CONSERVATION LAWS FOR PARTIAL DIFFERENTIAL EQUATIONS BASED ON THE POLYNOMIAL CHARACTERISTIC METHOD

by

***Yi TIAN*[a] *and Kang-Le WANG*[b*]**

[a] College of Data Science and Application,
Inner Mongolia University of Technology, Hohhot, China
[b] School of Mathematics and Information Science,
Henan Polytechnic University, Jiaozuo, China

Original scientific paper
https://doi.org/10.2298/TSCI2004529T

In this paper, the direct construction method combined with the differential polynomial characteristic set algorithm is used to complete conservation laws of PDE. The process of the direct construction method is to solve a system of linear determining equations, which is not easy to be solved. This paper uses the differential polynomial characteristic set algorithm to overcome the shortcoming, and constructs an explicit conservation law.

Key words: *direct construction method, local conservation law, differential polynomial characteristic set algorithm, variational principle*

Introduction

In the field of PDE, conservation laws and variational formulations [1-20] have many important applications, particularly with regard to linearization and integrability of PDE. In this paper, the direct construction method [21, 22] is used to complete the conservation laws of PDE, this method holds when no variational principle exists, and its derivation process requires to solve an over-determined linear PDE (determining equations), which are usually large and not easy to be solved directly, so the differential polynomial characteristic set algorithm is used to overcome the shortcoming.

Direct construction method

Consider the following PDE with M dependent variables $u=(u^1,u^2,\cdots,u^M)$ and m independent variables $x=(x_1,x_2,\cdots,x_m)$:

$$F_\beta(x,u,\partial_x u,\cdots,\partial_x^n u)=0, \quad (\beta=1,2,\cdots,N) \tag{1}$$

Definition 1. A local conservation law of (1) has the following expression:

$$D_i\Theta^i[u]=0 \tag{2}$$

holding for all solutions of eq. (1).

Definition 2. If functions $\{\Lambda^\gamma[U]\}$ and $\{\Theta^i[U]\}$ for system (1) satisfy:

* Corresponding author, e-mail: kangle83718@163.com; ttxsun@163.com

$$\Lambda^{\gamma}[U]F_{\gamma}[U]=D_i\Theta^i[U] \tag{3}$$

then $\{\Lambda^{\gamma}[U]\}$ is called multipliers of a conservation law.

The function $\{\Lambda^{\gamma}[U]\}$ yields a set of multipliers for conservation law of system (1) if:

$$E_{U^o}(\Lambda^{\gamma}[U]F_{\gamma}[U])\equiv 0, \quad o=1,2,\cdots,M \tag{4}$$

where Euler operator is denoted by

$$E_{U^{\sigma}}=\frac{\partial}{\partial U^{\sigma}}-D_i\frac{\partial}{\partial U_i^{\sigma}}+D_iD_j\frac{\partial}{\partial U_{ij}^{\sigma}}+\cdots$$

and total derivative is defined:

$$D_i=\frac{\partial}{\partial x_i}+U_i^{\sigma}\frac{\partial}{\partial U^{\sigma}}+U_{ij}^{\sigma}\frac{\partial}{\partial U_j^{\sigma}}+\cdots+U_{ii_1i_2\cdots i_l}^{\sigma}\frac{\partial}{\partial U_{i_1i_2\cdots i_l}^{\sigma}}+\cdots$$

Differential polynomial characteristic set algorithm

Suppose *DTE* and *IP* are differential polynomial system (dps), we frequently use the following notations [23-25]:

$Zero(DTE)$ = all solutions of DTE = 0;

$Zero(DTE/IP)$ = all solutions of DTE = 0 such that $IP\neq 0$;

$Zero(DTE,IP)$ = all solutions of DTE = 0 such that $IP=0$;

The *IP* denotes the initials and separate products for differential polynomials, called *IP* products;

$\mathrm{P}remd(DTE/DCS)$ = the set of pseudo-remainders of *DTE* w.r.t chain *DCS*.

Theorem 1 Let *DPS* be a finite differential polynomials system, then the differential polynomial characteristic set algorithm yields a finite number characteristic set DCS_k of *DPS* with initials and separate products IP_k such that the following zero decomposition:

$$Zero(DPS)=\sum_k Zero(DCS_k\ /\ IP_k)$$

is held.

The algorithm to determine characteristic set DCS listed below.

Input A dps DPS.

Output A differential characteristic set DCS.

Start Let $i=0$;

Step 1 Select a basic set DBS_i from DPS_i;

Step 2 Compute all the nonzero coherent dps of DBS_i, and put them in set from $IT_i^{'}$;

Step 3 $\forall IP$ (nonzero coherent differential polynomial) $\in IT_i^{'}$, compute $\mathrm{P}remd(IP/DBS_i)\setminus\{0\}$, and put the results in set IT_i;

Step 4 Compute $R_i=\mathrm{P}remd((DPS_i\setminus DBS_i)/DBS_i)\setminus\{0\}$, let $RJ_i=IT_i\cup R_i$;

Step 5 If $RJ_i=\phi$ (Empty), then $DCS=DBS_i$, and stop, else $i=i+1$, and $DPS_i=DPS_{i-1}\cup RJ_{i-1}$, go to **Step 1**.

Characteristic set has fine properties. On the one hand, it is equivalent to the original system under $IS \neq 0$, and on the other hand, it is triangular which is more important for differential problems. For example, in this paper, the determining equations of conservation law is hard to solve, we use differential polynomial characteristic set algorithm to deal with it, and obtain characteristic set which is triangular and easy to be solved.

Local conservation law computation

Example 1. Consider the following equation [26]:

$$u_t + F(u)u_x + u_{xxx} = 0 \tag{5}$$

for simplicity of computation, we specify $F(u) = u$, and eq. (5) is changed:

$$u_t + uu_x + u_{xxx} = 0 \tag{6}$$

Suppose that $\xi(x,t,u,u_x)$ is a multiplier of a conservation law of eq. (6), then:

$$\xi(x,t,u,u_x)(u_t + uu_x + u_{xxx}) = D_x X + D_t T \tag{7}$$

The determining equations derived from (4) are:

$$DTE = \begin{cases} \xi_{u_x} = 0 \\ u_x \xi_{uu} + \xi_{xu} = 0 \\ \xi_t + \xi_{xxx} + u_x^3 \xi_{uuu} + u\xi_x + 3u_x \xi_{xxu} + 3u_x^2 \xi_{xuu} = 0 \end{cases} \tag{8}$$

Taking left hand side as differential polynomial in ξ and their derivatives for each equation, we have following differential polynomial system:

$$DPS = \begin{cases} \xi_{u_x} \\ u_x \xi_{uu} + \xi_{xu} \\ \xi_t + \xi_{xxx} + u_x^3 \xi_{uuu} + u\xi_x + 3u_x \xi_{xxu} + 3u_x^2 \xi_{xuu} \end{cases} \tag{9}$$

Using the differential polynomial characteristic set algorithm, we obtain the following characteristic set *DCS* of *DPS:*

$$DCS = \begin{cases} \xi_{u_x}, \xi_{uu} \\ u_t + u\xi_x, \xi_{xu}, \xi_{xx} \end{cases} \tag{10}$$

Solving the characteristic set corresponding equations *DCS* = 0, we have:

$$Zero(DPS) = Zero(DCS) = \{\xi = (x - ut)c_1 + c_2 u + c_3\} \tag{11}$$

where c_1, c_2, c_3 are arbitrary constants. The corresponding conserved densities is:

$$\begin{cases} T = \dfrac{1}{2}u(-c_1 ut + c_2 u) + (c_1 x + c_3)u \\ X = \dfrac{1}{2}u^2(c_1 x + c_3) + \dfrac{1}{3}u^3(c_2 - c_1 t) + uu_{xx}(c_2 - c_1 t) + c_1 u_x\left(\dfrac{1}{2}u_x t - 1\right) - \dfrac{1}{2}u_x^2 c_2 + c_1 x u_{xx} + c_3 u_{xx} \end{cases} \tag{12}$$

Example 2. Consider the following non-linear telegraph equation [27]:

$$u_{tt} - [F(u)u_x]_x - [G(u)]_x = 0 \tag{13}$$

For simplicity of computation, we specify $F(u) = G(u) = u$, and eq. (13) becomes:

$$u_{tt} - (uu_x)_x - u_x = 0 \tag{14}$$

Suppose $\xi(x,t,u,u_x,u_{xx})$ is a multiplier of a conservation law of (14), then:

$$\xi(x,t,u,u_x,u_{xx})[u_{tt} - (uu_x)_x - u_x] = D_x X + D_t T \tag{15}$$

The determining equations derived from (4) are:

$$DTE = \begin{cases} \xi_{uu} = 0, \xi_{u_x u_x} = 0, \xi_{uu_x} = 0, \xi_{ut} = 0, \xi_{u_x t} = 0 \\ -\xi_{u_x x} + 2\xi_u = 0 \\ -u\xi_{u_x x} + 3u_x \xi_{u_x} - 2u\xi_u + 2\xi_{u_x} = 0 \\ -u_x^2 \xi_u - 2uu_x \xi_{ux} + u_x^2 \xi_{xu_x} + u_x \xi_{xu_x} + u_x^2 \xi_{uu_x} - u\xi_{xx} + u_x^3 \xi_{uu_x} + \xi_x + \xi_{tt} - uu_x^2 \xi_{uu} = 0 \end{cases} \tag{16}$$

Taking left hand side as differential polynomial in ξ and their derivatives for each equation, we have following differential polynomial system:

$$DPS = \begin{cases} \xi_{uu}, \xi_{u_x u_x}, \xi_{uu_x}, \xi_{ut}, \xi_{u_x t}, \\ -\xi_{u_x x} + 2\xi_u, \\ -u\xi_{u_x x} + 3u_x \xi_{u_x} - 2u\xi_u + 2\xi_{u_x} \\ -u_x^2 \xi_u - 2uu_x \xi_{ux} + u_x^2 \xi_{xu_x} + u_x \xi_{xu_x} + u_x^2 \xi_{uu_x} - u\xi_{xx} + u_x^3 \xi_{uu_x} + \xi_x + \xi_{tt} - uu_x^2 \xi_{uu} \end{cases} \tag{17}$$

Using the differential polynomial characteristic set algorithm, we obtain the following characteristic set *DCS* of *DPS*:

$$DCS = \left\{ \xi_{u_x}, \xi_u, \xi_{xx}, \xi_{tt} + \xi_x \right\} \tag{18}$$

Solving the characteristic set corresponding equations *DCS* = 0, we obtain the following solutions:

$$Zero(DCS) = \left\{ \xi = c_3 xt + c_4 x - \frac{1}{6} c_3 t^3 - \frac{1}{2} c_4 t^2 + c_1 t + c_2 \right\} \tag{19}$$

The corresponding conserved density is:

$$\begin{cases} T = u_t \left(c_3 xt + c_4 x - \frac{1}{6} c_3 t^3 - \frac{1}{2} c_4 t^2 + c_1 t + c_2 \right) + u \left(c_4 t - c_1 - c_3 x + \frac{1}{2} c_3 t^2 \right) \\ X = \frac{1}{2} u^2 (c_3 t + c_4) + \frac{1}{6} uc_3 t^3 (u_x + 1) + \frac{1}{2} c_4 t^2 u(u_x + 1) - u_x t(c_1 u + c_3 u_x) \\ \quad -tu(c_1 + c_3 x) - c_2 uu_x - u_x^2 c_4 - c_2 u - xuc_4 \end{cases} \tag{20}$$

Conclusion

In this article, conservation laws of PDE are obtained by using the direct construction method. Firstly, an over-determined linear PDE (determining equations) is obtained, which is always difficult to be solved, then, we use differential polynomial characteristic set algorithm to process this problem, and get characteristic set of the determining equations, which is triangular and easy to be solved. The examples show that this method is effective to compute the local conservation laws of PDE and can be applicable to different cases of non-linear PDE.

Acknowledgment

The work is supported by National Natural Science Foundation of China (Grant No. 11561051, 61862048), the Natural Science Foundation of Inner Mongolia (2019MS05068), the Scientific Research Project Foundation of Inner Mongolia University of Technology (ZZ201820), and the Inner Mongolia Key Laboratory of Statistical Analysis Theory for Life Data and Neural Network Modeling.

References

[1] He, J. H., Generalized Equilibrium Equations for Shell Derived from a Generalized Variational Principle, *Applied Mathematics Letters*, *64* (2017), Feb., pp. 94-100
[2] He, J. H., Hamilton's Principle for Dynamical Elasticity, *Applied Mathematics Letters*, *72* (2017), Oct., pp. 65-69
[3] He, J. H., Variational Principles for some Nonlinear Partial Differential Equations with Variable Coefficients, *Chaos, Solitons & Fractals*, *19* (2004), 4, pp. 847-851
[4] Li, X. W., *et al.*,, On the Semi-Inverse Method and Variational Principle, *Thermal Science*, *17* (2013), 5, pp. 1565-1568
[5] He, J. H., Semi-Inverse Method and Generalized Variational Principles with Multi-Variables in Elasticity, *Applied Mathematics and Mechanics*, *21* (2000), 7, pp. 797-808
[6] He, J. H., A Modified Li-He's Variational Principle for Plasma, *International Journal of Numerical Methods for Heat and Fluid Flow*, On-line first, https://doi.org/10.1108/HFF-06-2019-0523, 2019
[7] He, J. H., Lagrange Crisis and Generalized Variational Principle for 3D unsteady flow, *International Journal of Numerical Methods for Heat and Fluid Flow*, On-line first, https://doi.org/10.1108/HFF-07-2019-0577, 2019
[8] He, J. H., Sun, C., A Variational Principle for a Thin Film Equation, *Journal of Mathematical Chemistry*, *57* (2019), 9, pp. 2075-2081
[9] Wang, Y., *et al.*, A Variational Formulation for Anisotropic Wave Traveling in a Porous medium, *Fractals*, *27* (2019), 4, 1950047
[10] Wang, K. L., He, C. H., A Remark on Wang's Fractal Variational Principle, *Fractals*, *27* (2019), 8, ID 1950134
[11] He, J. H., Generalized Variational Principles for Buckling Analysis of Circular Cylinders, *Acta Mechanica*, *231* (2019), pp. 899-906
[12] He, J. H., A Fractal Variational Theory for One-Dimensional Compressible Flow in a Microgravity Space, *Fractals*, On-line first, https://doi.org/10.1142/S0218348X20500243, 2019
[13] He, J. H., A Simple Approach to One-Dimensional Convection-Diffusion Equation and its Fractional Modification for E Reaction Arising in Rotating Disk Electrodes, *Journal of Electroanalytical Chemistry*, *854* (2019), Dec., ID 113565
[14] He, J. H.,Variational Principle for the Generalized KdV-Burgers Equation with Fractal Derivatives for Shallow Water Waves, *J. Appl. Comput. Mech.*, 6 (2020), 4, pp. 735-740
[15] Li, X. J., He, J. H., Variational Multi-Scale Finite Element Method for the Twophase Flow of Polymer Melt Filling Process, *International Journal of Numerical Methods for Heat & Fluid Flow*, On-line first, https://doi.org/10.1108/HFF-07-2019-0599
[16] Zhang, J. J., *et al.*, Some Analytical Methods for Singular Boundary Value Problem in a Fractal Space, *Appl. Comput. Math*., *18* (2019), 3, pp. 225-235
[17] He, C. H., *et al*., Taylor Series Solution for Fractal Bratu-Type Equation Arising in Electrospinning Process, *Fractals*, *28* (2020), 1, ID 2050011

[18] Zhang, Y., Tian, X., Conservation Laws of Nonconservative Nonholonomic System Based on Herglotz Variational Problem, *Physics Letters A*, *383* (2019), 8, pp. 691-696
[19] Muatjetjeja, B., Mogorosi, T. E., Variational Principle and Conservation Laws of a Generalized Hyperbolic Lane-Emden System, *Journal of Computational and Nonlinear Dynamics*, *13* (2018), 12, 121002
[20] Sugama, H., *et al.*, Eulerian Variational Formulations and Momentum Conservation Laws for Kinetic Plasma Systems, *Physics of Plasmas*, *25* (2018), 10, 102506
[21] Anco, S. C., Bluman, G., Direct Construction Method for Conservation Laws of Partial Differential Equations, Part I: Examples of Conservation Law Classifications, *European Journal of Applied Mathematics*, *13* (2002), 5, pp. 545-566
[22] Anco, S. C., Bluman, G., Direct Construction Method for Conservation Laws of Partial Differential Equations, Part II: General treatment, *European Journal of Applied Mathematics*, *13* (2002), 5, pp. 567-585
[23] Wu, W. T., *Mathematics Mechanization* (in Chinese), Science Press, Beijing, 2000
[24] Tian, Y., Symmetry Reduction a Promising Method for Heat Conduction Equations, *Thermal Science*, *23* (2019), 4, pp. 2219-2227
[25] Tian, Y., Diffusion-Convection Equations and Classical Symmetry Classification, *Thermal Science*, *23* (2019), 4, pp. 2151-2156
[26] Tian, Y., Temuer, C. L., Conservation Law Classification of Partial Differential Equations and the Application of Wu' s Method, (in Chinese), *Journal of Inner Mongolia University*, *41* (2010), 2, pp. 138-143
[27] Bluman, G. W., *et al.*, Local and Nonlocal Symmetries for Nonlinear Telegraph Equation, *Journal of Mathematical Physics*, *46* (2005), 2, ID 023505

Paper submitted: March 23, 2019
Paper revised: October 20, 2019
Paper accepted: October 21, 2019