LOCAL FRACTIONAL HEAT AND WAVE EQUATIONS
WITH LAGUERRE TYPE DERIVATIVES

by

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In this paper, we investigate a local fractional PDE with Laguerre type derivative. The considered equation represents a general extension of the classical heat and wave equations. The method of separation of variables is used to solve the differential equation defined in a bounded domain.

Key words: Laguerre type derivatives, the method of separation of variables, local fractional derivative

Introduction

Consider the following local fractional PDE with Laguerre type derivative:

\[
k(x)\frac{\partial^{2\alpha} u(x,t)}{\partial t^{2\alpha}} = a^2 D_{t^\alpha} u(x,t), \quad t > 0
\]

defined in a bounded domain \(0 \leq x \leq d\), with initial conditions:

\[
u(x,0) = f(x), \quad \frac{\partial u(x,0)}{\partial t} = 0
\]

and boundary condition:

\[
\frac{\partial u(d,t)}{\partial x} = 0
\]

where \(a\) is a constant, \((\partial^{2\alpha})/(\partial t^{2\alpha})\) – the local fractional derivative of order \(2\alpha\), \(0 < \alpha \leq 1\), and \(D_{t^\alpha} = (\partial/\partial x)k(x)(\partial/\partial x)\) is the Laguerre type derivative.

The proposed equation is a generalization of the classical heat and wave equations. When \(k(x) \equiv \text{constant}\), we note that in case of \(\alpha = 1/2\), eq. (1) coincides with the classical heat equation and in case of \(\alpha = 1\), it becomes the classical wave equation [1-4].

The local fractional derivative has been successfully applied in the fractal elasticity and fractal wave equation [5-10]. The Laguerre type derivative could be used in order to substitute the classical derivative operators in many frameworks, including the models for heat and wave phenomena [11-14].

Preliminaries

In this section, we recall some definitions and properties of local fractional derivative [5-10] and Laguerre type derivative [11-14].

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Definition 1. Assume the following relation exists:

\[ |f(x) - f(x_0)| < \varepsilon^\alpha \]  \quad (4)

with \( |x - x_0| < \delta \) for \( \varepsilon, \delta > 0 \). Then \( f(x) \) is local fractional continuous at \( x_0 \) which is denoted by \( \lim_{x \to x_0} f(x) = f(x_0) \). If \( f(x) \) is local fractional continuous on the interval \((a, b)\), it is denoted by \( f(x) \in C_\alpha(a, b) \).

Definition 2. In a fractal space, let \( f(x) \in C_\alpha(a, b) \), the local fractional derivative of \( f(x) \) of order \( \alpha \) at the point \( x = x_0 \) is given by:

\[ D_\alpha^x f(x_0) = \frac{d^\alpha}{dx^\alpha} f(x) \bigg|_{x=x_0} = f^{(\alpha)}(x_0) = \lim_{x \to x_0} \frac{\Delta^\alpha [f(x) - f(x_0)]}{(x - x_0)^\alpha} \]  \quad (5)

where \( \Delta [f(x) - f(x_0)] \equiv \Gamma(\alpha + 1)|f(x) - f(x_0)| \).

Local fractional derivative of high order is defined in the form:

\[ f^{(k\alpha)}(x) = \underbrace{D_\alpha^x D_\alpha^x \cdots D_\alpha^x}_{k \text{ times}} f(x) \]  \quad (6)

and local fractional partial derivative of high order is written in the form:

\[ \frac{\partial^{k\alpha}}{\partial x^{k\alpha}} f(x, t) = \frac{\partial^\alpha}{\partial x^\alpha} \frac{\partial^\alpha}{\partial x^\alpha} \cdots \frac{\partial^\alpha}{\partial x^\alpha} f(x, t) \]  \quad (7)

The following formulas on local fractional derivative will play very important role in next section.

\[ \frac{d^\alpha (x^{\alpha x})}{dx^\alpha} = \frac{\Gamma(1 + n\alpha)x^{(n-1)\alpha}}{\Gamma[1 + (n-1)\alpha]} \]  \quad (8)

\[ \frac{d^\alpha [g(x)]}{dx^\alpha} = f^{(\alpha)}(x)g^{(\alpha)}(x) \]  \quad (9)

where there exists \( f^{(\alpha)}(g(x)) \) and \( g^{(\alpha)}(x) \).

Definition 3. For every positive integer, \( n \), the operator:

\[ D_{nL} = \underbrace{Dx \cdots Dx}_{n+1} D \]  (containing ordinary derivatives) \quad (10)

is called the \( n \)-order Laguerre derivatives and the \( nL \) – exponential function is defined by:

\[ e_{\lambda x} = \sum_{k=0}^{\infty} \frac{\lambda^k x^k}{(k!)^{n+1}} \]  \quad (11)

Obviously, the function \( e_{\lambda x} \) is an eigenfunction of the operator \( D_{nL} \), where \( \lambda \) be an arbitrary real or complex number.
Solutions of the problem of eqs. (1)-(3)

In this work we consider the case \( k(x) = x \):

\[
\begin{cases}
  x \frac{\partial^2 u}{\partial t^{2\alpha}} = a^2 \frac{\partial}{\partial x} \frac{\partial u}{\partial x} \\
  \frac{\partial u(d, t)}{\partial x} = 0, \quad \frac{\partial u(x, 0)}{\partial t} = 0, \quad u(x, 0) = f(x)
\end{cases}
\]  
(12)

This equation can be solved by the Taylor series method [15], the variational iteration method [16, 17], and the homotopy perturbation method [18-26]. In this paper we will apply the method of separation of variables [27].

First, by using the following two-scale transform [28-30]:

\[ T = \frac{t^\alpha}{\Gamma(1+\alpha)} \]  
(13)

we get

\[
\begin{align*}
  &\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial u}{\partial T} \frac{\partial T}{\partial t^\alpha} = \frac{\partial u}{\partial T} \\
  &\frac{\partial^2 u}{\partial t^{2\alpha}} = \frac{\partial^2 u}{\partial T^2} \frac{\partial T}{\partial t^\alpha} = \frac{\partial^2 u}{\partial T^2}
\end{align*}
\]  
(14)

Then, we can transfer the problem (4) into the following problem:

\[
\begin{cases}
  x \frac{\partial^2 u(x, T)}{\partial T^2} = a^2 \frac{\partial}{\partial x} \frac{\partial u(x, T)}{\partial x} \\
  \frac{\partial u(d, T)}{\partial x} = 0, \quad \frac{\partial u(x, 0)}{\partial t} = 0, \quad u(x, 0) = f(x)
\end{cases}
\]  
(16)

Applying the method of separation of variables, i. e.:

\[ u(x, T) = X(x)Y(T) \]  
(17)

one obtains:

\[
\begin{align*}
  Y'' + \lambda^2 a^2 Y &= 0 \\
  x^2 X'' + x X' + \lambda^2 x^2 X &= 0
\end{align*}
\]  
(18)

(19)

where \( \lambda \) is a separation constant and \( X(x) \) satisfies the boundary condition:

\[ X'(d) = 0 \]  
(20)

Note that the eq. (19) is a Bessel differential equation for \( X(x) \). So the solution which is finite at \( x = 0 \) is:

\[ X(x) = J_0(\lambda x) \]  
(21)

where \( J_0(x) \) is the solution of Bessel equation of order zero [27].

Thus the eigenvalues are:

\[ \lambda_n = \frac{x_n^{(0)}}{d}, \quad (n = 0, 1, 2, \cdots) \]  
(22)
and the corresponding eigenfunctions are:

\[ X_n(x) = J_0 \left( \frac{x^{(0)}_n}{d} x \right) \]  

(23)

where the \( x^{(0)}_n \) is the nth positive root of \( J'(x) \).

From eq. (22), we get:

\[ Y_0(T) = A_0 + B_0T \]  

(24)

\[ Y_n(T) = A_n \cos \frac{x^{(0)}_n}{d} T + B_n \sin \frac{x^{(0)}_n}{d} T \]  

(25)

Since the system (23) forms a complete orthogonal basis, we can expand the solution of the problem (1)-(3) by the following series:

\[ u(x, T) = A_0 + B_0T + \sum_{n=1}^{\infty} A_n \cos \frac{x^{(0)}_n}{d} T + B_n \sin \frac{x^{(0)}_n}{d} T J_0 \left( \frac{x^{(0)}_n}{d} x \right) \]  

(26)

Inserting eq. (26) into eq. (16), we find that:

\[ A_0 + \sum_{n=1}^{\infty} A_n J_0 \left( \frac{x^{(0)}_n}{d} x \right) = f(x) \]  

(27)

\[ B_0 + \sum_{n=1}^{\infty} B_n \frac{x^{(0)}_n}{d} J_0 \left( \frac{x^{(0)}_n}{d} x \right) = 0 \]  

(28)

So we obtain:

\[ B_0 = B_n = 0 \]  

(29)

\[ A_0 = \frac{2}{d^2} \int_0^d f(x) dx \]  

(30)

\[ A_n = \frac{2}{d^2 J_0^2(x^{(0)}_n)} \int_0^d \left( \frac{x^{(0)}_n}{d} \right) f(x) dx \]  

(31)

Hence:

\[ u(x, T) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{x^{(0)}_n}{d} T J_0 \left( \frac{x^{(0)}_n}{d} x \right) \]  

Finally, by (13), we get:

\[ u(x, t) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{x^{(0)}_n}{d^\alpha} T \Gamma(1 + \alpha) J_0 \left( \frac{x^{(0)}_n}{d} x \right) \]  

where \( A_0, A_n \) were determined by eqs. (30) and (31).
Conclusion

The method of separation of variables is used to solve a local fractional differential equation defined involving Laguerre type derivatives in a bounded domain. The considered equation represents a general extension of the classical heat and wave equation. The explicit solutions are obtained, and the present method can be extended to fractal calculus [31-37].

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References


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