LOCAL FRACTIONAL HEAT AND WAVE EQUATIONS WITH LAGUERRE TYPE DERIVATIVES

by

Chun-Fu WEI *

School of Mathematics and Information Science, Henan Polytechnic University, Jiaozuo, China

Original scientific paper
https://doi.org/10.2298/TSCI2004575W

In this paper, we investigate a local fractional PDE with Laguerre type derivative. The considered equation represents a general extension of the classical heat and wave equations. The method of separation of variables is used to solve the differential equation defined in a bounded domain.

Key words: *Laguerre type derivatives, the method of separation of variables, local fractional derivative*

Introduction

Consider the following local fractional PDE with Laguerre type derivative:

$$k(x)\frac{\partial^{2\alpha}u(x,t)}{\partial t^{2\alpha}} = a^2 D_{L_x} u(x,t), \quad t>0 \tag{1}$$

defined in a bounded domain $0 \le x \le d$, with initial conditions:

$$u(x,0) = f(x), \quad \frac{\partial u(x,0)}{\partial t} = 0 \tag{2}$$

and boundary condition:

$$\frac{\partial u(d,t)}{\partial x} = 0 \tag{3}$$

where a is a constant, $(\partial^{2\alpha})/(\partial t^{2\alpha})$ – the local fractional derivative of order 2α, $0 < \alpha \le 1$, and $D_{L_x} = (\partial/\partial x)k(x)(\partial/\partial x)$ is the Laguerre type derivative.

The proposed equation is a generalization of the classical heat and wave equations. When $k(x) \equiv$ constant, we note that in case of $\alpha = 1/2$, eq. (1) coincides with the classical heat equation and in case of $\alpha = 1$, it becomes the classical wave equation [1-4].

The local fractional derivative has been successfully applied in the fractal elasticity and fractal wave equation [5-10]. The Laguerre type derivative could be used in order to substitute the classical derivative operators in many frameworks, including the models for heat and wave phenomena [11-14].

Preliminaries

In this section, we recall some definitions and properties of local fractional derivative [5-10] and Laguerre type derivative [11-14].

* Author's e-mail: mathwcf@163.com

Definition 1. Assume the following relation exists:

$$\left| f(x) - f(x_0) \right| < \varepsilon^{\alpha} \tag{4}$$

with $|x - x_0| < \delta$ for ε, $\delta > 0$. Then $f(x)$ is local fractional continuous at x_0 which is denoted by $\lim_{x \to x_0} f(x) = f(x_0)$. If $f(x)$ is local fractional continuous on the interval (a, b), it is denoted by $f(x) \in C_\alpha(a, b)$.

Definition 2. In a fractal space, let $f(x) \in C_\alpha(a, b)$**,** the local fractional derivative of $f(x)$ of order α at the point $x = x_0$ is given by:

$$\mathrm{D}_x^{\alpha} f(x_0) = \frac{\mathrm{d}^{\alpha}}{\mathrm{d}x^{\alpha}} f(x) \bigg|_{x=x_0} = f^{(\alpha)}(x_0) = \lim_{x \to x_0} \frac{\Delta^{\alpha}[f(x) - f(x_0)]}{(x - x_0)^{\alpha}} \tag{5}$$

where $\Delta[f(x) - f(x_0)] \cong \Gamma(\alpha+1)[f(x) - f(x_0)]$.

Local fractional derivative of high order is defined in the form:

$$f^{(k\alpha)}(x) = \overbrace{\mathrm{D}_x^{\alpha} \mathrm{D}_x^{\alpha} \cdots \mathrm{D}_x^{\alpha}}^{k \text{ times}} f(x) \tag{6}$$

and local fractional partial derivative of high order is written in the form:

$$\frac{\partial^{k\alpha} f(x,t)}{\partial x^{k\alpha}} = \overbrace{\frac{\partial^{\alpha}}{\partial x^{\alpha}} \frac{\partial^{\alpha}}{\partial x^{\alpha}} \cdots \frac{\partial^{\alpha}}{\partial x^{\alpha}}}^{k \text{ times}} f(x,t) \tag{7}$$

The following formulas on local fractional derivative will play very important role in next section.

$$\frac{\mathrm{d}^{\alpha}(x^{n\alpha})}{\mathrm{d}x^{\alpha}} = \frac{\Gamma(1+n\alpha) x^{(n-1)\alpha}}{\Gamma[1+(n-1)\alpha]} \tag{8}$$

$$\frac{\mathrm{d}^{\alpha} f[g(x)]}{\mathrm{d}x^{\alpha}} = f'[g(x)] g^{(\alpha)}(x) \tag{9}$$

where there exists $f'[g(x)]$ and $g^{(\alpha)}(x)$.

Definition 3. For every positive integer, n, the operator:

$$D_{nL_x} = \overbrace{Dx \cdots DxD}^{n+1\, D} \text{ (containing ordinary derivatives)} \tag{10}$$

is called the n-order Laguerre derivatives and the nL – exponential function is defined by:

$$e_n(x) = \sum_{k=0}^{\infty} \frac{x^k}{(k!)^{n+1}} \tag{11}$$

Obviously, the function $e_n(\lambda x)$ is an eigenfunction of the operator D_{nL}, where λ be an arbitrary real or complex number.

Solutions of the problem of eqs. (1)-(3)

In this work we consider the case $k(x) = x$:

$$\begin{cases} x\dfrac{\partial^{2\alpha}u}{\partial t^{2\alpha}} = a^2 \dfrac{\partial}{\partial x} x \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u(d,t)}{\partial x} = 0, \quad \dfrac{\partial u(x,0)}{\partial t} = 0, \quad u(x,0) = f(x) \end{cases} \tag{12}$$

This equation can be solved by the Taylor series method [15], the variational iteration method [16, 17], and the homotopy perturbation method [18-26]. In this paper we will apply the method of separation of variables [27].

First, by using the following two-scale transform [28-30]:

$$T = \frac{t^\alpha}{\Gamma(1+\alpha)} \tag{13}$$

we get

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial u}{\partial T}\frac{\partial^\alpha T}{\partial t^\alpha} = \frac{\partial u}{\partial T} \tag{14}$$

$$\frac{\partial^{2\alpha} u}{\partial t^{2\alpha}} = \frac{\partial^2 u}{\partial T^2}\frac{\partial^\alpha T}{\partial t^\alpha} = \frac{\partial^2 u}{\partial T^2} \tag{15}$$

Then, we can transfer the problem (4) into the following problem:

$$\begin{cases} x\dfrac{\partial^2 u(x,T)}{\partial T^2} = a^2 \dfrac{\partial}{\partial x} x \dfrac{\partial u(x,T)}{\partial x} \\ \dfrac{\partial u(d,T)}{\partial x} = 0, \quad \dfrac{\partial u(x,0)}{\partial t} = 0, \quad u(x,0) = f(x) \end{cases} \tag{16}$$

Applying the method of separation of variables, *i. e.*:

$$u(x,T) = X(x)Y(T) \tag{17}$$

one obtains:

$$Y'' + \lambda^2 a^2 Y = 0 \tag{18}$$

$$x^2 X'' + xX' + \lambda^2 x^2 X = 0 \tag{19}$$

where λ is a separation constant and $X(x)$ satisfies the boundary condition:

$$X'(d) = 0 \tag{20}$$

Note that the eq. (19) is a Bessel differential equation for $X(x)$. So the solution which is finite at $x = 0$ is:

$$X(x) = J_0(\lambda x) \tag{21}$$

where $J_0(x)$ is the solution of Bessel equation of order zero [27].

Thus the eigenvalues are:

$$\lambda_n = \frac{x_n^{(0)}}{d}, \quad (n = 0, 1, 2, \cdots) \tag{22}$$

and the corresponding eigenfunctions are:

$$X_n(x) = J_0\left(\frac{x_n^{(0)}}{d}x\right) \tag{23}$$

where the $x_n^{(0)}$ is the nth positive root of $J'(x)$.

From eq. (22), we get:

$$Y_0(T) = A_0 + B_0 T \tag{24}$$

$$Y_n(T) = A_n \cos\frac{x_n^{(0)}a}{d}T + B_n \sin\frac{x_n^{(0)}a}{d}T \tag{25}$$

Since the system (23) forms a complete orthogonal basis, we can expand the solution of the problem (1)-(3) by the following series:

$$u(x,T) = A_0 + B_0 T + \sum_{n=1}^{\infty}\left[A_n \cos\frac{x_n^{(0)}aT}{d} + B_n \sin\frac{x_n^{(0)}aT}{d}\right]J_0\left(\frac{x_n^{(0)}}{d}x\right) \tag{26}$$

Inserting eq. (26) into eq. (16), we find that:

$$A_0 + \sum_{n=1}^{\infty} A_n J_0\left(\frac{x_n^{(0)}}{d}x\right) = f(x) \tag{27}$$

$$B_0 + \sum_{n=1}^{\infty} B_n \frac{x_n^{(0)}}{d} J_0\left(\frac{x_n^{(0)}}{d}x\right) = 0 \tag{28}$$

So we obtain:

$$B_0 = B_n = 0 \tag{29}$$

$$A_0 = \frac{2}{d^2}\int_0^d f(x)x\mathrm{d}x \tag{30}$$

$$A_n = \frac{2}{d^2 J_0^2(x_n^{(0)})}\int_0^d J_0\left(\frac{x_n^{(0)}}{d}\right) f(x)x\mathrm{d}x \tag{31}$$

Hence:

$$u(x,T) = A_0 + \sum_{n=1}^{\infty}\left[A_n \cos\frac{x_n^{(0)}aT}{d}\right]J_0\left(\frac{x_n^{(0)}}{d}x\right)$$

Finally, by (13), we get:

$$u(x,t) = A_0 + \sum_{n=1}^{\infty}\left[A_n \cos\frac{x_n^{(0)}a}{d\,\Gamma(1+\alpha)}t^{\alpha}\right]J_0\left(\frac{x_n^{(0)}}{d}x\right)$$

where A_0, A_n were determined by eqs. (30) and (31).

Conclusion

The method of separation of variables is used to solve a local fractional differential equation defined involving Laguerre type derivatives in a bounded domain. The considered equation represents a general extension of the classical heat and wave equation. The explicit solutions are obtained, and the present method can be extended to fractal calculus [31-37].

Acknowledgment

This paper, was partially supported by Henan Natural Science Foundation in China under Grant No. 182300410105, and by the Doctoral Foundation at Henan Polytechnic University in China under Grant No. B2015-52.

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Paper submitted: December 26, 2018
Paper revised: June 29, 2019
Paper accepted: August 8, 2019