FRACTIONAL DERIVATIVE OF INVERSE MATRIX AND ITS APPLICATIONS TO SOLITON THEORY

by

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In this paper, a formula of the local fractional partial derivative of inverse matrix is presented and proved. With the help of the derived formula, two new non-linear PDE are derived including the local fractional non-isospectral self-dual Yang-Mills equation and the local fractional principal chiral field equation. It is shown that the formula of the local fractional partial derivative of inverse matrix can be used to derive some other local fractional non-linear PDE in soliton theory.

Key words: *local fractional non-isospectral self-dual Yang-Mills equation, local fractional derivative, inverse matrix, local fractional principal chiral field equation*

Introduction

With the development of fractional calculus and fractal calculus, more and more problems described by classical differential equations with integer orders have been generalized to fractional order models [1-5] which can span multiple scales [6], such as nanoscale, microscale, mesoscale, and macroscale, and two-scale thermodynamics [7, 8] can be adopted. Especially when the dimensions of flow systems approach to the molecular size, continuum models are no longer valid [7-9]. It is worth mentioning that human skin as highly ordered multilayer organ is particularly suitable for a fractional model [9]. Recently, there are some interesting studies on fractal calculus and its application. He [10] gave a geometrical explanation of the fractal calculus. Wang *et al.* [11, 12] employed fractal calculus to explain the biomechanism of polar bear hairs. Wang and Deng [13] established a fractal derivative-based tsunami model. Li, *et al*. [14] solved a paradox in the surface coverage model for an electrochemical arsenic sensor by a fractal modification. Wang *et al.* [15] studied the snow's thermal property by fractal calculus with great success. Wang *et al.* [16] established for the first time a variational formulation for the fractal differential equation. Wang and He [17] extended Wang's fractal variational principle to fractal time. Ain and He [7] shed a new light on applications of fractal theory to real problems.

In the field of non-linear mathematical physics, the derivative of inverse matrix plays important role, which has been used to derive some well-known non-linear PDE like the non-isospectral self-dual Yang-Mills equation and the principal chiral field equation [18]. In this paper, we shall present and verify a formula of the local fractional partial derivative of inverse

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matrix. Considering the applications of the presented formula, we shall employ it and the properties of local fractional derivative to derive the local fractional non-isospectral self-dual Yang-Mills equation and the local fractional principal chiral field equation. As far as we know, the derived local fractional equations have not been reported in literature.

Definitions and properties

Definition 1. The local fractional derivative using the fractal geometry $u(x)$ of order $\alpha(0 < \alpha \leq 1)$ is defined [1]:

$$u^{(\alpha)}(x) = \left.\frac{\mathrm{d}^{\alpha}u(x)}{\mathrm{d}x^{\alpha}}\right|_{x=x_0} = \lim_{x \to x_0} \frac{\Delta^{\alpha}[u(x)-u(x_0)]}{(x-x_0)^{\alpha}} \tag{1}$$

where $\Delta^{\alpha}[u(x)-u(x_0)] \cong \Gamma(1+\alpha)[u(x)-u(x_0)]$ with the Euler's Gamma function:

$$\Gamma(1+\alpha) = \int_0^{\infty} x^{\alpha-1}\mathrm{e}^{-\alpha}\mathrm{d}x \tag{2}$$

Definition 2. Let $u(x, t)$ be defined in the domain $\wp$ of the xt-plane, the local fractional partial derivative of order $\alpha(0 < \alpha \leq 1)$ with respective to x in the domain $\wp$ is defined [1]:

$$u_x^{(\alpha)}(x_0,t) = \left.\frac{\partial^{\alpha}u(x,t)}{\partial x^{\alpha}}\right|_{x=x_0} = \lim_{x \to x_0} \frac{\Delta^{\alpha}[u(x,t)-u(x_0,t)]}{(x-x_0)^{\alpha}} \tag{3}$$

where $\Delta^{\alpha}[u(x,t)-u(x_0,t)] \cong \Gamma(1+\alpha)[u(x,t)-u(x_0,t)]$.

Definition 3. The local fractional partial derivative operator of $u(x, t)$ of high order $(m + n)\alpha(0 < \alpha \leq 1)$ with respective to x and t in the domain $\wp$ is defined [1]:

$$u_{x^{m\alpha}t^{n\alpha}}^{[(m+n)\alpha]}(x,t) = \frac{\partial^{[(m+n)\alpha]}u(x,t)}{\partial x^{m\alpha}\partial t^{n\alpha}} \tag{4}$$

Definition 4. A function $u(x, t)$ is said to be the local fractional continuous at the point (x_0, t_0) if there is [1]:

$$\lim_{(x,t)\to(x_0,t_0)} u(x,t) = u(x_0,t_0) \tag{5}$$

Definition 5. Let $a_{ij}(x, t)$ $(i, j = 1, 2, \ldots, n)$ be fractional differentiable functions of x, then the n-order matrix $\mathrm{A}(x, t) = [a_{ij}(x, t)]_{n\times n}$ is called local fractional differentiable with respect to x, note as $A_x^{(\alpha)}(x,t) = [a_{ij,x}^{(\alpha)}(x,t)]_{n\times n}$.

Property 1. Suppose c, λ, and μ are arbitrary constants, k is a positive integer, then we have:

$$\frac{\mathrm{d}^{\alpha}(x+c)^{k\alpha}}{\mathrm{d}x^{\alpha}\Gamma(1+k\alpha)} = \frac{(x+c)^{(k-1)\alpha}}{\Gamma[1+(k-1)\alpha]} \tag{6}$$

$$[\lambda f(x,t)+\mu g(x,t)]_x^{(\alpha)} = \lambda f_x^{(\alpha)}(x,t)+\mu g_x^{(\alpha)}(x,t) \tag{7}$$

$$[f(x,t)g(x,t)]_x^{(\alpha)} = f_x^{(\alpha)}(x,t)g(x,t) + f(x,t)g_x^{(\alpha)}(x,t) \tag{8}$$

$$\left[\frac{f(x,t)}{g(x,t)}\right]_x^{(\alpha)} = \frac{f_x^{(\alpha)}(x,t)g(x,t) - f(x,t)g_x^{(\alpha)}(x,t)}{g^2(x,t)} \tag{9}$$

Proof. We here give only the proof of eq. (4), the rest of proofs can be referred to [1]. When $k = 1$, from *Definition 1* we have:

$$\frac{\mathrm{d}^\alpha (x+c)^\alpha}{\mathrm{d}x^\alpha \Gamma(1+\alpha)} = \lim_{\Delta x\to 0} \frac{[(x+c)+\Delta x]^\alpha - (x+c)^\alpha}{(\Delta x)^\alpha} = \lim_{\Delta x\to 0} \frac{[(x+c)+\Delta x]^\alpha - (x+c)^\alpha}{(\Delta x)^\alpha} = 1 \tag{10}$$

For any $k > 1$, we always have:

$$\frac{\mathrm{d}^\alpha (x+c)^{k\alpha}}{\mathrm{d}x^\alpha \Gamma(1+k\alpha)} = \lim_{\Delta x\to 0} \frac{\Gamma(1+\alpha)\{[(x+c)+\Delta x]^{k\alpha} - (x+c)^{k\alpha}\}}{\Gamma(1+k\alpha)(\Delta x)^\alpha} =$$

$$= \lim_{\Delta x\to 0} \frac{\Gamma(1+\alpha)\left\{(x+c)^{k\alpha} + \dfrac{\Gamma(1+k\alpha)}{\Gamma(1+\alpha)\Gamma[1+(k-1)\alpha]}(x+c)^{(k-1)\alpha}(\Delta x)^\alpha + \cdots - (x+c)^{k\alpha}\right\}}{\Gamma(1+k\alpha)(\Delta x)^\alpha} =$$

$$= \frac{(x+c)^{(k-1)\alpha}}{\Gamma[1+(k-1)\alpha]} \tag{11}$$

The proof of eq. (1) is therefore end.

Property 2. Suppose $a_{ij}(x, t)$ and $b_{ij}(x, t)$ are two local fractional differentiable matrixes of x for any $i, j = 1, 2, \ldots, n$¸ then the matrixes $\mathrm{A}(x, t) = [a_{ij}(x, t)]_{n\times n}$ and $\mathrm{B}(x, t) = [b_{ij}(x, t)]_{n\times n}$ are differentiable with respect to x, and:

$$[\mathrm{A}(x,t) + \mathrm{B}(x,t)]_x^{(\alpha)} = \mathrm{A}_x^{(\alpha)}(x,t) + \mathrm{B}_x^{(\alpha)}(x,t) \tag{12}$$

$$[\mathrm{A}(x,t)\mathrm{B}(x,t)]_x^{(\alpha)} = \mathrm{A}_x^{(\alpha)}(x,t)\mathrm{B}(x,t) + \mathrm{A}(x,t)\mathrm{B}_x^{(\alpha)}(x,t) \tag{13}$$

Proof. Since $a_{ij}(x, t)$ and $b_{ij}(x, t)$ are local fractional differentiable functions of x for $i, j = 1, 2, \ldots, n$¸ then the functions $a_{ij}(x, t) + b_{ij}(x, t)$ ($j = 1, 2, \ldots, n$) are all local fractional differentiable with respective to x [1], and hence the matrix $\mathrm{A}(x, t) + \mathrm{B}(x, t)$ are differentiable with respect to x. At the same time, we have:

$$[\mathrm{A}(x,t) + \mathrm{B}(x,t)]_x^{(\alpha)} = a_{ij,x}^{(\alpha)}(x,t) + b_{ij,x}^{(\alpha)}(x,t) = \mathrm{A}_x^{(\alpha)}(x,t) + \mathrm{B}_x^{(\alpha)}(x,t) \tag{14}$$

We can easily verify eq. (13) by employing eq. (8). Thus, the proof of *Property 2* is complete.

Local fractional derivative of inverse matrix

Theorem 1. (Local fractional Rolle's theorem [1]). Suppose that $f(x) \in C_\alpha[a, b]$, $f(x) \in D_\alpha(a, b)$, and $f(a) = f(b)$, here $C_\alpha[a, b]$ and $D_\alpha(a, b)$ are called local fractional derivative set on closed interval $[a, b]$ and open interval (a, b), respectively. Then, there exists a point $x_0 \in (a, b)$ and $\alpha \in (0, 1]$ such that $f^{(\alpha)}(x_0) = 0$.

Theorem 2. (Local fractional Lagrange's mean value theorem). Suppose that $f(x) \in C_\alpha[a, b]$ and $f(x) \in D_\alpha(a, b)$. Then, there exists a point $\xi \in (a, b)$ and $\alpha \in (0, 1]$ such that:

$$f(b) - f(a) = f^{(\alpha)}(\xi)\frac{(b-a)^\alpha}{\Gamma(1+\alpha)} \tag{15}$$

Proof. We introduce the local fractional differentiable function:

$$F(x) = f(x) - f(a) - [f(b) - f(a)]\frac{(x-a)^\alpha}{(b-a)^\alpha}, \quad \alpha \in (0,1] \tag{16}$$

It is easy to see that $F(x) \in C_\alpha[a,b]$, $F(x) \in D_\alpha(a,b)$, and $F(a) = F(b) = 0$. Applying *Theorem 1* to the function $F(x)$, we have:

$$F(b) - F(a) = f^{(\alpha)}(\xi) - [f(b) - f(a)]\frac{\Gamma(1+\alpha)}{(b-a)^\alpha} = 0, \quad \xi \in (a,b) \tag{17}$$

which can be rewritten as eq. (15). Thus, the proof of *Theorem 2* is over.

Theorem 3. If the functions $a_{ij,xt}^{(\alpha)}(x,t)$ and $a_{ij,tx}^{(\alpha)}(x,t)$ are all continuous at the point (x_0, t_0) for any $i, j = 1, 2, \ldots, n$, then n-order matrix $\mathrm{A}(x, t) = [a_{ij}(x, t)]_{n\times n}$ there is:

$$\mathrm{A}_{xt}^{(2\alpha)}(x_0,t_0)_{n\times n} = \mathrm{A}_{tx}^{(2\alpha)}(x_0,t_0)_{n\times n} \tag{18}$$

Proof. For any $i, j = 1, 2, \ldots, n$, letting:

$$F_{ij}(\Delta x, \Delta y) = a_{ij}(x_0 + \Delta x, y_0 + \Delta y) - a_{ij}(x_0 + \Delta x, y_0) - a_{ij}(x_0, y_0 + \Delta y) + a_{ij}(x_0, y_0) \tag{19}$$

$$\phi_{ij}(x) = a_{ij}(x, y_0 + \Delta y) - a_{ij}(x, y_0) \tag{20}$$

we have:

$$F_{ij}(\Delta x, \Delta y) = \phi_{ij}(x_0 + \Delta x) - \phi_{ij}(x_0) \tag{21}$$

Since $a_{ij,x}^{(\alpha)}(x,t)$ exists, the function $\phi_{ij}(x)$ is local fractional differentiable. Applying *Theorem 2* yields:

$$\phi_{ij}(x_0 + \Delta x) - \phi_{ij}(\Delta x) = \phi_{ij,x}^{(\alpha)}(x_0 + \theta_1 \Delta x)\frac{(\Delta x)^\alpha}{\Gamma(1+\alpha)}, \quad 0 < \theta_1 < 1 \tag{22}$$

which can be written:

$$\phi_{ij}(x_0 + \Delta x) - \phi_{ij}(\Delta x) = [a_{ij,x}^{(\alpha)}(x_0 + \theta_1 \Delta x, y_0 + \Delta y) - a_{ij,x}^{(\alpha)}(x_0 + \theta_1 \Delta x, y_0)]\frac{(\Delta x)^\alpha}{\Gamma(1+\alpha)} \tag{23}$$

In view of the existence of $a_{ij,xy}^{(2\alpha)}(x,t)$, we apply *Theorem 2* to the function $a_{ij,x}(x_0 + \theta_1 \Delta x, y)$ and then derive:

$$\phi_{ij}(x_0 + \Delta x) - \phi_{ij}(\Delta x) = a_{ij,xy}^{(2\alpha)}(x_0 + \theta_1 \Delta x, y_0 + \theta_2 \Delta y)\frac{(\Delta x)^\alpha}{\Gamma(1+\alpha)}\frac{(\Delta y)^\alpha}{\Gamma(1+\alpha)}, \quad 0 < \theta_1, \quad \theta_2 < 1 \tag{24}$$

which is namely:

$$F_{ij}(\Delta x, \Delta y) = a_{ij,xy}^{(2\alpha)}(x_0 + \theta_1 \Delta x, y_0 + \theta_2 \Delta y)\frac{(\Delta x)^\alpha}{\Gamma(1+\alpha)}\frac{(\Delta y)^\alpha}{\Gamma(1+\alpha)} \tag{25}$$

If setting:

$$\varphi_{ij}(x) = a_{ij}(x + \Delta x, y_0) - a_{ij}(x_0, y) \tag{26}$$

we have:

$$F_{ij}(\Delta x, \Delta y) = \varphi_{ij}(y_0 + \Delta y) - \varphi_{ij}(y_0) \tag{27}$$

Similarly, we derive:

$$F_{ij}(\Delta x, \Delta y) = a_{ij,yx}^{(2\alpha)}(x_0 + \theta_3 \Delta x, y_0 + \theta_4 \Delta y)\frac{(\Delta x)^\alpha}{\Gamma(1+\alpha)}\frac{(\Delta y)^\alpha}{\Gamma(1+\alpha)}, \quad 0 < \theta_3, \quad \theta_4 < 1 \tag{28}$$

When $\Delta x \neq 0$ and $\Delta y \neq 0$, eqs. (25) and (28) give:

$$a_{ij,xy}^{(2\alpha)}(x_0 + \theta_1 \Delta x, y_0 + \theta_2 \Delta y) = a_{ij,yx}^{(2\alpha)}(x_0 + \theta_3 \Delta x, y_0 + \theta_4 \Delta y), \; 0 < \theta_1, \quad \theta_2, \theta_3, \theta_4 < 1 \tag{29}$$

Using the continuity of the functions $a_{ij,xt}^{(\alpha)}(x,t)$ and $a_{ij,tx}^{(\alpha)}(x,t)$ at the point (x_0, t_0), from eq. (29) we have:

$$\lim_{(\Delta x, \Delta y)\to(0,0)} a_{ij,xy}^{(2\alpha)}(x_0 + \theta_1 \Delta x, y_0 + \theta_2 \Delta y) = a_{ij,xy}^{(2\alpha)}(x_0, y_0) \tag{30}$$

$$\lim_{(\Delta x, \Delta y)\to(0,0)} a \lim_{(\Delta x, \Delta y)\to(0,0)} a_{ij,yx}^{(2\alpha)}(x_0 + \theta_3 \Delta x, y_0 + \theta_4 \Delta y) = a_{ij,yx}^{(2\alpha)}(x_0, y_0) \tag{31}$$

and hence arrive at $a_{ij,xy}^{(2\alpha)}(x_0, y_0) = a_{ij,yx}^{(2\alpha)}(x_0, y_0)$, by which eq. (16) is verified. The proof of *Theorem 3* is finished.

Theorem 4. Suppose that the inverse matrix $\mathrm{A}^{-1}(x,t)$ of the matrix A(x, t) = = $[a_{ij}(x, t)]_{n\times n}$ is local fractional differentiable with respective x, then there is:

$$\mathrm{A}_x^{(\alpha)}(x,t) = -\mathrm{A}^{-1}(x,t)\,\mathrm{A}_x^{(\alpha)}(x,t)\,\mathrm{A}^{-1}(x,t) \tag{32}$$

Proof. We let B(x, t) = $\mathrm{A}^{-1}(x, t)$, then A(x, t)B(x, t) = E, here E is the n-order unit matrix. With the help of *Theorem 2*, we have:

$$[\mathrm{A}(x,t)\,\mathrm{B}(x,t)]_x^{(\alpha)} = \mathrm{A}_x^{(\alpha)}(x,t)\,\mathrm{B}(x,t) + \mathrm{A}(x,t)\,\mathrm{B}_x^{(\alpha)}(x,t) = 0 \tag{33}$$

From eq. (31), we have:

$$\mathrm{B}_x^{(\alpha)}(x,t) = -\mathrm{A}^{-1}(x,t)\,\mathrm{A}_x^{(\alpha)}(x,t)\,\mathrm{B}(x,t) = -\mathrm{A}^{-1}(x,t)\,\mathrm{A}_x^{(\alpha)}(x,t)\,\mathrm{A}^{-1}(x,t) \tag{34}$$

which is namely eq. (32). We, therefore, finish the proof of *Theorem 4.*

Applications in soliton theory

Example 1. Derivation of local fractional non-isospectral self-dual Yang-Mills equation:

$$[D_{\bar{\mu}}^{(\alpha)}, D_{\bar{\vartheta}}^{(\alpha)}]=0, \quad [D_{\mu}^{(\alpha)}, D_{\bar{\mu}}^{(\alpha)}]+[D_{\vartheta}^{(\alpha)}, D_{\bar{\vartheta}}^{(\alpha)}]=0, \quad [D_{\mu}^{(\alpha)}, D_{\vartheta}^{(\alpha)}]=0 \tag{35}$$

and its reduction. Here:

$$D_{\rho}^{(\alpha)}=\frac{\partial^{(\alpha)}}{\partial \rho^{\alpha}}+A_{\rho}, \quad \rho=\mu, \bar{\mu}, \vartheta, \bar{\vartheta} \tag{36}$$

where $\bar{\mu}$ and $\bar{\vartheta}$ are complex conjugations of μ and ϑ, respectively, and the canonical potential $A_{\rho} \in gl(n)$ is local fractional differentiable with respect to $\mu, \vartheta, \bar{\mu},$ and $\bar{\vartheta}$.

We consider the following local fractional linear non-isospectral problem:

$$(D_{\mu}^{(\alpha)}-\lambda D_{\bar{\vartheta}}^{(\alpha)})\psi=0 \tag{37}$$

$$(D_{\vartheta}^{(\alpha)}-\lambda D_{\bar{\mu}}^{(\alpha)})\psi=0 \tag{38}$$

where the non-isospectral parameter, λ, is continuous function of $\mu, \vartheta, \bar{\mu},$ and $\bar{\vartheta}$ and satisfies the local fractional non-linear PDE of:

$$\frac{\partial^{(\alpha)}\lambda}{\partial \mu^{\alpha}}-\lambda\frac{\partial^{(\alpha)}\lambda}{\partial \bar{\vartheta}^{\alpha}}=0, \quad \frac{\partial^{(\alpha)}\lambda}{\partial \vartheta^{\alpha}}-\lambda\frac{\partial^{(\alpha)}\lambda}{\partial \bar{\mu}^{\alpha}}=0 \tag{39}$$

From the compatibility condition of eqs. (37) and (38), we have:

$$(D_{\mu}^{(\alpha)}-\lambda D_{\bar{\vartheta}}^{(\alpha)})(D_{\vartheta}^{(\alpha)}-\lambda D_{\bar{\mu}}^{(\alpha)})\psi=(D_{\vartheta}^{(\alpha)}-\lambda D_{\bar{\mu}}^{(\alpha)})(D_{\mu}^{(\alpha)}-\lambda D_{\bar{\vartheta}}^{(\alpha)})\psi \tag{40}$$

the arbitrariness of which gives eqs. (35).

Setting:

$$A_{\mu}=H^{-1}H_{\mu}, \quad A_{\vartheta}=H^{-1}H_{\vartheta}, \quad A_{\bar{\mu}}=K^{-1}K_{\bar{\mu}}, \quad A_{\bar{\vartheta}}=K^{-1}K_{\bar{\vartheta}}, \quad J=HK^{-1} \tag{41}$$

where H and K are non-degenerate matrixes, the local fractional partial derivatives of A, K and H with respective to $\mu, \vartheta, \bar{\mu},$ and $\bar{\vartheta}$ are commutative.

A direct computation tells:

$$\begin{aligned}&[D_{\mu}^{(\alpha)}, D_{\vartheta}^{(\alpha)}]=(\partial_{\mu}^{(\alpha)}+A_{\mu})(\partial_{\vartheta}^{(\alpha)}+A_{\vartheta})-(\partial_{\vartheta}^{(\alpha)}+A_{\vartheta})(\partial_{\mu}^{(\alpha)}+A_{\mu})=\\&=-H^{-1}H_{\mu}^{(\alpha)}H^{-1}H_{\vartheta}^{(\alpha)}+H^{-1}H_{\vartheta\mu}^{(2\alpha)}+H^{-1}H_{\mu}^{(\alpha)}H^{-1}H_{\vartheta}^{(\alpha)}+H^{-1}H_{\vartheta}^{(\alpha)}H^{-1}H_{\mu}^{(\alpha)}-\\&-H^{-1}H_{\mu\vartheta}^{(2\alpha)}-H^{-1}H_{\vartheta}^{(\alpha)}H^{-1}H_{\mu}^{(\alpha)}=0\end{aligned} \tag{42}$$

By a similar way, we have $[D_{\bar{\mu}}^{(\alpha)}, D_{\bar{\vartheta}}^{(\alpha)}]=0.$

On the other hand, we have:

$$\begin{aligned}&[D_{\mu}^{(\alpha)}, D_{\bar{\mu}}^{(\alpha)}]=-K^{-1}K_{\mu}^{(\alpha)}K^{-1}K_{\bar{\mu}}^{(\alpha)}+K^{-1}K_{\bar{\mu}\mu}^{(2\alpha)}+H^{-1}H_{\bar{\mu}}^{(\alpha)}H^{-1}H_{\mu}^{(\alpha)}-H^{-1}H_{\mu\bar{\mu}}^{(2\alpha)}+\\&+H^{-1}H_{\mu}^{(\alpha)}K^{-1}K_{\bar{\mu}}^{(\alpha)}-K^{-1}K_{\bar{\mu}}^{(\alpha)}H^{-1}H_{\mu}^{(\alpha)}=\\&=-K^{-1}J^{-1}J_{\bar{\mu}}^{(\alpha)}J^{-1}J_{\mu}^{(\alpha)}K-K^{-1}J^{-1}J_{\mu\bar{\mu}}^{(\alpha)}K=-K^{-1}[J^{-1}J_{\mu}^{(\alpha)}]_{\bar{\mu}}^{(\alpha)}K\end{aligned} \tag{43}$$

Similarly, we derive:

$$[\mathrm{D}_{\vartheta}^{(\alpha)},\mathrm{D}_{\vartheta}^{(\alpha)}]=-\mathrm{K}^{-1}(\mathrm{J}^{-1}\mathrm{J}_{\vartheta}^{(\alpha)})_{\bar{\vartheta}}^{(\alpha)}\mathrm{K} \tag{44}$$

With the help of eqs. (43) and (44), we reduce eq. (35) to:

$$[J^{-1}J_{\mu}^{(\alpha)}]_{\bar{\mu}}^{(\alpha)}+[J^{-1}J_{\vartheta}^{(\alpha)}]_{\bar{\vartheta}}^{(\alpha)}=0 \tag{45}$$

If we set $\alpha = 1$ and select

$$\mathrm{J}=\frac{1}{u}\begin{pmatrix}1 & w\\ v & u^2+vw\end{pmatrix} \tag{46}$$

then eq. (45) degenerates into the earliest Yang's equation for R-gauge fields.

Example 2. Derivation of local fractional principal chiral field equation:

$$[\mathrm{g}_x^{(\alpha)}\mathrm{g}^{-1}]_t^{(\alpha)}+[\mathrm{g}_t^{(\alpha)}\mathrm{g}^{-1}]_x^{(\alpha)}=0 \tag{47}$$

Selecting the *n*-order matrices:

$$\mathrm{M}=\frac{-\mathrm{A}}{\lambda-1},\quad \mathrm{N}=\frac{\mathrm{B}}{\lambda+1} \tag{48}$$

which are local fractional differentiable with respect to *x* and *t*, and substituting them into the local fractional zero curvature equation:

$$\mathrm{M}_t^{(\alpha)}-\mathrm{N}_x^{(\alpha)}+[\mathrm{M},\mathrm{N}]=0 \tag{49}$$

we have:

$$\frac{-\mathrm{A}_t^{(\alpha)}}{\lambda-1}-\frac{\mathrm{B}_x^{(\alpha)}}{\lambda+1}-\frac{[\mathrm{A},\mathrm{B}]}{(\lambda+1)(\lambda-1)}=0 \tag{50}$$

which gives:

$$(\lambda+1)\mathrm{A}_t^{(\alpha)}+(\lambda-1)\mathrm{B}_x^{(\alpha)}+[\mathrm{A},\mathrm{B}]=0 \tag{51}$$

Setting $\lambda = 1$ and $\lambda = -1$, respectively, from eq. (51) we have:

$$\mathrm{A}_t^{(\alpha)}+\frac{1}{2}[\mathrm{A},\mathrm{B}]=0,\quad \mathrm{B}_x^{(\alpha)}-\frac{1}{2}[\mathrm{A},\mathrm{B}]=0, \tag{52}$$

which are equivalent to:

$$\mathrm{A}_t^{(\alpha)}+\mathrm{B}_x^{(\alpha)}=0,\quad \mathrm{A}_t^{(\alpha)}-\mathrm{B}_x^{(\alpha)}+[\mathrm{A},\mathrm{B}]=0 \tag{53}$$

Taking $\mathrm{A}=\mathrm{g}_x^{(\alpha)}\mathrm{g}^{-1}$ and $\mathrm{B}=\mathrm{g}_t^{(\alpha)}\mathrm{g}^{-1}$, here the *n*-order matrixes g and g^{-1} are local fractional differentiable with respect to *x* and *t*, from eq. (32) we have:

$$\mathrm{A}_t^{(\alpha)}=\mathrm{g}_{xt}^{(2\alpha)}\mathrm{g}^{-1}-\mathrm{g}_x^{(\alpha)}[\mathrm{g}^{-1}\mathrm{g}_t^{(\alpha)}\mathrm{g}^{-1}]=\mathrm{g}_{xt}^{(2\alpha)}\mathrm{g}^{-1}-\mathrm{A}\mathrm{B} \tag{54}$$

Similarly, we have:

$$\mathrm{B}_x^{(\alpha)}=\mathrm{g}_{tx}^{(2\alpha)}\mathrm{g}^{-1}-\mathrm{B}\mathrm{A} \tag{55}$$

In view of eqs. (54) and (55), we reduce eqs. (53) to eq. (47).

Conclusion

This paper applies the local fractional partial derivative of inverse matrix to derive two fractional partners of Yang-Mills equation and Yang-Mills equation. The present paper gives an example to derive fractional equations for two-scale thermodynamics [7, 8]: one scale is for the traditional equation, while the other scale for fractional partner.

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