HEAT CONDUCTION IN RECTANGULAR SOLIDS WITH INTERNAL HEAT GENERATION

by

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A representative steady-state heat conduction problem in rectangular solids with uniformly distributed heat generation has been investigated analytically. An analytical solution is provided by solving a non-homogeneous PDE. A simple and accurate model is proposed to predict the dimensionless shape factor parameter for the first time. The dimensionless shape factor is obtained in the light of the solution of Poisson equation with constant wall temperature boundary conditions. The area-mean temperature is found by integration on the rectangular cross-section. The model is very concise and nice for quick real world approximations, and it provides acceptable accuracy for engineering practice.

Key words*: heat conduction, rectangular solids, dimensionless shape factor, internal heat sources, method of eigenfunction expansions*

Introduction

Heat conduction problems with internal energy sources are frequently encountered in various applications. For example, an electric current flowing through a body has the effect of an external energy addition (power input) to the internal portions of the body because of the dissipation due to electrical resistance. Since the dissipated energy is to be transferred out of the body by some heat transfer mechanism, the effect is said to be a heat source distributed throughout the body. Processes that produce similar effects are chemical reactions distributed throughout a body, nuclear reactions in a fissionable material exposed to a neutron flux, change of phase, and biological problems in fermentation. Some efforts have been made to solve analytically heat conduction problems in engineering. Rogie *et al.* [1] analytically modeled heat conduction of printed wired board with heat generation. Xu and Wang [2] analytically studied the temperature fields in a slab for laser heating. Forslund *et al.* [3] analytically modeled the moving Gaussian heat flux problem appeared in powder bed fusion application. Franca and Orlande [4] carried out an investigation on heat conduction in a transistor employing a Bayesian Approach. Shen *et al.* [5] performed an investigation on the effect of non-condensable gas on heat conduction in pressure steam sterilization.

In general, exact solutions of common and simple physical models are utilized to validate the accuracy of numerical solutions in fluid mechanics and heat transfer. Significant contributions have been made by Haji-Sheikh *et al*. [6], Aviles-Ramos *et al.* [7], Beck *et al*. [8, 9], and Beck and Cole [10] for multidimensional heat conduction problems. Beck *et al*. [9]

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discussed convergence problems for heat conduction in rectangular plates. Wang [11] presented an analytical investigation on the steady heat-conduction problem via local fractional derivative. Laraqi *et al*. [12] pointed out that heat conduction in solids subjected to non-homogenous boundary conditions are difficult to solve by using the classical methods such as integral transforms or separation of variables (SOV). Gao and Yang [13] proposed the local fractional Euler's method to consider the steady heat-conduction problem. EI Maakoul *et al*. [14] pointed out that the resistance and quadruple concepts may be utilized to work out heat conduction problems.

Deng and Ge [15] considered a generalized local fractional 2-D Helmholtz equation in steady heat transfer process. Yilmazer and Kocar [16] solved analytically heat conduction equation for an eccentric spherical annulus. Some efforts have been made to solve differential equations in fluid-flow and heat conduction problems [17-22]. Tian [23] presented a symmetry analysis of non-linear heat conduction equations. Harfash [24] focused on penetrative convection porous media with internal heat generation. Bennett and Hohmann [25] highlighted the shear stress have a significant mitigating effect on heat exchanger fouling. Monsivais *et al*. [26] focused on the conjugate heat transfer problem in a thin micro-channel. Li *et al*. [27] performed an investigation on the optimization of heat transmission paths. A heat transmission problem in the human head was considered by Mohsenyzadeh *et al*. [28]. Maitama and Zhao [29] proposed a semi-analytic method for solving non-homogeneous heat transmission problems. Wang *et al*. [30] presented an optimization study of the heat source identification.

It is difficult and time-consuming to obtain analytical solutions of heat conduction equation. The method of SOV is commonly employed to find exact solutions of heat transmission problems. Though the SOV is a powerful approach, it may not acquire exact solutions at non-homogeneous boundary conditions. The standard SOV procedure for steady heat transmission is the means recommended in heat transmission textbooks. For instance, Carslaw and Jaeger [31], Arpaci [32], Ozisik [33], Schneider [34], Bejan [35], and Kakac *et al.* [36] point out that the SOV is the appropriate way to work out heat transmission problems. However, the SOV often produces analytical solutions having low convergence. The problem was clearly identified in [8, 37, 38]. Hayat *et al.* [39] solved the convergence of the temperature equations employing the homotopy technique. The convergence of the SOV solutions has not been detailedly discussed in heat conduction textbooks published over the last six decades. Therefore, solving the area-mean temperature of an object by integration is extremely difficult.

Selecting the proper form of solutions can reduce significantly the number of terms in the summations. This paper deals with a typical 2-D heat conduction problem in the rectangle with consistently distributed heat generation. The series solution having rapid convergence is obtained by solving a non-homogeneous PDE utilizing the method of eigenfunction expansions. A model is first proposed for precise prediction of a dimensionless shape factor parameter. The conduction shape factor which is composed of area-average temperature difference, internal heat generation, thermal conductivity of the solid media, and a representative dimension of the cross-section. The developed solution can be easily applied to engineering practice and verification of the accuracy of numerical solutions.

Theoretical analysis

In this work, it is considered that typical 2-D steady-state heat conduction problems with consistently distributed heat generation which are idealizations of more involved problems frequently encountered in practice. For instance, thermal management system controls lithium batteries to work in a suitable temperature range for electric vehicle application.

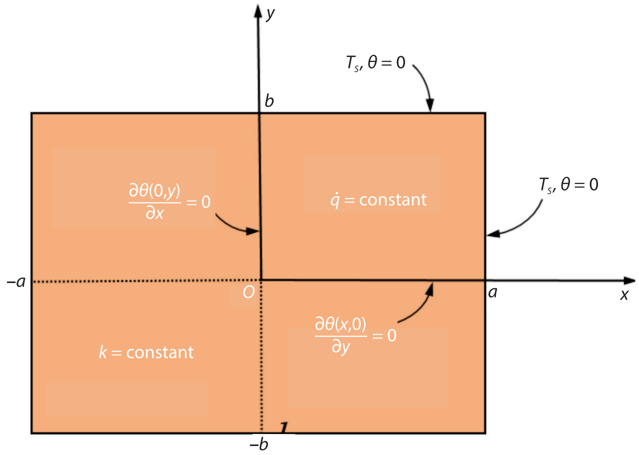

Figure 1. Heat conduction in a rectangular solid

The steady heat transmission in the rectangular solid media $2a \times 2b \times L$ with uniform surface temperature T_s is shown in fig. 1. The surface of the solid media is cooled by external systems so that its temperature is nearly constant. The power of the heat source in the solid is $\dot{q}$ and the thermal conductivity of the solid media is k. It is assumed that the dimension of the rectangular solid in the z-direction is sufficiently large so that heat flow may be considered as 2-D.

We are interested in the temperature distribution $T(x, y)$ of the rectangle, in order to simplify the solution we introduce the temperature difference expressed as $\theta = T(x, y) - T_s$. The governing equation of heat transmission in this solid with heat generation can be expressed:

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = -\frac{\dot{q}}{k} \tag{1}$$

where k represents the thermal conductivity of the rectangular solid media and $\dot{q}$ – the uniformly distributed volumetric heat generation. Due to symmetry, the boundary conditions can be written:

$$\theta(x,b) = 0,\ \ 0 \le x < a \tag{2}$$

$$\theta(a,y) = 0,\ \ 0 \le y < b \tag{3}$$

$$\left.\frac{\partial\theta}{\partial y}\right|_{y=0} = 0,\ \ 0 \le x \le a \tag{4}$$

$$\left.\frac{\partial\theta}{\partial x}\right|_{x=0} = 0,\ \ 0 \le y \le b \tag{5}$$

The transformed Poisson equation and the corresponding boundary conditions may be expressed:

$$\varepsilon_2 \frac{\partial^2\theta}{\partial\left(\dfrac{x}{a}\right)^2} + \frac{\partial^2\theta}{\partial\left(\dfrac{x}{a}\right)^2} = \frac{\dot{q}b^2}{k} \tag{6}$$

where $\varepsilon = b/a$, the ratio of width to height of the rectangular section:

$$\theta\left(\frac{x}{a},1\right)=0,\quad 0\le\frac{x}{a}<1 \tag{7}$$

$$\theta\left(1,\frac{y}{b}\right)=0,\quad 0\le\frac{y}{b}<1 \tag{8}$$

$$\left.\frac{\partial\theta}{\partial\left(\dfrac{y}{b}\right)}\right|_{\frac{y}{b}}=0,\quad 0\le\frac{x}{a}\le 1 \tag{9}$$

$$\left.\frac{\partial\theta}{\partial\left(\dfrac{x}{a}\right)}\right|\frac{x}{a}=0,\quad 0\le\frac{y}{b}\le 1 \tag{10}$$

Employing the method of eigenfunction expansions, it is assumed that the solution of the temperature distribution is:

$$\theta(x,y)=-\frac{\dot{q}b^2}{k}\sum_{n=1}^{\infty}X_n\left(\frac{x}{a}\right)\cos\left(\delta_n\frac{y}{b}\right) \tag{11}$$

where δ_n is a cluster of eigenvalues, X_n – the cluster of functions of x/a, and $\cos(\delta_n y/b)$ are the cluster of eigenfunctions. The temperature distribution satisfies the boundary condition, eq. (9). Further, substituting the solution into the boundary condition, eq. (7), the eigenvalues δ_n are determined:

$$\delta_n=(2n-1)\frac{\pi}{2},\ n=1,2\ldots \tag{12}$$

Then, the temperature distribution is obtained:

$$\theta(x,y)=\frac{2\dot{q}b^2}{k}\sum_{n=1}^{\infty}\frac{\sin\delta_n\cos\left(\delta_n\dfrac{y}{b}\right)}{\delta_n^3}\left[1-\frac{\cosh\left(\dfrac{\delta_n}{\varepsilon}\dfrac{x}{a}\right)}{\cosh\left(\dfrac{\delta_n}{\varepsilon}\right)}\right] \tag{13}$$

The method of SOV is generally used to determine analytical solutions of heat conduction problems, however, the SOV often produces analytical solutions having low convergence [8]. To obtain the average temperature of a body, it is critical and desired to utilize a method of evaluation with fast convergence. However, the issue of the convergence of analytical solutions has not been discussed in advanced heat conduction books published over the last six decades. Hence, further solving for the area-mean temperature by integration is extremely difficult. As eq. (13) consists of a rapidly convergent series, the value of any point in the cross-section can be easily and quickly determined using this series solution.

The average temperature rise of the rectangular solid is obtained by integration of eq. (13) for the rectangular cross-section:

$$\theta m=\frac{1}{A}\int_A\theta\mathrm{d}A=\int_{-1}^{1}\int_{-1}^{1}\theta\mathrm{d}\frac{x}{a}\mathrm{d}\frac{y}{b}=\frac{2\dot{q}b^2}{k}\sum_{n=1}^{\infty}\frac{\varepsilon}{\delta_n^5}\left[\frac{\delta_n}{\varepsilon}-\tanh\left(\frac{\delta_n}{\varepsilon}\right)\right] \tag{14}$$

where θ_m is the area-mean temperature variation. Take a rectangular solid with $\varepsilon = 1$ as an example, the variation of the values of the series solution with the number of terms is demonstrated in fig. 2. In details, $\dot{q}$, k are constants, here taken as unity. The convergence analysis of the developed solution demonstrates that the single term solution gives acceptable accuracy for engineering practice.

Further, absolute values of the relative errors δ in the average temperature rise of rectangular solids with various aspect ratios ($0.1 \le \varepsilon \le 1$) for the single term solution and two terms solution are showed in fig. 3. In details, the largest error compared to the convergence value occurs when the aspect ratio is equal to 1, and this error of the single term solution is less than 0.7%. When higher accuracy is required, the two terms approximation is accurate enough because of the fast convergence performance (the largest difference is in this case less than 0.05%) [40].

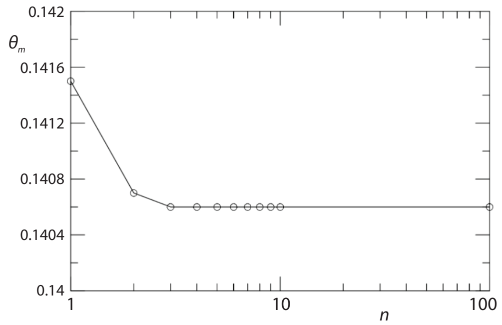

Figure 2. The values of the developed solution of the average temperature rise *vs.* the number of terms in the series

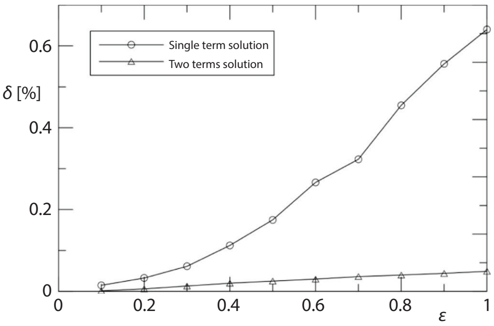

Figure 3. Absolute values of the relative errors in the average temperature rise *vs.* the number of terms in the series

Thus, the two terms approximation is recommended:

$$\theta_m = \left\{ \frac{1}{3} - \frac{64\varepsilon}{\pi^5} \left[\tanh\left(\frac{\pi}{2\varepsilon}\right) + \frac{1}{243} \tanh\left(\frac{3\pi}{2\varepsilon}\right) \right] \right\} \frac{\dot{q} b^2}{k} \tag{15}$$

Since eq. (15) is based on theoretical analysis, it has greater accuracy compared to those solutions relying on the curve fitting method. In addition, the solution of the average temperature provides a good insight into the underlying heat conduction mechanism and it can be recast:

$$\dot{q} = \left\{ \frac{1}{3} - \frac{64\varepsilon}{\pi^5} \left[\tanh\left(\frac{\pi}{2\varepsilon}\right) + \frac{1}{243} \tanh\left(\frac{3\pi}{2\varepsilon}\right) \right] \right\}^{-1} \frac{k\theta_m}{b^2} \tag{16}$$

For steady heat transmission problems in solid materials, it is often handled on the basis of conduction shape factor parameter. In numerous cases, heat conduction problems can be quite rapid worked out by employing existing solutions to the heat conduction equation. However, these available solutions are presented in terms of a dimensional shape factor in heat and mass transfer textbooks [41] and are not easy to utilize.

It is well known that engineering practice is based on the combination of theoretical analysis and experimental research, especially for heat transmission which is heavily dependent on experimental means. In general, so far researchers and engineers followed the standard practice of providing practical results on a simple diagram in terms of dimensionless groups which are quite convenient to use. The graphic description of these existing results avoided a lot of difficulties related to dealing with engineering problems on the slide rule.

For complex engineering problems, it is important and necessary to introduce a useful dimensionless group and describe existing solutions in dimensionless form. The determination of the proper dimensionless parameters provides a strong means for simplifying and solving engineering problems. Moreover, the shape factor of heat transmission problems with internal heat generation is rarely reported. Therefore, using scale analysis of the heat conduction equation, a new dimensionless shape factor parameter of interest is proposed herein and defined:

$$\Theta = \frac{k\theta_m}{\overline{q}_s \ell} \tag{17}$$

where $\overline{q}_s$ is the average surface heat flux and ℓ – the arbitrary scaling parameter that is related to the length scales of the cross-section. This may be associated with the temperature gradient by means of the integration for the perimeter, P, of the domain, and defined:

$$\overline{q}_s = \frac{1}{P}\int_P \left(-k\frac{\partial\theta}{\partial n}\right)\mathrm{d}s \tag{18}$$

where $\partial/\partial n$ is the directional derivative normal to the perimeter and ds – the arc length of the perimeter.

The average surface heat flux is given:

$$\overline{q}_s = \dot{q}\frac{A}{P} \tag{19}$$

The ratio of area to perimeter can be expressed in terms of b and aspect ratio:

$$\frac{A}{P} = \frac{b}{1+\varepsilon} \tag{20}$$

When the scaling parameter is selected as $\ell = D_h$, the dimensionless shape factor becomes:

$$\Theta_{D_h} = \frac{k\theta_m}{\overline{q}_s D_h} \tag{21}$$

For rectangular solids, we can obtain the dimensionless shape factor from the previous equation. Substituting eqs. (16) and (20) into eq. (19), the average surface heat flux can be expressed:

$$\overline{q}_s = \frac{12}{(1+\varepsilon)^2\left\{1-\dfrac{192\varepsilon}{\pi^5}\left[\tanh\left(\dfrac{\pi}{2\varepsilon}\right)+\dfrac{1}{243}\tanh\left(\dfrac{3\pi}{2\varepsilon}\right)\right]\right\}}\frac{k\theta_m}{D_h} = \frac{1}{\Theta(\varepsilon)}\frac{k\theta_m}{D_h} \tag{22}$$

This relation confirms that the average surface heat flux scales as $k\theta_m/D_h$. Hence, the shape factor for the rectangle with internal heat source can be written:

$$\Theta(\varepsilon) = \frac{1}{12}(1+\varepsilon)^2\left\{1-\frac{192\varepsilon}{\pi^5}\left[\tanh\left(\frac{\pi}{2\varepsilon}\right)+\frac{1}{243}\tanh\left(\frac{3\pi}{2\varepsilon}\right)\right]\right\} \tag{23}$$

The results of dimensionless shape factor for the rectangle are obtained from eq. (23) and presented in fig. 4 for the aspect ratio range from 0.01-1.

For rectangular solids, the model of the dimensionless shape factor, Θ, is developed by means of a least-square fit of the results calculated from eq. (23) and expressed:

$$\Theta(\varepsilon) = 0.0829 + 0.1256\varepsilon - 0.0707\varepsilon^2 + 0.0026\varepsilon^3 \tag{24}$$

The conduction shape factor, Θ, is only a function of the aspect ratio and the model is very concise and nice for quick real world approximations, and it provides acceptable accuracy for engineering practice.

We can observe from fig. 4 that the shape factor curve is rather steep at low aspect ratios but flattens out starting with an aspect ratio value of about 0.8. The increase in shape factor with the aspect ratio is not so pronounced at high aspect ratios. It is well understood that heat conduction is less influenced by the aspect ratio of the geometry as the aspect ratio increases.

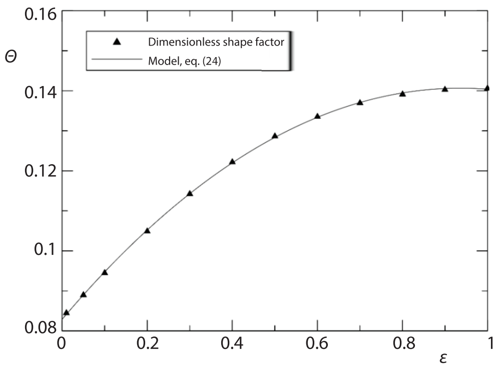

Figure 4. Dimensionless shape factor as a function of aspect ratio

Results and discussion

The reliability and performance of electronic chips and batteries are significantly affected by their operating temperature [42-44]. Determining accurately of the temperature distribution and analyzing the influence of aspect ratio on temperature distribution are critically paramount for designing an effective cooling scheme of the thermal management for electronic applications and electric vehicle applications. We investigated analytically steady-state heat conduction with uniformly distributed volumetric heat sources in long rectangular solids. Temperature distributions from eq. (13) for three typical aspect ratios 1, 0.5, and 0.2 are presented in this section.

Temperature distribution from eq. (13) for ε = 1 is shown in fig. 5. A crucial feature of the solution that should be captured is axial symmetry and diagonal symmetry duo to $\theta = 0$ at all boundaries of the square cross-section. Figure 5(a) demonstrates the axial and diagonal symmetries of the temperature distribution. Another significant feature of the solution that should be captured is that the temperature gradient gradually increases from the origin to edges of the rectangle. This trend is clearly observed in fig. 5, and fig. 5(b) demonstrates the 3-D structure of the temperature distribution which is a parabolic profile with the maximum temperature in the core region and minimum temperature at the walls.

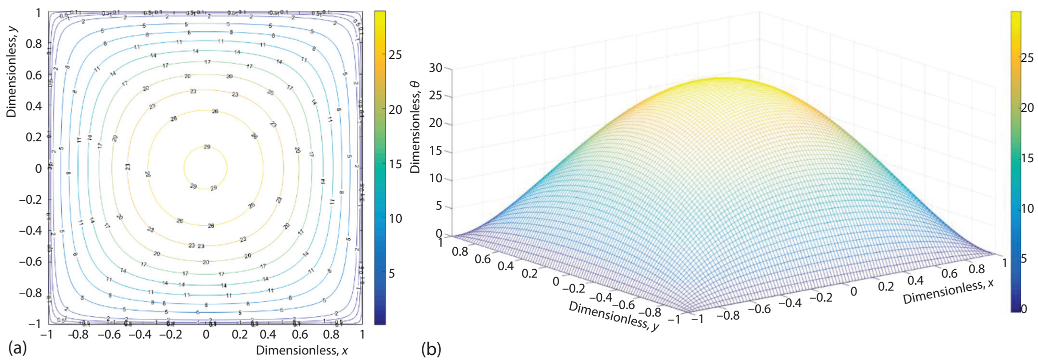

Figure 5. Temperature distribution from eq. (13) for ε = 1; (a) computational domain and (b) the 3-D structure of the temperature distribution

Figures 6 and 7 demonstrate the computed results of the temperature distribution on the cross-section using eq. (13) for aspect ratios 0.5 and 0.2, respectively. The temperature dis-

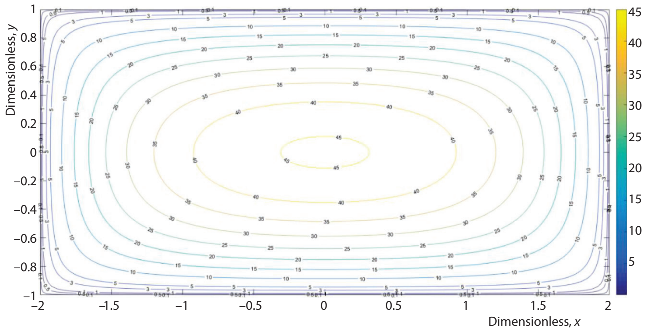

Figure 6. Temperature distribution from eq. (13) for ε = 0.5

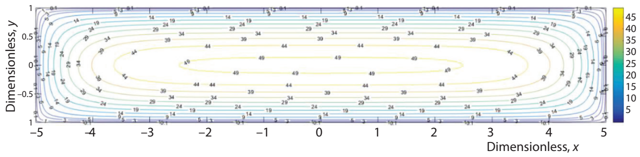

Figure 7. Temperature distribution from eq. (13) for ε = 0.2

tribution corresponding to a value of θ may be obtained, and components of the heat flux can be obtained by utilizing eq. (13) with Fourier's law. The axial symmetry of the obtained results can be observed in figs. 6(a) and 7(a). The temperature distribution is symmetric about $x = 0$ and $y = 0$, with $\partial T/\partial x = 0$ at $x = 0$ and $\partial T/\partial y = 0$ at $y = 0$ in detail. Hence, by Fourier's law, we could know the symmetry planes at $x = 0$ and $y = 0$ are adiabatic and therefore, are heat flow lines.

An interesting variation trend of the temperature gradient can be found that it increases from the origin to edges of the rectangle, and larger temperature gradient appears near the walls. As can be seen from figs. 5-7, compared with numerical methods, it is a great advantage that more accurate values can be obtained from eq. (13) at any position of the rectangle, especially near the walls and at the corners. Compared with numerical means, the developed solution is a more appropriate choice because its cost is negligible. In case of numerical solution, the computational grid should be refined near the region of high temperature gradients in order to achieve grid independency. This is not required when the present analytical approach is used. In addition, it is visually observed for all figures that the maximum temperature occurs in the core region and the minimum temperature at the walls. Although not shown here, similar trends are also observed for other aspect ratios.

Conclusion

This paper provides an exact analytical solution for steady-state heat transmission with uniformly distributed volumetric heat sources in long rectangular solids. A dimensionless shape factor parameter was introduced and its solution was first obtained. This factor depends on the area-average temperature difference, the average wall heat flux, the thermal conductivity and a characteristic scaling parameter of the cross-section. The analytical solution provides highly precise values of the temperature distribution, and the area-average temperature difference is found by integration on the rectangular cross-section. The obtained solution can be easily applied to engineering practice and verification of numerical solutions.

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Nomenclature

A	– cross-sectional area, [m^2]
a	– major semi-axis of rectangle, [m]
b	– minor semi-axis of rectangle, [m]
D_h	– hydraulic diameter (= $4A/P$), [m]
k	– thermal conductivity, [Wm^{-1}K^{-1}]
L	– length of solid, [m]
P	– perimeter, [m]
$\dot{q}$	– internal volumetric heat sources, [Wm^{-3}]
q_s, $\overline{q}_s$	– heat flux and mean surface heat flux, [Wm^{-2}]
T, T_s	– temperature and surface temperature, [K]
X_n	– function of x/a, [–]
x, y, z	– Cartesian co-ordinates, [m]

Greek symbols

δ_n	– eigenvalues, [–]
ε	– aspect ratio (= b/a), [–]
Θ	– dimensionless shape factor, [–]
θ	– temperature rise [= $T(x, y) - T_s$], [K]
θ_m	– area-average temperature rise, [K]

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