HE'S FRACTIONAL DERIVATIVE FOR THE EVOLUTION EQUATION

by

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In this paper, He's fractional derivative is adopted to establish fractional evolution equations in a fractal space. He's fractional complex transform is used to convent the fractional evolution equation into its traditional partner, and the homotopy perturbation method is used to solve the equations. Some illustrative examples are presented to show that the proposed technology is very excellent.

Key words: *He's fractional derivative, fractional complex transform, two-scale transform, homotopy perturbation method, fractional evolution equation, variational principle*

Introduction

Evolution processes arise everywhere from life to a fractal pattern of a lightning, its basic property is similarities with remarkable diversity at different periods. The traditional evolution equation can be written in the form:

$$H_t + H_x = 2H_{xxt} \tag{1}$$

where H can be some a thermal parameter.

Consider the memory property of an evolution process, which cannot be modeled by eq. (1), we give a modification of eq. (1) in the form:

$$H_t^\beta + H_x = 2H_{xxt}, \quad H(x,0) = \mathrm{e}^{-x}, \quad t > 0 \tag{2}$$

where H_t^β is He's fractional derivative defined [1]:

$$\mathrm{D}_t^\alpha f = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_{t_0}^{t} (s-t)^{n-\alpha-1} [f_0(s) - f(s)] \mathrm{d}s$$

where $f_0(t)$ is a known a function.

He's fractional derivative is defined through the variational iteration algorithm [2-5], and it has been caught much attention in practical applications [6-9].

Equation (2) can effectively describe the similarity pattern at different periods, it can be solved by the variational iteration method [2-5], the Taylor series method [10, 11], the ho-

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motopy perturbation method [12-20] and the variational principle [21, 22]. In this paper we will study eq. (2) by He's fractional complex transform (HFCT) [23-25] to convert the fractional equation into a common differential equation, and solve the resultant equations by the homotopy perturbation method [12-20].

Fractional complex transform

The fractional complex transform [23-25] is a very good mathematical tool to converting a fractional differential equation in a fractal space into its traditional partner in a continuous space. The dimension and scale are very important things in modeling a practical problem, the different scales or dimensions will lead to different results or properties for the same phenomenon. The fractional complex transform can convert a discontinuous space on a small scale into a continuous space in a large scale, so it is also called as a two-scale transform [26, 27].

Consider the following general fractional differential equation:

$$\begin{cases} f(u,u_t^{\alpha},u_x^{\beta},u_y^{\gamma},u_z^{\lambda},u_t^{2\alpha},u_x^{2\beta},u_y^{2\gamma},u_z^{2\lambda}...)=0 \\ 0<\alpha\le 1,\quad 0<\beta\le 1,\quad 0<\gamma\le 1,\quad 0<\lambda\le 1 \end{cases}$$

where $u_t^{\alpha}=(\partial^{\alpha}u)/(\partial t^{\alpha})$ is He's fractional derivation [1], u is continuous (but not necessarily differentiable) function.

The fractional complex transform reads [23, 24]:

$$T=\frac{pt^{\alpha}}{\Gamma(1+\alpha)},\quad X=\frac{qx^{\beta}}{\Gamma(1+\beta)},\quad Y=\frac{ky^{\gamma}}{\Gamma(1+\gamma)},\quad Z=\frac{lz^{\lambda}}{\Gamma(1+\lambda)}$$

where *p, q, k, and l* are unknown constants. Using the basic properties of fractional derivation and above transform, we have:

$$\frac{\partial^{\alpha}u}{\partial t^{\alpha}}=p\frac{\partial u}{\partial T},\quad \frac{\partial^{\beta}u}{\partial x^{\beta}}=q\frac{\partial u}{\partial X},\quad \frac{\partial^{\gamma}u}{\partial y^{\gamma}}=k\frac{\partial u}{\partial Y},\quad \frac{\partial^{\lambda}u}{\partial z^{\lambda}}=k\frac{\partial u}{\partial Z}$$

Therefore, we can easily convert the fractional differential equations into ODE.

The homotopy perturbation method [12-20]

Consider the following equation:

$$A(u)-f(r)=0,\quad r\in\Omega \tag{3}$$

with the boundary condition of:

$$B\left(u,\frac{\partial u}{n}\right)=0,\quad r\in\Gamma \tag{4}$$

where A is a common differential operator, B is a boundary operator, $f(r)$ is a known function, Γ is the boundary of the domain Ω.

We divide operator A into N and L. The operator N is a non-linear operator. The operator L is a linear operator.

So, eq. (3) can be written into the following form:

$$L(u)+N(u)-f(r)=0 \tag{5}$$

Using the He's homotopy technique, we construct a homotopy as $\mu(r, q):\Omega\times \times[0, 1] \to R$ which satisfies:

$$H(\mu,q)=(1-q)[L(\mu)-L(u_0)]+q[A(\mu)-f(r)]=0 \tag{6}$$

or

$$H(\mu,q)=L(\mu)-L(u_0)+qL(u_0)+q[N(\mu)-f(r)]=0 \tag{7}$$

where $q\in[0, 1]$ is a parameter, u_0 is an initial condition of eq. (2), which satisfies the boundary conditions. According to eqs. (5) and (6), we have:

$$H(\mu,0)=L(\mu)-L(u_0)=0 \tag{8}$$

$$H(\mu,1)=A(\mu)-f(r)=0 \tag{9}$$

According to the homotopy perturbation method (HPM), we can first adopt the parameter q as a small parameter. We assume the solution of eqs. (5) and (6) can be written into a power series:

$$\mu=\mu_0+q\mu_1+q^2\mu_2+q^3\mu_3+q^4\mu_4+\cdots \tag{10}$$

Setting $q = 1$ in eq. (10), we obtain:

$$u=\lim_{q\to 1}\mu=\mu_0+\mu_1+\mu_2+\mu_3+\mu_4+\cdots \tag{11}$$

Numerical applications

First, we adopt He's fractional complex transform [23, 24]:

$$T=\frac{t^{\beta}}{\Gamma(1+\beta)} \tag{12}$$

We can convert eq. (2) into a common differential equation:

$$H_T+H_x=2H_{xxT} \tag{13}$$

$$H(x,0)=\mathrm{e}^{-x},\quad t>0 \tag{14}$$

By HPM, we have:

$$H_{0T}-\mathrm{e}^{-x}=0,\quad H_0(x,0)=\mathrm{e}^{-x}$$

$$H_{1T}+H_{0T}-2H_{0xxT}+\mathrm{e}^{-x}=0,\quad H_1(x,0)=0$$

$$H_{2T}+H_{1T}-2H_{1xxT}=0,\quad H_2(x,0)=0$$

$$H_{3T}+H_{2T}-2H_{2xxT}=0,\quad H_3(x,0)=0$$

$$H_{4T}+H_{3T}-2H_{3xxT}=0,\quad H_4(x,0)=0$$

$$H_{5T}+H_{4T}-2H_{4xxT}=0,\quad H_5(x,0)=0$$

$$H_{6T}+H_{5T}-2H_{5xxT}=0,\quad H_6(x,0)=0$$

By the previous equations, we have:

$$H_0(x,T)=\exp\{-x\}(T+1)$$

$$H_1(x,T)=\frac{\exp\{-x\}(T^2+4T)}{2}$$

$$H_2(x,T)=\frac{\exp\{-x\}(T^3+12T^2+24T)}{6}$$

$$H_3(x,T)=\frac{\exp\{-x\}(T^4+24T^3+144T^2+192T)}{24}$$

$$H_4(x,T)=\frac{\exp\{-x\}(T^5+40T^4+480T^3+1920T^2+1920T)}{120}$$

$$H_5(x,T)=\frac{\exp\{-x\}(T^6+60T^5+1200T^4+9600T^3+28800T^2+23040T)}{720}$$

Therefore, the 5-order approximate solution of eq. (13) can be written into the following form:

$$H(x,T)=H_0(x,T)+H_1(x,T)+H_2(x,T)+H_3(x,T)+H_4(x,T)+H_5(x,T)=$$

$$=\frac{[T^6+66T^5+1470T^4+13320T^3+46440T^2+45360T+720]\exp\{-x\}}{720}$$

So, the approximate solution of eq. (2) can be written:

$$H(x,t)=H_0(x,t)+H_1(x,t)+H_2(x,t)+H_3(x,t)+H_4(x,t)+H_5(x,t)=$$

$$=\exp\{-x\}\left[\frac{t^\beta}{\Gamma(1+\beta)}+1\right]+\frac{\exp\{-x\}}{6}\left\{\left[\frac{t^\beta}{\Gamma(1+\beta)}\right]^3+12\left[\frac{t^\beta}{\Gamma(1+\beta)}\right]^2+24\left[\frac{t^\beta}{\Gamma(1+\beta)}\right]\right\}+$$

$$+\frac{\exp\{-x\}}{24}\left\{\left[\frac{t^\beta}{\Gamma(1+\beta)}\right]^4+24\left[\frac{t^\beta}{\Gamma(1+\beta)}\right]^3+144\left[\frac{t^\beta}{\Gamma(1+\beta)}\right]^2+192\left[\frac{t^\beta}{\Gamma(1+\beta)}\right]\right\}+\frac{\exp\{-x\}}{120}\cdot$$

$$\cdot\left\{\left[\frac{t^\beta}{\Gamma(1+\beta)}\right]^5+40\left[\frac{t^\beta}{\Gamma(1+\beta)}\right]^4+480\left[\frac{t^\beta}{\Gamma(1+\beta)}\right]^3+1920\left[\frac{t^\beta}{\Gamma(1+\beta)}\right]^2+1920\left[\frac{t^\beta}{\Gamma(1+\beta)}\right]\right\}+$$

$$+\frac{\exp\{-x\}}{720}\left\{\left[\frac{t^\beta}{\Gamma(1+\beta)}\right]^6+60\left[\frac{t^\beta}{\Gamma(1+\beta)}\right]^5+1200\left[\frac{t^\beta}{\Gamma(1+\beta)}\right]^4+\right.$$

$$+9600\left[\frac{t^{\beta}}{\Gamma(1+\beta)}\right]^{3}+28800\left[\frac{t^{\beta}}{\Gamma(1+\beta)}\right]^{2}+23040\left[\frac{t^{\beta}}{\Gamma(1+\beta)}\right]\Bigg\}=$$

$$=\frac{\exp\{-x\}}{720}\left\{\left[\frac{t^{\beta}}{\Gamma(1+\beta)}\right]^{6}+66\left[\frac{t^{\beta}}{\Gamma(1+\beta)}\right]^{5}+\left[\frac{t^{\beta}}{\Gamma(1+\beta)}\right]^{4}+\right.$$

$$\left.+13320\left[\frac{t^{\beta}}{\Gamma(1+\beta)}\right]^{3}+46440\left[\frac{t^{\beta}}{\Gamma(1+\beta)}\right]^{2}+45360\left[\frac{t^{\beta}}{\Gamma(1+\beta)}\right]+720\right\}$$

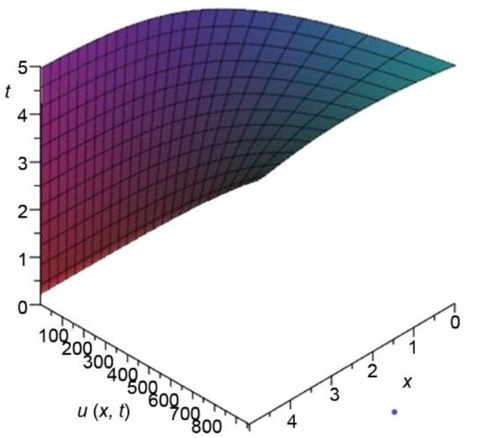

Figure 1. When β = 0.5, the 5-order approximate solution of eq. (2)

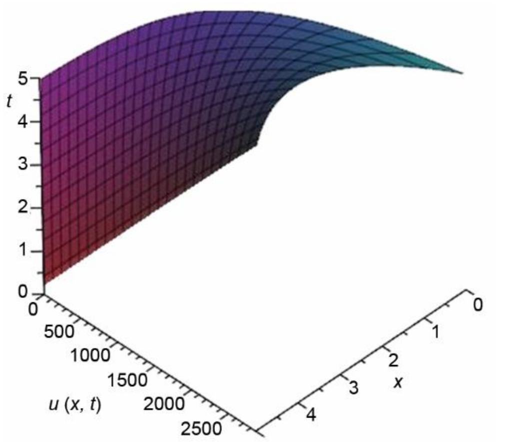

Figure 2. When β = 0.8, the 5-order approximate solution of eq. (2)

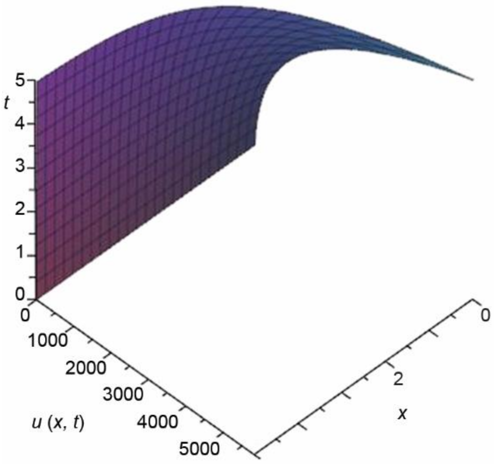

Figure 3. When β = 1, the 5-order approximate solution of eq. (2)

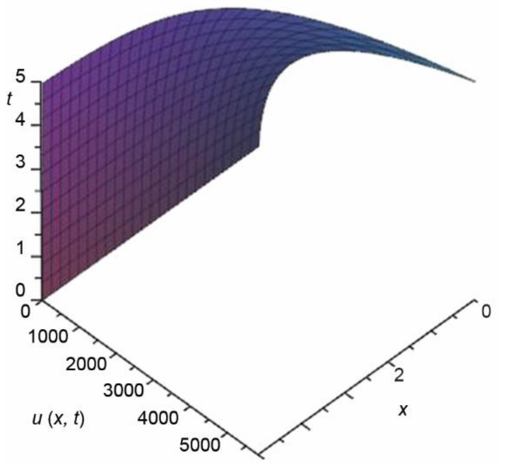

Figure 4. When β = 1, the exact solution of eq (2)

Figures 1-3 show the 5-order approximate solution for different values of α. Figure 4 shows the exact solution of eq. (2) when β = 1. We clearly find that the approximate solutions of eq. (2) continuously trend to its exact solution when $\beta \to 1$. In this paper, we only use five terms to approximate the solution of eq. (2). If we adopt more terms, its accuracy of the solutions will be greatly improved.

Discussion and conclusion

Variational principles in a fractal space has been caught much attention recently. Wang *et al.* [28] shed a bright light on effective establishment of a variational formulation in a fractal space by the semi-inverse method [28-30]. There might be a variational principle for eq. (2), we will discuss it in a forthcoming paper.

In this paper, the fractional evolution equation is described by He's fractional derivative in a fractal space. We successfully use HFCT and HHPM to find the approximate analyti cal solution of the fractional evolution equation. The numerical example is presented to show that the proposed method is very excellent. The present method can be easily extended to fractal calculus [31-40].

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