NATUREL MATES OF EQUIAFFINE SPACE CURVES IN AFFINE 3-SPACE

by

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In this study, we investigated the natural mates of equiaffine curves with constant equiaffine curvatures, associated to equiaffine frame in affine 3-space and we gave the position vectors under some conditions.

Key words: *affine space, equiaffine frame, natural mate curve*

Introduction

The affine differential geometry is the study of differential invariants with respect to the group of affine transformation. The group of affine motion's special linear transformation namely the group of equiaffine or unimodular transformations consist of volume preserving, $\det(a_{jk}) = 1$, linear transformations together with translation such:

$$x_j^* = \sum_{k=1}^{3} a_{jk} x_k + c_j$$

where $j = 1, 2, 3$. This transformations group denoted by $ASL(3, IR) := SL(3, IR) \times IR^3$ and comprising diffeomorphisms of IR^3 that preserve some important invariants such as curvatures in curve theory as well. An equiaffine group is also called an Euclidean group [1].

Salkowski and Schells [2] gave the equiaffine frame, Kreyszig and Pendl [1] gave the characterization of spherical curves in both Euclidean and affine 3-spaces. Su [3] classified the curves in affine 3-space by using equiaffine frame.

A set of points that corresponds to a vector of vector space, V, constructed on a field is called an *affine space* associated with vector space V. We denote A_3 as affine 3-space associated with IR^3. Let:

$$\alpha : J \longrightarrow A_3$$

represent a curve in A_3, where $J = (t_1, t_2) \subset IR$ is fixed and open interval. Regularity of a curve in A_3 is defined as $|\dot{\alpha}\ \ddot{\alpha}\ \dddot{\alpha}| \neq 0$ on J, where $\dot{\alpha} = \mathrm{d}\alpha/\mathrm{d}t$, *etc.* Then, we may associate α with the invariant parameter:

$$s = \sigma(t) = \int_{t_1}^{t_2} \left|\dot{\alpha} \quad \ddot{\alpha} \quad \dddot{\alpha}\right|^{1/6} \mathrm{d}t$$

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which is called the *affine arc length* of $\alpha(s)$. The co-ordinates of a curve are given by three linearly independent solutions of the equation:

$$\alpha^{(iv)}(s)+\kappa_\alpha(s)\alpha''(s)+\tau_\alpha(s)\alpha'(s)=0 \tag{1}$$

under the condition:

$$\left|\alpha'(s) \quad \alpha''(s) \quad \alpha'''(s)\right|=1 \tag{2}$$

where $\kappa_\alpha(s)$ and $\tau_\alpha(s)$ are differentiable functions of s and $\alpha' = \mathrm{d}\alpha/\mathrm{d}s$. The vectors $\alpha'(s)$, $\alpha''(s)$, and $\alpha'''(s)$ are called tangent, affine normal and affine binormal vectors, respectively, and the planes $sp\{\alpha'(s), \alpha''(s)\}$, $sp\{\alpha'(s), \alpha'''(s)\}$, $sp\{\alpha''(s), \alpha'''(s)\}$ are called osculating, rectifying, and normal planes of the curve $\alpha(s)$. Thus, the frame:

$$T'(s)=N(s),\ \ N'(s)=B(s),\ \ B'(s)=-\tau_\alpha T(s)-\kappa_\alpha T(s) \tag{3}$$

is called equiaffine frame, where $\kappa_\alpha(s)$ and $\tau_\alpha(s)$ are called equiaffine curvature and equiaffine torsion, which are given:

$$\kappa_\alpha=\left|\alpha'(s) \quad \alpha'''(s) \quad \alpha^{(iv)}(s)\right| \tag{4}$$

$$\tau_\alpha=-\left|\alpha''(s) \quad \alpha'''(s) \quad \alpha^{(iv)}(s)\right| \tag{5}$$

[2-6].

Hu obtained some equiaffine curves with constant equiaffine affine curvatures by using Shengjin's formulae and by solving ODE (1) for each cases and gave the following classification [5]. We separated this classification into proper six types.

Any non-degenerate equiaffine space curve $\alpha(s)$ with constant equiaffine curvatures $\kappa_\alpha(s) := \kappa_\alpha$ and $\tau_\alpha(s) := \tau_\alpha$, is equiaffinely equivalent to one of the following curves:

Type 1:

$$\alpha(s)=\left(s,\frac{1}{2}s^2,\frac{1}{2}s^3\right)$$

Type 2:

$$\alpha(s)=\left(\mathrm{e}^{\phi s},\ s\mathrm{e}^{\phi s},\ \frac{-1}{18\phi^5}\mathrm{e}^{-2\phi s}\right)$$

where $\phi = (\kappa_\alpha/2)^{1/3}$.

Type 3: For $\kappa_\alpha = 0$ and $\tau_\alpha > 0$:

$$\alpha(s)=\frac{1}{\sqrt{\tau_\alpha}}\left(-s,\ \frac{\sin(\sqrt{\tau_\alpha}s)}{\sqrt{\tau_\alpha}},\ \frac{\cos(\sqrt{\tau_\alpha}s)}{\sqrt{\tau_\alpha}}\right)$$

Type 4: For $\kappa_\alpha \neq 0$:

$$\alpha(s)=\left(\frac{\mathrm{e}^{-2as}}{2ab(9a^2+b^2)(a^2+b^2)},\ \mathrm{e}^{as}\sin(bs),\ \mathrm{e}^{as}\cos(bs)\right)$$

where

$$a=\frac{1}{6}\left\{\sqrt[3]{\frac{3\left[9\kappa_\alpha+\sqrt{12(\tau_\alpha)^3+81(\kappa_\alpha)^2}\right]}{2}}+\sqrt[3]{\frac{3\left[9\kappa_\alpha-\sqrt{12(\tau_\alpha)^3+81(\kappa_\alpha)^2}\right]}{2}}\right\} \tag{6}$$

$$b=\frac{\sqrt{3}}{6}\left\{\sqrt[3]{\frac{3\left[9\kappa_\alpha+\sqrt{12(\tau_\alpha)^3+81(\kappa_\alpha)^2}\right]}{2}}-\sqrt[3]{\frac{3\left[9\kappa_\alpha-\sqrt{12(\tau_\alpha)^3+81(\kappa_\alpha)^2}\right]}{2}}\right\} \tag{7}$$

Type 5: For $\kappa_\alpha = 0$ and $\tau_\alpha < 0$:

$$\alpha(s)=\frac{-1}{\tau_\alpha}\left[-\sqrt{-\tau_\alpha}s,\sinh\left(\sqrt{-\tau_\alpha}s\right),\cosh\left(\sqrt{-\tau_\alpha}s\right)\right]$$

Type 6: For $(27/2)\kappa_\alpha(-3\tau_\alpha)^{-3/2} \in (-1, 1)$ and $\tau_\alpha < 0$

$$\alpha(s)=\left(\frac{\mathrm{e}^{-2ms}}{4mn(9m^2-n^2)(m^2-n^2)},\ \mathrm{e}^{(m+n)s},\ \mathrm{e}^{(m-n)s}\right)$$

where

$$m=\frac{1}{3}\sqrt{-3\tau_\alpha}\cos\left\{\frac{1}{3}\arccos\left[\frac{27}{2}\kappa_\alpha(-3\tau_\alpha)^{-3/2}\right]\right\} \tag{8}$$

$$n=\sqrt{-\tau_\alpha}\sin\left\{\frac{1}{3}\arccos\left[\frac{27}{2}\kappa_\alpha(-3\tau_\alpha)^{-3/2}\right]\right\} \tag{9}$$

On the other hand, there are a lot of studies under the concepts of binormal-direction curve, *W*-direction curve, conjugate mate curve, principal normal adjoint curve and binormal adjoint curve, *etc.* in different spaces denoted with metric, by using a curve on a region consisting of that curve but Deshmukh, Chen, and Alghanemi defined the natural mate of a space curve in Euclidean 3-space as tangent to the principal normal vector field of given space curve, along the curve not for a curve on a region [7-11]. We use this definition and extend affine tangent, affine normal and affine binormal mates of space curves in affine 3-space, by using equiaffine frame and giving equiaffine curvatures.

Natural mate curves of space curves

In this section, we introduce the affine tangent, affine normal and affine binormal mates of space curves in affine 3-space and obtain position vectors of those curves with constant equiaffine curvatures. Let $\alpha(s)$ be a regular curve with affine arc-length parameter, s, we define the curve $\beta(s^*)$ as affine tangent mate curve of $\alpha(s)$ with the relation:

$$\beta(s^*)=\int T_\alpha(s)\mathrm{d}s \tag{10}$$

The tangent vectors of $\beta(s^*)$ are common with the tangent vectors of the curve $\alpha(s)$ at each corresponding points along $\alpha(s)$. We take the parameter s^* as equiaffine arc-length parameter of $\beta(s^*)$ and we denote the derivation with respect to s^* as previously doted. By differentiating eq. (10) with respect to s and by using eq. (3), it is easily obtained:

$$T_\beta(s^*)=\frac{1}{\sigma}T_\alpha(s) \tag{11}$$

$$N_\beta(s^*)=\frac{-\sigma'}{\sigma^3}T_\alpha(s)+\frac{1}{\sigma^2}N_\alpha(s) \tag{12}$$

$$B_\beta(s^*)=PT_\alpha(s)+QN_\alpha(s)+RB_\alpha(s) \tag{13}$$

where $\sigma = \mathrm{d}s^*/\mathrm{d}s$ and:

$$P = -\frac{\sigma\sigma'' - 3(\sigma')^2}{\sigma^5}, \quad Q = \frac{-3(\sigma')^2}{\sigma^4}, \quad R = \frac{1}{\sigma^3}$$

Since s^* is the equiaffine arc-length parameter, then $|\dot{\beta}(s^*)\ \ddot{\beta}(s^*)\ \dddot{\beta}(s^*)| = 1$ should satisfy, so we get:

$$\sigma = \frac{\mathrm{d}s^*}{\mathrm{d}s} = \pm 1$$

that is:

$$s^* = \varepsilon s + c_0$$

where $\varepsilon = \pm 1$. Thus $P = Q = 0$, $R = \pm 1$, and eqs. (11), (12), and (13) turn to:

$$T_\beta(s^*) = \varepsilon T_\alpha(s), \;\; N_\beta(s^*) = N_\alpha(s), \;\; B_\beta(s^*) = \varepsilon B_\alpha(s) \tag{14}$$

By differentiating eq. (14) with respect to s, we get:

$$\dddot{\beta}(s^*) = -\tau_\alpha T(s) - \kappa_\alpha N(s)$$

and from the definition of equiaffine curvatures, we obtain $\kappa_\beta = \kappa_\alpha$ and $\tau_\beta = \varepsilon\tau_\alpha$. It is obvious that the curve $\alpha(s)$ and its affine tangent mate curve have the same shape. By using main classification of curves with constant equiaffine affine curvatures, we give the following theorem.

Theorem 1. Let $\alpha(s)$ be any non-degenerate equiaffine space curve with constant equiaffine curvatures $\kappa_\alpha(s) := \kappa_\alpha$ and $\tau_\alpha(s) := \tau_\alpha$, then its affine tangent mate curves:

– if $\alpha(s)$ is *type 1* curve, then its affine tangent mate curve:

$$\beta(s^*) = \left[\varepsilon s^* + c, \frac{1}{2}\left(\varepsilon s^* + c\right)^2, \frac{1}{2}\left(\varepsilon s^* + c\right)^3\right]$$

– if $\alpha(s)$ is *type 2* curve, then its affine tangent mate curve:

$$\beta(s^*) = \left[\mathrm{e}^{\phi(\varepsilon s^* + c)}, (\varepsilon s^* + c)\mathrm{e}^{\phi(\varepsilon s^* + c)}, \frac{-1}{18\phi^5}\mathrm{e}^{-2\phi(\varepsilon s^* + c)}\right]$$

where $(\kappa_\alpha/2)^{1/3}$,

– if $\alpha(s)$ is *type 3* curve, with $\tau_\alpha > 0$, then its affine tangent mate curve:

$$\beta(s^*) = \frac{1}{\sqrt{\tau_\alpha}}\left\{-(\varepsilon s^* + c), \frac{\sin\left[\sqrt{\tau_\alpha}(\varepsilon s^* + c)\right]}{\sqrt{\tau_\alpha}}, \frac{\cos\left[\sqrt{\tau_\alpha}(\varepsilon s^* + c)\right]}{\sqrt{\tau_\alpha}}\right\}$$

– if $\alpha(s)$ is type 4 curve, then its affine tangent mate curve:

$$\beta(s^*) = \left\{\frac{\mathrm{e}^{-2a(\varepsilon s^* + c)}}{2ab(9a^2 + b^2)(a^2 + b^2)}, \mathrm{e}^{a(\varepsilon s^* + c)}\sin[b(\varepsilon s^* + c)], \mathrm{e}^{a(\varepsilon s^* + c)}\cos[b(\varepsilon s^* + c)]\right\}$$

where a and b are as in eqs. (6) and (7), respectively.

– if $\alpha(s)$ is type 5 curve with $\tau_\alpha < 0$ then its affine tangent mate curve:

$$\beta(s^*) = \left\{-(-\tau_\alpha)^{-1/2}(\varepsilon s^* + c), (-\tau_\alpha)^{-1}\sinh\left[\sqrt{-\tau_\alpha}(\varepsilon s^* + c)\right], (-\tau_\alpha)^{-1}\cosh\left[\sqrt{-\tau_\alpha}(\varepsilon s^* + c)\right]\right\}$$

– if $\alpha(s)$ is type 6 curve then its affine tangent mate curve:

$$\beta(s^*) = \left[\frac{\mathrm{e}^{-2m(\varepsilon s^* + c)}}{4mn(9m^2 - n^2)(m^2 - n^2)}, \mathrm{e}^{(m+n)(\varepsilon s^* + c)}, \mathrm{e}^{(m-n)(\varepsilon s^* + c)}\right]$$

where m and n are as in eqs. (8) and (9), respectively.

Assume that $\beta(s^*)$ be affine normal mate curve of $\alpha(s)$ then $\beta(s^*)$ is tangent to normal vectors of $\alpha(s)$ at each corresponding point, along $\alpha(s)$, we have:

$$\beta(s^*) = \int N_\alpha(s)\mathrm{d}s \tag{15}$$

and by differentiating eq. (15) with respect to s and by using eq. (3), we obtain the first, the second and the third derivations of $\beta(s^*)$:

$$T_\beta(s^*) = \frac{1}{\sigma} N_\alpha(s) \tag{16}$$

$$N_\beta(s^*) = \frac{-\sigma'}{\sigma^3} N_\alpha(s) + \frac{1}{\sigma^2} B_\alpha(s) \tag{17}$$

$$B_\beta(s^*) = PT_\alpha(s) + QN_\alpha(s) + RB_\alpha(s) \tag{18}$$

where

$$\sigma = \frac{\mathrm{d}s^*}{\mathrm{d}s} = \sqrt[6]{-\tau_\alpha} \tag{19}$$

and

$$P = -\frac{\tau_\alpha}{\sigma^3}, \quad Q = \frac{-1}{\sigma^5}\left\{\kappa_\alpha \sigma^2 + \sigma\sigma'' - 3(\sigma')^2\right\}, \quad R = \frac{-3(\sigma')}{\sigma^4} \tag{20}$$

Equation (19) restricts us in the case that $\tau_\alpha < 0$ and also the condition of regularity of $\beta(s^*)$ requires that $\alpha(s)$ has to be the curve with non-zero equiaffine torsion. From eq. (19), the fourth derivation of $\beta(s^*)$:

$$\ddot{B}_\beta(s^*) = \frac{(P' - R\tau_\alpha)}{\sigma} T_\alpha(s) + \frac{(P + Q' - R\kappa_\alpha)}{\sigma} N_\alpha(s) + \frac{(Q + R')}{\sigma} B_\alpha(s) \tag{21}$$

and also, from the definition of equiaffine affine curvature and affine torsion, we obtain:

$$\kappa_\beta(s^*) = \frac{1}{\sigma^2}\left\{RP' - PR' - PQ - R^2\tau_\alpha\right\} \tag{22}$$

and

$$\tau_\beta(s^*) = \frac{1}{\sigma^4}\left\{\sigma\left\{QP' - PQ' - P^2 + PR\kappa_\alpha - QR\tau_\alpha\right\} + \sigma'\left\{RP' - PR' - PQ - R^2\tau_\alpha\right\}\right\} \tag{23}$$

From (20), eqs. (22) and (24) turn:

$$\kappa_\beta(s^*) = \frac{-1}{\sigma^{10}}\left\{\kappa_\alpha\tau_\alpha\sigma^2 + 4\tau_\alpha\sigma\sigma'' + 3\tau_\alpha(\sigma')^2 - 3\tau'_\alpha\sigma\sigma'\right\}$$

and

$$\tau_\beta(s^*) = \frac{-1}{\sigma^{10}}\left\{\sigma\left\{(\tau_\alpha)^2 + \kappa'_\alpha\tau_\alpha - \kappa_\alpha\tau'_\alpha\right\} + \kappa_\alpha\tau_\alpha\sigma' - \tau'_\alpha\sigma'' + \tau_\alpha\sigma'''\right\}$$

If the curve $\alpha(s)$ has the constant affine curvature and non-zero constant affine torsion, then its affine normal mate curve has also constant affine curvature and affine torsion:

$$\kappa_\beta(s^*) = \frac{\kappa_\alpha}{\sqrt[3]{-\tau_\alpha}}$$

and

$$\tau_\beta(s^*) = -\sqrt{-\tau_\alpha}$$

with the parameterization

$$s=\frac{s^{*}+c}{\sqrt[6]{-\tau_{\alpha}}}$$

Since the *type 1* curves have the curvatures $\kappa_\alpha = \tau_\alpha = 0$, then they do not have any normal mate curves. It is clear that if equiaffine curve $\alpha(s)$ with constant equiaffine curvatures has a normal mate curve, then it has to be a curve with negative equiaffine torsion. Due to this situation, *type 3* curves do not have real normal mate curves. Thus, we give the following theorem by using the main classification of curves with constant equiaffine affine curvatures.

Theorem 2. Let $\alpha(s)$ be any non-degenerate equiaffine space curve with constant equiaffine curvatures such that $\kappa_\alpha(s) := \kappa_\alpha$ and $\tau_\alpha(s) := \tau_\alpha < 0$, then its affine normal mate curves:

– if $\alpha(s)$ is *type 2* curve, then its affine normal mate curve:

$$\beta(s^{*})=\left[\phi \mathrm{e}^{-\frac{\phi(-s^{*}+c)}{\sqrt[6]{-\tau_{\alpha}}}},\ \frac{(s^{*}-\phi c+\sqrt[6]{-\tau_{\alpha}})}{\sqrt[6]{-\tau_{\alpha}}}\mathrm{e}^{-\frac{\phi(-s^{*}+c)}{\sqrt[6]{-\tau_{\alpha}}}},\ \frac{1}{9\phi^{4}}\mathrm{e}^{\frac{2\phi(-s^{*}+c)}{\sqrt[6]{-\tau_{\alpha}}}}\right]$$

where $\phi = (\kappa_\alpha/2)^{1/3}$.

– if $\alpha(s)$ is *type 4* curve, then its affine normal mate curve:

$$\beta(s^{*})=\mathrm{e}^{-\frac{a(-s^{*}+c)}{\sqrt[6]{-\tau_{\alpha}}}}\left[\frac{-\mathrm{e}^{-\frac{a(-s^{*}+c)}{\sqrt[6]{-\tau_{\alpha}}}}}{b(9a^{2}+b^{2})(a^{2}+b^{2})},\ b\cos(w)-\sin(w),\ b\cos(w)+\sin(w)\right]$$

where

$$w=b(-s^{*}+c)/\sqrt[6]{-\tau_{\alpha}}$$

a and b are as in eqs. (6) and (7), respectively.

– if $\alpha(s)$ is *type 5* curve, then its affine normal mate curve:

$$\beta(s^{*})=\frac{}{\sqrt{}}\left\{-1,\cosh\left[(-\tau_{\alpha})^{1/3}(-s^{*}+c)\right],-\sinh\left[(-\tau_{\alpha})^{1/3}(-s^{*}+c)\right]\right\}$$

– if $\alpha(s)$ is *type 6* curve, then its affine normal mate curve:

$$\beta(s^{*})=\left[\frac{-\mathrm{e}^{-\frac{2m(-s^{*}+c)}{\sqrt[6]{-\tau_{\alpha}}}}}{2n(9m^{2}-n^{2})(m^{2}-n^{2})},\ (m+n)\mathrm{e}^{-\frac{(m+n)(-s^{*}+c)}{\sqrt[6]{-\tau_{\alpha}}}},\ (m-n)\mathrm{e}^{-\frac{(m-n)(-s^{*}+c)}{\sqrt[6]{-\tau_{\alpha}}}}\right]$$

where m and n are as in eqs. (8) and (9), respectively, for which $(27/2)\kappa_\alpha(-3\tau_\alpha)^{-3/2} \in (-1, 1)$.

When $\alpha(s)$ is the curve with zero equiaffine torsion then it does not have normal mate curve, in this case, these types of curves satisfy the differential equation:

$$\frac{\mathrm{d}^{4}\alpha(s)}{\mathrm{d}s^{4}}+\kappa_{\alpha}\frac{\mathrm{d}^{2}\alpha(s)}{\mathrm{d}s^{2}}=0 \tag{24}$$

We get the solution of ODE (24) for positive constant equiaffine curvature such:

$$\alpha(s)=c_{1}+c_{2}s+c_{3}\sin\left(\sqrt{\kappa_{\alpha}}s\right)+c_{4}\cos\left(\sqrt{\kappa_{\alpha}}s\right) \tag{25}$$

where $c_1 \in IR^3$ is any vector, $c_2, c_3, c_4 \in IR^3$ satisfy. Thus we get the following.

Corollary 1. The curves with $\tau_\alpha = 0$ and the curves with $\tau_\alpha > 0$, do not have affine normal mates.

Now we define the curve $\beta(s^*)$ as affine binormal mate curve of $\alpha(s)$ such that, $\beta(s^*)$ is tangent to binormal vectors of $\alpha(s)$ at each corresponding point along $\alpha(s)$, then we have the relation:

$$\beta(s^*) = \int B_\alpha(s)\mathrm{d}s \tag{26}$$

By differentiating eq. (26) with respect to s and by using eq. (3), we obtain the affine tangent, affine normal and affine binormal vectors of $\beta(s^*)$:

$$T_\beta(s^*) = \frac{1}{\sigma} B_\alpha(s) \tag{27}$$

$$N_\beta(s^*) = \frac{-\tau_\alpha}{\sigma^2} T_\alpha(s) - \frac{\kappa_\alpha}{\sigma^2} N_\alpha(s) - \frac{\sigma'}{\sigma^2} B_\alpha(s) \tag{28}$$

$$B_\beta(s^*) = PT_\alpha(s) + QN_\alpha(s) + RB_\alpha(s) \tag{29}$$

where

$$\sigma = \frac{\mathrm{d}s^*}{\mathrm{d}s} = \left\{\kappa'_\alpha \tau_\alpha - \tau'_\alpha \kappa_\alpha + (\tau_\alpha)^2\right\}^{1/6} \tag{30}$$

and

$$P = \frac{1}{\sigma^4}\left\{3\tau_\alpha \sigma' - \tau'_\alpha \sigma\right\},\ Q = \frac{1}{\sigma^4}\left\{3\kappa_\alpha \sigma' - \tau_\alpha \sigma - \kappa'_\alpha \sigma\right\},\ R = \frac{-1}{\sigma^4}\left\{\sigma\sigma'' - 3(\sigma')^2 + \kappa_\alpha \sigma\right\} \tag{31}$$

The condition of regularity of $\beta(s^*)$ requires that $\alpha(s)$ has to be the curve with non-zero equiaffine torsion. By differentiating eq. (29) with respect to s, we obtain:

$$\dddot{B}_\beta(s^*) = \frac{\left(P' - R\tau_\alpha\right)}{\sigma} T_\alpha(s) + \frac{\left(P + Q' - R\kappa_\alpha\right)}{\sigma} N_\alpha(s) + \frac{\left(Q + R'\right)}{\sigma} B_\alpha(s) \tag{32}$$

From the definition of equiaffine curvature and torsion \ and by using the eqs. (27)-(32), we get the curvatures for the curves with non-zero equiaffine torsion:

$$\kappa_\beta(s^*) = \frac{\left\{\begin{matrix}\left(2(\tau'_\alpha)^2 + \kappa''_\alpha \tau'_\alpha - \tau_\alpha \tau''_\alpha - \kappa'_\alpha \tau''_\alpha\right)\sigma^2 + \kappa_\alpha \sigma^8 + \sigma^8 \sigma'' - 3\sigma^7 (\sigma')^2 \\ +3\sigma^7 \sigma'' - 3\left(2\tau_\alpha \tau'_\alpha + \kappa''_\alpha \tau_\alpha\right)\sigma' \sigma'' + 3\kappa_\alpha \tau''_\alpha \sigma\sigma' + 6\sigma^6 (\sigma')^2\end{matrix}\right\}}{\sigma^{10}} \tag{33}$$

and

$$\tau_\beta(s^*) = \frac{\left\{\begin{matrix}\sigma^4\left\{\sigma^3 \sigma''' + 3\sigma\sigma'\sigma'' - 2\sigma^2 \sigma'\sigma'' + 6(\sigma')^3 - 9\sigma(\sigma')^3 + \kappa_\alpha \sigma^2 \sigma' + 2\kappa'_\alpha \sigma^3 + \tau_\alpha \sigma^3\right\} \\ +\sigma'\left\{2(\tau'_\alpha)^2 - \tau_\alpha \tau''_\alpha + \kappa''_\alpha \tau'_\alpha - \kappa'_\alpha \tau''_\alpha\right\} - 6\sigma^4 \sigma'\left\{\sigma^2 \sigma'' - 3\sigma(\sigma')^2 + 3(\sigma')^2\right\} + \kappa_\alpha \sigma^3\end{matrix}\right\}}{\sigma^{10}} \tag{34}$$

If the curve $\alpha(s)$ has the constant affine curvatures, then from eq. (30), $\sigma = (\tau_\alpha)^{1/3}$ is constant and from the eqs. (33) and (34):

$$\kappa_\beta(s^*) = \kappa_\alpha$$

and

$$\tau_\beta(s^*) = 1$$

with the parameterization:

$$s = \frac{s^* + c}{(\tau_\alpha)^{1/3}}$$

Regularity of $\beta(s^*)$ requires $\tau_\alpha \neq 0$, so the curves satisfy eq. (24) do not have any binormal mate curves. Two of them are type 1 curve and the curve given in eq. (25). Thus, we give the following.

Corollary 2. The curves with $\tau_\alpha = 0$ do not have affine binormal mates.

From the main classification of curves with constant equiaffine curvatures, we give following theorem.

Theorem 3. Let $\alpha(s)$ be any non-degenerate equiaffine space curve with constant equiaffine curvatures such that $\kappa_\alpha(s) := \kappa_\alpha$ and $\tau_\alpha(s) := \tau_\alpha \neq 0$, then its affine binormal mate curves:

– if $\alpha(s)$ is *type 2* curve, then its affine binormal mate curve:

$$\beta(s^*) = \left[\phi^2 e^{-\frac{\phi(-s^*+c)}{(\tau_\alpha)^{1/3}}}, \frac{\phi(\phi s^* - \phi c + 2(\tau_\alpha)^{1/3})e^{-\frac{\phi(-s^*+c)}{(\tau_\alpha)^{1/3}}}}{(\tau_\alpha)^{1/3}}, \frac{-2}{9\phi^3} e^{\frac{2\phi(-s^*+c)}{(\tau_\alpha)^{1/3}}} \right]$$

where $\phi = (\kappa_\alpha/2)^{1/3}$,

– if $\alpha(s)$ is *type 3* curve with $\tau_\alpha > 0$, then its affine binormal mate curve:

$$\beta(s^*) = \left\{0,\ \sin\left[(\tau_\alpha)^{1/6}(-s^* + c)\right],\ -\cos\left[(\tau_\alpha)^{1/6}(-s^* + c)\right]\right\}$$

– if $\alpha(s)$ is *type 4* curve, then its affine binormal mate curve:

$$\beta(s^*) = \left[\frac{2a e^{(\tau_\alpha)^{1/3}}}{b(9a^2 + b^2)(a^2 + b^2)}, -\gamma e^{-\frac{a(-s^*+c)}{(\tau_\alpha)^{1/3}}}, \eta e^{-\frac{a(-s^*+c)}{(\tau_\alpha)^{1/3}}} \right]$$

where

$$\gamma = (a^2 - b^2)\sin\left[\frac{b(-s^* + c)}{(\tau_\alpha)^{1/3}}\right] - 2ab\cos\left[\frac{b(-s^* + c)}{(\tau_\alpha)^{1/3}}\right]$$

$$\eta = (a^2 - b^2)\cos\left[\frac{b(-s^* + c)}{(\tau_\alpha)^{1/3}}\right] + 2ab\sin\left[\frac{b(-s^* + c)}{(\tau_\alpha)^{1/3}}\right]$$

a and b are as in eqs. (6) and (7), respectively.

– if $\alpha(s)$ is *type 5* curve with $\tau_\alpha < 0$ then its affine binormal mate curve:

$$\beta(s^*) = \left\{0, -\sinh\left[\frac{(-\tau_\alpha)^{1/2}(-s^* + c)}{(\tau_\alpha)^{1/3}}\right], \cosh\left[\frac{(-\tau_\alpha)^{1/2}(-s^* + c)}{(\tau_\alpha)^{1/3}}\right]\right\}$$

– if $\alpha(s)$ is type 6 curve with $\tau_\alpha < 0$ then its affine binormal mate curve:

$$\beta(s^*) = \left[\frac{m e^{\frac{2m(-s^*+c)}{(\tau_\alpha)^{1/3}}}}{n(9m^2 - n^2)(m^2 - n^2)}, (m+n)^2 e^{-\frac{(m+n)(-s^*+c)}{(\tau_\alpha)^{1/3}}}, (m-n)^2 e^{-\frac{(m-n)(-s^*+c)}{(\tau_\alpha)^{1/3}}} \right]$$

where m and n are as in eqs. (8) and (9), respectively, for which $(27/2)\kappa_\alpha(-3\tau_\alpha)^{-3/2} \in (-1, 1)$.

Here we give some examples of natural mates of equiaffine space curves such as the main curve $\alpha(s)$ in black, its affine normal mate in red and its affine binormal mate in blue, figs. 1 and 2.

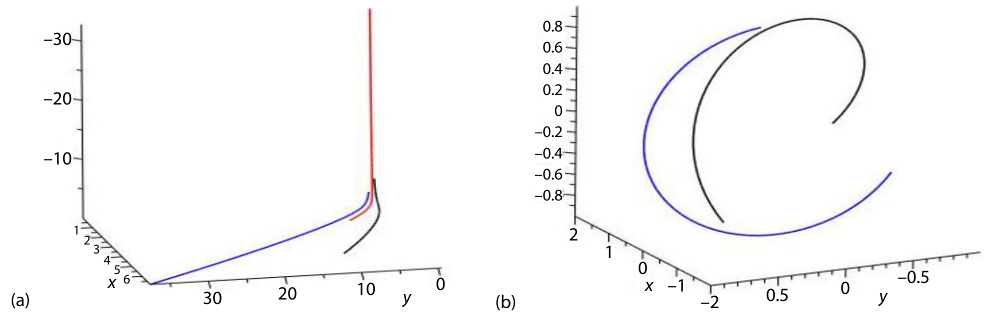

Figure 1. (a) Natural mates of type 2 curves for $\kappa_\alpha = 1$, $\tau_\alpha = 0$, $c = 1$, and $-2 < s < 2$, (b) natural mates of type 3 curves for $\kappa_\alpha = 0$, $\tau_\alpha = 1$, $c = 1$, and $-2 < s < 2$

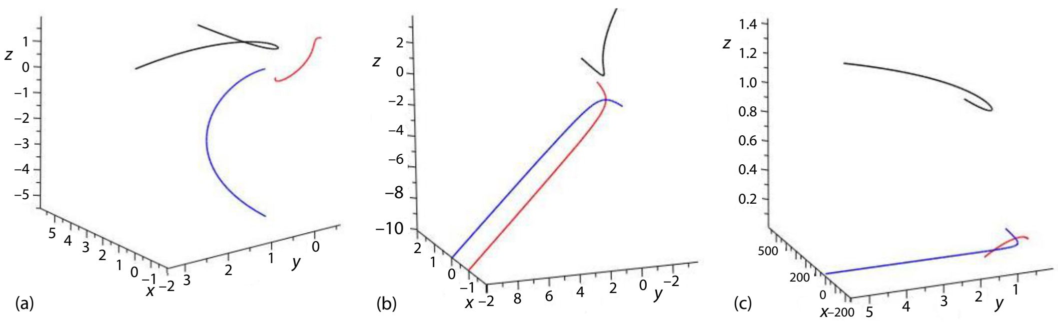

Figure 2. (a) Natural mates of type 4 curves for $\kappa_\alpha = 1$, $\tau_\alpha = -1$, $c = 1$, and $-2 < s < 2$, (b) natural mates of type 5 curves for $\kappa_\alpha = 0$, $\tau_\alpha = -1$, $c = 1$, and $-2 < s < 2$, and (c) natural mates of type 6 curves for $\kappa_\alpha = 1/27$, $\tau_\alpha = -1/3$, $c = 1$, and $-\pi < s < \pi$

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