0$ compute $Zero(\{DPS, I_1\})$

We obtain characteristic set:

$$DCS_2 = \begin{cases} \xi_u, \xi_t, \eta_t, \eta_x, \tau_u, \tau_t, \tau_x \\ 2G(u)\xi_x - \eta G'(u) \\ G(u)\eta_u G'(u) - \eta G'(u)^2 + G(u)\eta G''(u) \end{cases} \tag{11}$$

with *IP* products $I_2 * I_3 \neq 0$, and:

$$I_2 = 7G'(u)^2 - 4G(u)G''(u)$$

$$I_3 = -3G'(u)^2 + 2G(u)G''(u)$$

then, we get

$$Zero(\{DPS, I_1\}) = ZeroDCS_2 / I_2 * I_3 + Zero(\{DPS, I_1, I_2\}) + Zero(\{DPS, I_1, I_3\})$$

in which:

$$ZeroDCS_2 / I_2 * I_3 = \begin{cases} \xi = \dfrac{c_1 x}{2} + c_2 \\ \eta = u - c_2 \\ \tau = c_4 \end{cases} \tag{12}$$

with the corresponding function $G(u) = (c_2 - u)^{c_1} c_3$, , and $c_1, \cdots, c_4$ are arbitrary constants.

Step 3. $I_2 = 0$ compute $Zero(\{DPS, I_1, I_2\})$

We obtain characteristic set:

$$DCS_3 = \begin{cases} \xi_u, \xi_t, \eta_t, \eta_{xx}, \tau_u, \tau_t, \tau_x \\ 2G(u)\xi_x - \eta G'(u) \\ 4G(u)\eta_u + 3\eta G'(u) \end{cases} \tag{14}$$

with *IP* products $I_4 \neq 0$, and:

$$I_4 = f'(u)$$

then, we get:

$$Zero(\{DPS, I_1, I_2\}) = Zero\frac{DCS_3}{I_4} + Zero(\{DPS, I_1, I_2, I_4\})$$

in which:

$$Zero\frac{DCS_3}{I_4} = \begin{cases} \xi = -\dfrac{c_1c_3x^2}{3} - \dfrac{2}{3}c_1c_4x + c_5 \\ \eta = (c_1u + c_2)(c_3x + c_4) \\ \tau = c_6 \end{cases} \tag{15}$$

with the corresponding function $G(u) = \left[-\dfrac{4}{3(c_1u + c_2)}\right]^{4/3}$ and $c_1,\ldots,c_6$ are arbitrary constants.

Step 4. $I_4 = 0$ compute $Zero(\{DPS, I_1, I_2, I_4\})$

$I_4 = 0$ *i. e.* $f(u) = \text{constant}$,we ignore this situation, then, we get:

$$Zero(\{DPS, I_1, I_2\}) = Zero\frac{DCS_3}{I_4}$$

Step 5. $I_3 = 0$ compute $Zero(\{DPS, I_1, I_3\})$

We obtain characteristic set:

$$DCS_4 = \begin{cases} \xi_u, \xi_t, \xi_{xx}, \eta_x, \tau_u, \tau_x \\ 2G(u)\xi_x - 2G(u)\tau_t - \eta G'(u) \\ 2G(u)\eta_u + \eta G'(u) \\ 2\varepsilon G(u)\xi_x - \varepsilon\eta G'(u) - \eta_t G'(u) \end{cases}$$

with *IP* products $I_4 \neq 0$, then, we get:

$$Zero(\{DPS, I_1, I_3\}) = Zero\frac{DCS_4}{I_4}$$

in which:

$$Zero\frac{DCS_4}{I_4} = \begin{cases} \xi = -c_1xF(t) - \dfrac{c_1xF'(t)}{\varepsilon} + c_2 \\ \eta = F(t)(c_1u + c_2) \\ \tau = -\dfrac{c_1F(t)}{\varepsilon} + c_2 \end{cases} \tag{15}$$

with the corresponding function:

$$G(u)=\frac{4}{(c_1u+c_2)^2}$$

and c_1, c_2 are arbitrary constants, $F(t)$ is a arbitrary function.

Consider *Step 1* to *Step 5,* we have:

$$Zero(DPS)=Zero\frac{DCS_1}{I_1}+ZeroDCS_2/I_2*I_3+Zero\frac{DCS_3}{I_4}+Zero\frac{DCS_4}{I_4}$$

Conclusion

In this paper, we use Lie algorithm to determine symmetry classification of a given PDE. Firstly, we obtain an over-determined linear PDE (determining equations), which is always difficult to solve, then, we use differential polynomial characteristic set algorithm to decompose the determining equations into a series of equations (characteristic set), which are easy to be solved. The example shows that this method is effective to get symmetry classification of PDE with parameters and is applicable to different cases non-linear PDE.

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