INVERSE SCATTERING TRANSFORM FOR NEW MIXED SPECTRAL ABLOWITZ-KAUP-NEWELL-SEGUR EQUATIONS

by

Sheng ZHANG [a,b*] ***and Caihong YOU*** [a]

[a] School of Mathematics and Physics, Bohai University, Jinzhou, China
[b] Department of Mathematics, Hohhot Minzu College, Hohhot, China

Original scientific paper
https://doi.org/10.2298/TSCI2004437Z

The inverse scattering transform plays a very important role in promoting the development of analytical methods to solve non-linear PDE exactly. In this paper, new and more general mixed spectral Ablowitz-Kaup-Newell-Segur equations are derived and solved by embedding a novel time-varying spectral parameter into an associated linear problem and the inverse scattering transform. As a result, new exact solutions and n-soliton solutions are obtained. To gain more insights into the embedded spectral parameter and the obtained solutions, some dynamical evolutions, and spatial structures are simulated. It is shown that the derived Ablowitz-Kaup-Newell-Segur equations are Lax integrable and the obtained soliton solutions possess time-varying amplitudes.

Key words: *mixed spectral Ablowitz-Kaup-Newell-Segur equations, inverse scattering transform, n-soliton solution, dynamical evolution, spatial structure, Lax integrability*

Introduction

In 1967, Gardner *et al.* [1] discovered a wonderful relationship between the celebrated Korteweg-de Vries (KdV) equation and the scattering theory of Schroedinger equation and hence obtained the exact and explicit *n*-soliton solution of the KdV equation for the first time. Gardner *et al.*'s [1] discovery has brought far-reaching influence, it not only brings about the climax of soliton research, but also develops into a basic method the well-known inverse scattering transform (IST) method for constructing soliton solutions of non-linear PDE. Since the steps of IST method solving the initial value problem (IVP) of non-linear PDE are very similar to those of Fourier transform used to deal with the IVP of linear equations, the IST method is often referred to as non-linear Fourier analysis [2]. With the development of the IST method, more and more analytical methods have been developed for solving non-linear PDE, such as Hirota's bilinear method [3-7], Painleve truncation expansion [8-10], homogeneous balance method [11, 12], auxiliary equation methods [13-15], variational iteration method (VIM) [16], homotopy perturbation method (HPM) [17], exp-function (EXP) method [18-20], and so on. Thanks to the powerful VIM, HPM, and EXP method proposed by Ji-Huan He, a large number of exact and approximate solutions have been obtained in different fields. Recently, Zhang and Hong [21] extended the IST method to the generalized isospectral Ablowitz-Kaup-Newell-Segur (AKNS) hierarchy, the variable-coefficient non-isospectral

* Corresponding author, e-mail: szhangchina@126.com

Toda lattice hierarchy [22], the generalized non-isospectral AKNS hierarchies [23-27], the mixed spectral KdV hierarchy with self-consistent sources [28] and the mixed spectral AKNS hierarchies [29-33], supersymmetric KdV equation [34], respectively.

In this paper, we shall derive and solve the following new and more general mixed spectral AKNS equations:

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = L^3 \begin{pmatrix} -q \\ r \end{pmatrix} + \sum_{n=0}^{5} L^n \begin{pmatrix} -xq \\ xr \end{pmatrix} \tag{1}$$

in the framework of IST method, and then simulate not only the spatial structures of the obtained soliton solutions but also the dynamical evolutions of the non-isospectral parameter:

$$ik_t = \frac{1}{2}\sum_{n=0}^{5}(2ik)^n \tag{2}$$

which is embedded into the associated linear spectral problem:

$$\phi_x = M\phi, \quad M = \begin{pmatrix} -ik & q \\ r & ik \end{pmatrix}, \quad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \tag{3}$$

and its evolution equation:

$$\phi_t = N\phi, \quad N = \begin{pmatrix} A & B \\ C & -A \end{pmatrix} \tag{4}$$

Here the operator L is defined as:

$$L = \begin{pmatrix} -\partial & 0 \\ 0 & \partial \end{pmatrix} + 2\begin{pmatrix} q \\ -r \end{pmatrix}\partial^{-1}(r,q), \quad \partial = \frac{\partial}{\partial x}, \quad \partial^{-1} = \frac{1}{2}\left(\int_{-\infty}^{x} - \int_{x}^{+\infty}\right)\mathrm{d}x \tag{5}$$

where $i^2 = -1$, $q = q(x,t)$, and $r = r(x,t)$ and their derivatives of any order with respect to x and t are smooth functions which vanish as x tends to infinity, and *A*, *B*, and *C* are undetermined functions of *x*, *t*, *q*, *r*, and *k*.

Derivation and Lax integrability

In view of eqs. (2)-(4), we first suppose that:

$$A = \partial^{-1}(r,q)\begin{pmatrix} -B \\ C \end{pmatrix} - \frac{1}{2}(2ik)^3 - \frac{1}{2}\left[\sum_{n=0}^{5}(2ik)^n\right]x \tag{6}$$

Then the compatibility condition of eqs. (3) and (4) are equivalent to:

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = L\begin{pmatrix} -B \\ C \end{pmatrix} - 2ik\begin{pmatrix} -B \\ C \end{pmatrix} + (2ik)^3\begin{pmatrix} -q \\ r \end{pmatrix} + \sum_{n=0}^{5}(2ik)^n\begin{pmatrix} -xq \\ xr \end{pmatrix} \tag{7}$$

Next, further letting:

$$\begin{pmatrix} -B \\ C \end{pmatrix} = \sum_{n=1}^{5}\begin{pmatrix} -b_n \\ c_n \end{pmatrix}(2ik)^{5-n} \tag{8}$$

and substituting eq. (8) into eq. (7), then comparing the coefficients of $2ik$ in eq. (7) yields:

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = L\begin{pmatrix} -b_5 \\ c_5 \end{pmatrix} + \begin{pmatrix} -xq \\ xr \end{pmatrix}, \quad \begin{pmatrix} -b_5 \\ c_5 \end{pmatrix} = L\begin{pmatrix} -b_4 \\ c_4 \end{pmatrix} + \begin{pmatrix} -xq \\ xr \end{pmatrix} \tag{9}$$

$$\begin{pmatrix} -b_4 \\ c_4 \end{pmatrix} = L\begin{pmatrix} -b_3 \\ c_3 \end{pmatrix} + \begin{pmatrix} -xq \\ xr \end{pmatrix}, \quad \begin{pmatrix} -b_3 \\ c_3 \end{pmatrix} = L\begin{pmatrix} -b_2 \\ c_2 \end{pmatrix} + \begin{pmatrix} -xq \\ xr \end{pmatrix} + \begin{pmatrix} -q \\ r \end{pmatrix} \tag{10}$$

$$\begin{pmatrix} -b_2 \\ c_2 \end{pmatrix} = L\begin{pmatrix} -b_1 \\ c_1 \end{pmatrix} + \begin{pmatrix} -xq \\ xr \end{pmatrix}, \quad \begin{pmatrix} -b_1 \\ c_1 \end{pmatrix} = \begin{pmatrix} -xq \\ xr \end{pmatrix} \tag{11}$$

Finally, from eqs. (9)-(11) we arrive at eq. (1). This process of derivation shows that the mixed spectral AKNS equations are Lax integrable.

Time-dependence of scattering data

Making use of the asymptotic property of the matrix:

$$N \to \begin{pmatrix} -\frac{1}{2}\left\{\sum_{n=0}^{5}[2i\kappa_j(t)]^n\right\}x - \frac{1}{2}[2i\kappa_j(t)]^3 & 0 \\ 0 & \frac{1}{2}\left\{\sum_{n=0}^{5}[2i\kappa_j(t)]^n\right\}x + \frac{1}{2}[2i\kappa_j(t)]^3 \end{pmatrix}, x \to +\infty \tag{12}$$

and the similar manipulations [31, 33], we can determine the time-dependence of the scattering data of the spectral problem (3):

$$\left[\kappa_j(t), \quad c_j(t), \quad R(k,t) = \frac{a(k,t)}{b(k,t)}, \quad j = 1, 2, \cdots, n\right] \tag{13}$$

$$\left[\bar{\kappa}_m(t), \quad \bar{c}_m(t), \quad \bar{R}(k,t) = \frac{\bar{a}(k,t)}{\bar{b}(k,t)}, \quad m = 1, 2, \cdots, \bar{n}\right] \tag{14}$$

as follows:

$$\kappa_{jt}(t) = -\frac{i}{2}\sum_{n=0}^{5}[2i\kappa_j(t)]^n, \quad \bar{\kappa}_{mt}(t) = -\frac{i}{2}\sum_{n=0}^{5}[2i\bar{\kappa}_m(t)]^n \tag{15}$$

$$c_j^2(t) = c_j^2(0)\mathrm{e}^{\int_0^t\left\{\sum_{n=1}^{5}n[2i\kappa_j(w)]^{n-1}+[2i\kappa_j(w)]^3\right\}\mathrm{d}w}, \quad \bar{c}_m^2(t) = \bar{c}_m^2(0)\mathrm{e}^{-\int_0^t\left\{\sum_{n=1}^{5}n[2i\bar{\kappa}_m(w)]^{n-1}+[2i\bar{\kappa}_m(w)]^3\right\}\mathrm{d}w} \tag{16}$$

$$\begin{gathered} a(k,t) = a(k,0), \quad b(k,t) = b(k,0)\mathrm{e}^{\int_0^t\{[2i\kappa_j(w)]^3\}\mathrm{d}w}, \quad \bar{a}(k,t) = \bar{a}(k,0), \\ \bar{b}(k,t) = \bar{b}(k,0)\mathrm{e}^{-\int_0^t[(2i\bar{\kappa}_m(w))^3]\mathrm{d}w} \end{gathered} \tag{17}$$

where $\kappa_j^2(0)$, $\bar{\kappa}_m^2(0)$, $c_j^2(0)$, $\bar{c}_m^2(0)$, $R(k,0) = b(k,0)/a(k,0)$, and $\bar{R}(k,0) = \bar{b}(k,0)/\bar{a}(k,0)$ are the scattering data of the spectral problem (3) in the case of $[q(x,0), r(x,0)]^T$, respectively.

Exact solutions and soliton solutions

From eqs. (2) and (13)-(17), we can obtain two formulae of exact solutions of the mixed spectral AKNS equations:

$$q(x,t) = -2K_1(x,x,t), \quad r(x,t) = \frac{K_{2x}(x,x,t)}{K_1(x,x,t)} \tag{18}$$

where $K(x,y,t) = [K_1(x,y,t), K_2(x,y,t)]^T$ satisfies Gel'fand-Levitan-Marchenko integral equation:

$$K(x,y,t) - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \bar{F}(x+y,t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \int_x^\infty F(z+x,t)\bar{F}(z+y,t)\mathrm{d}z$$
$$+ \int_x^\infty K(x,s,t) \int_x^\infty F(z+s,t)\bar{F}(z+y,t)\mathrm{d}z\mathrm{d}s = 0 \tag{19}$$

and

$$F(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R(k,t)\mathrm{e}^{ikx}\mathrm{d}k + \sum_{j=1}^{n} c_j^2 \mathrm{e}^{i\kappa_j x}, \quad \bar{F}(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{R}(k,t)\mathrm{e}^{-ikx}\mathrm{d}k - \sum_{j=1}^{\bar{n}} \bar{c}_j^2 \mathrm{e}^{-i\bar{\kappa}_j x} \tag{20}$$

are determined by the scattering data in eqs. (2) and (13)-(17).

Let $b(k,0) = \bar{b}(k,0) = 0$, we have $R(k,t) = \bar{R}(k,t) = 0$ and hence obtain n-soliton solutions of the mixed spectral AKNS equations:

$$q(x,t) = 2\mathrm{tr}[P^{-1}(x,t)\Lambda\bar{\Lambda}^T], \quad r(x,t) = -\frac{\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{tr}[P^{-1}(x,t)Q(x,t)\frac{\mathrm{d}}{\mathrm{d}x}Q^T(x,t)]}{\mathrm{tr}[P^{-1}(x,t)\Lambda\bar{\Lambda}^T]} \tag{21}$$

determined by:

$$P(x,t) = E + Q(x,t)Q^T(x,t), \quad Q(x,t) = \left[\frac{c_j(t)\bar{c}_m(t)}{\kappa_j(t) - \bar{\kappa}_m(t)}\mathrm{e}^{i[\kappa_j(t) - \bar{\kappa}_m(t)]x}\right]_{\bar{n}\times n} \tag{22}$$

$$\begin{aligned} \Lambda &= (c_1(t)\mathrm{e}^{i\kappa_1(t)x}, \quad c_2(t)\mathrm{e}^{i\kappa_2(t)x}, \cdots, \quad c_n(t)\mathrm{e}^{i\kappa_n(t)x})^T, \\ \bar{\Lambda} &= [\bar{c}_1(t)\mathrm{e}^{-i\bar{\kappa}_1(t)x}, \quad \bar{c}_2(t)\mathrm{e}^{-i\bar{\kappa}_2(t)x}, \cdots, \quad \bar{c}_n(t)\mathrm{e}^{-i\bar{\kappa}_{\bar{n}}(t)x}]^T \end{aligned} \tag{23}$$

where E is unit matrix, $\kappa_j(t)$, $c_j(t)$, $\bar{\kappa}_m(t)$, and $\bar{c}_m(t)$ are determined in eqs. (15) and (16).

Dynamical evolutions and spatial structures

To gain more insights into the non-isospectral parameter (2) and the obtained n-soliton solutions (21), we set $n = \bar{n} = 1$ and then consider their dynamical evolutions and spatial structures. In this case, eqs. (21) give one-soliton solutions:

$$q = \frac{2\bar{c}_1^2(0)\mathrm{e}^{-2i\bar{\kappa}_1(t)x - \int_0^t \left\{\sum_{n=1}^{5} n[2i\bar{\kappa}_m(w)]^{n-1} + [2\bar{\kappa}_1(w)]^3\right\}\mathrm{d}w}}{1 + \frac{c_1^2(0)\bar{c}_1^2(0)}{[\kappa_1(t) - \bar{\kappa}_1(t)]^2}\mathrm{e}^{2i[\kappa_1(t) - \bar{\kappa}_1(t)]x + \int_0^t \left\{\sum_{n=1}^{5} n[2i\kappa_m(w)]^{n-1} - \sum_{n=1}^{5} n[2i\bar{\kappa}_m(w)]^{n-1} + [2\kappa_1(w)]^3 - [2\bar{\kappa}_1(w)]^3\right\}\mathrm{d}w}} \tag{24}$$

$$r = \frac{2c_1^2(0)\mathrm{e}^{-2i\kappa_1(t)x - \int_0^t \left\{ \sum_{n=1}^{5} n[2i\kappa_m(w)]^{n-1} + [2\kappa_1(w)]^3 \right\} \mathrm{d}w}}{1 + \dfrac{c_1^2(0)\bar{c}_1^{\,2}(0)}{[\kappa_1(t)-\bar{\kappa}_1(t)]^2}\mathrm{e}^{2i[\kappa_1(t)-\bar{\kappa}_1(t)]x + \int_0^t \left\{ \sum_{n=1}^{5} n[2i\kappa_m(w)]^{n-1} - \sum_{n=1}^{5} n[2i\bar{\kappa}_m(w)]^{n-1} + [2\kappa_1(w)]^3 - [2\bar{\kappa}_1(w)]^3 \right\} \mathrm{d}w}} \tag{25}$$

where $\kappa_1(t)$ and $\bar{\kappa}_1(t)$ are determined by:

$$\kappa_{1t}(t) = -\frac{i}{2}\sum_{n=0}^{5}[2i\kappa_1(t)]^n\,, \quad \bar{\kappa}_{1t}(t) = -\frac{i}{2}\sum_{n=0}^{5}[2i\bar{\kappa}_1(t)]^n \tag{26}$$

Solving eqs. (26), we further determine:

$$\begin{aligned}\kappa_1(t) = \text{InverseFunction}\Bigg[\frac{1}{24}(4i\text{ArcTan}[2\#1] + 2i\text{ArcTan}[\frac{2\#1}{1-4\#1^2}] -\\ -2i\sqrt{3}\log[i+\sqrt{3}-4\#1] + 2i\sqrt{3}\log[-i+\sqrt{3}+4\#1] +\\ +2\log[1+4\#1^2] - \log[1-4\#1^2+16\#1^4])\&][\frac{t}{2}+C[1]\Bigg]\end{aligned} \tag{27}$$

$$\begin{aligned}\bar{\kappa}_1(t) = \text{InverseFunction}\Bigg[\frac{1}{24}(4i\text{ArcTan}[2\#1] + 2i\text{ArcTan}[\frac{2\#1}{1-4\#1^2}] -\\ -2i\sqrt{3}\log[i+\sqrt{3}-4\#1] + 2i\sqrt{3}\log[-i+\sqrt{3}+4\#1] +\\ +2\log[1+4\#1^2] - \log[1-4\#1^2+16\#1^4])\&][\frac{t}{2}+C[2]\Bigg]\end{aligned} \tag{28}$$

by introducing InverseFunction[$\cdot$]—a built-in function of MATHEMATICA 8 and two constants:

$$\begin{aligned}C[1] = \frac{1}{24}\Bigg(4i\text{ArcTan}[2\kappa_1[0]] + 2i\text{ArcTan}[\frac{2\kappa_1[0]}{1-4\kappa_1^2[0]}] -\\ -2i\sqrt{3}\log[i+\sqrt{3}-4\kappa_1[0]] + 2i\sqrt{3}\log[-i+\sqrt{3}+4\kappa_1[0]] +\\ +2\log[1+4\kappa_1^2[0]] - \log[1-4\kappa_1^2[0]+16\kappa_1^4[0]]\Bigg)\end{aligned} \tag{29}$$

$$\begin{aligned}C[2] = \frac{1}{24}\Bigg(4i\text{ArcTan}[2\bar{\kappa}_1[0]] + 2i\text{ArcTan}[\frac{2\bar{\kappa}_1[0]}{1-4\bar{\kappa}_1^2[0]}] -\\ -2i\sqrt{3}\log[i+\sqrt{3}-4\bar{\kappa}_1[0]] + 2i\sqrt{3}\log[-i+\sqrt{3}+4\bar{\kappa}_1[0]] +\\ +2\log[1+4\bar{\kappa}_1^2[0]] - \log[1-4\bar{\kappa}_1^2[0]+16\bar{\kappa}_1^4[0]]\Bigg)\end{aligned} \tag{30}$$

where #1 is the first parameter of the InverseFunction[·], & – the identification of an anonymous function.

In figs. 1 and 2, two novel simulations of dynamical evolutions of eqs. (27) and (28) are shown by selecting the parameters as $\kappa_1(0)=1$ and $\bar{\kappa}_1(0)=0$, respectively. Two spatial structures of the one-soliton solutions (24) and (25) are shown in figs. (3) and (4), where the parameters are selected as $\kappa_1(0)=0$, $\bar{\kappa}_1(0)=0.1$, $c_1(0)=0.1$ and $\bar{c}_1(0)=2$. It is easy to see from figs. 3 and 4 that the one-soliton solutions possess time-varying amplitudes in the process of propagations.

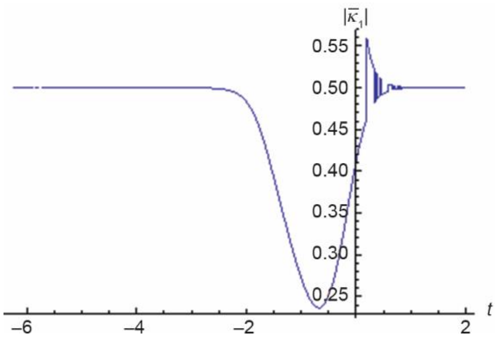

Figure 1. Dynamical evolutions of spectral parameter (27)

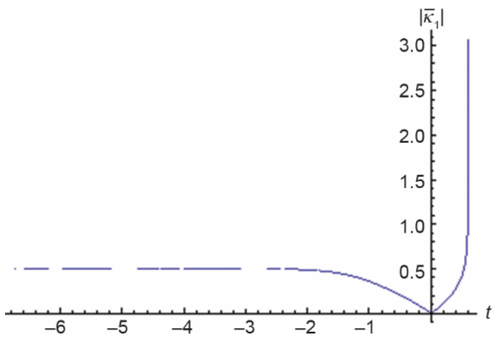

Figure 2. Dynamical evolutions of spectral parameter (28)

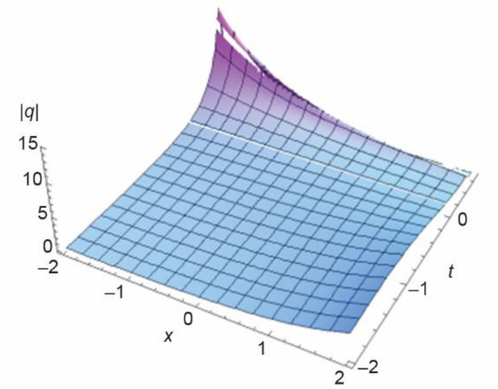

Figure 3. Spatial structure of one-soliton solution (24)

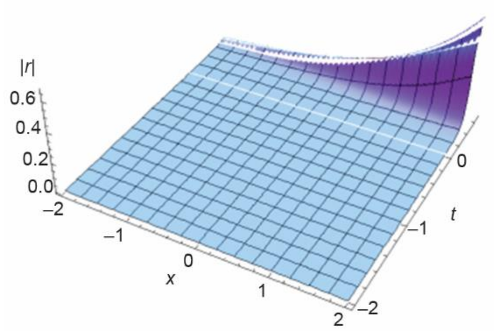

Figure 4. Spatial structure of one-soliton solution (25)

Conclusion

In summary, we have derived and solved a new and more general mixed spectral AKNS equations in the framework of IST method. This is due to the novel spectral parameter embedded into the linear spectral problem associated to the AKNS equations. Though there are similar studies [29-33], the AKNS equations, the spectral parameter and the results presented in this paper are different from those in literature. Like the Taylor series method [35, 36], the IST method, which is also called as the non-linear Fourier analysis, plays an important role in non-linear science and two scale thermodynamics [37, 38]. Variational formulations and conservation laws for AKNS equations are also helpful for studying the solution structures, and the semi-inverse method is a good tool to establishment of the needed variational formulations [39-43]**.**

Acknowledgment

This work was supported by the Natural Science Foundation of China (11547005), the Natural Science Foundation of Liaoning Province of China (20170540007), the Natural Science Foundation of Education Department of Liaoning Province of China (LZ2017002) and Innovative Talents Support Program in Colleges and Universities of Liaoning Province of China (LR2016021).

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Paper submitted: April 28, 2018
Paper revised: Septembe 1, 2019
Paper accepted: September 8, 2019